ON SURJECTIVITY IN TENSOR TRIANGULAR GEOMETRY

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Abstract. We prove that a jointly conservative family of geometric functors between rigidly-compactly generated tensor triangulated categories induces a surjective map on spectra. From this we deduce a fiberwise criterion for Balmer's comparison map to be a homeomorphism. This gives short alternative proofs of the Hopkins–Neeman theorem and Lau's theorem for the trivial action.

Throughout this note, we work in the context of rigidly-compactly generated tensor triangulated (*tt*) categories, usually denoted by $\mathcal{S}$ or $\mathcal{T}$. We write $\mathrm{Spc}(\mathcal{T}^c)$ for the associated Balmer spectrum of compact (=dualizable) objects and freely use basic constructions from tt-geometry [Bal05, Bal10]. A coproduct-preserving tensor triangulated functor $f^*\colon \mathcal{T} \to \mathcal{S}$ is called a *geometric functor*. Such a functor preserves compact objects and hence induces a continuous map $\mathrm{Spc}(\mathcal{S}^c) \to \mathrm{Spc}(\mathcal{T}^c)$. Following terminology introduced in [Bal20a, CSY22], a *weak ring* in $\mathcal{T}$ is an object $R \in \mathcal{T}$ equipped with a map $\eta\colon \mathbb{1} \to R$ from the unit object such that the induced map $R \otimes \eta\colon R \to R \otimes R$ is a split monomorphism.

1.1. *Definition.* Suppose $\{f_i^*\colon \mathcal{T} \to \mathcal{S}_i\}_{i\in I}$ is a family of geometric functors between rigidly-compactly generated tt-categories. We say the family is

- *jointly conservative* if for any $t \in \mathcal{T}$, $f_i^*(t) = 0$ for all $i \in I$ implies $t = 0$;
- *jointly nil-conservative* if for any weak ring $R \in \mathcal{T}$, $f_i^*(R) = 0$ for all $i \in I$ implies $R = 0$.

Note that any jointly conservative family is in particular jointly nil-conservative. The converse does not hold:

1.2. *Example.* The Morava K-theories $\{K(n) \otimes -\colon \mathrm{Sp} \to \mathrm{Mod}(K(n))\}_{n\in\mathbb{N}\cup\{\infty\}}$ are jointly nil-conservative as a consequence of the nilpotence theorem [HS98, Theorem 3] but they are not jointly conservative since they all annihilate the Brown–Comenetz dual of the sphere [HS99, Corollary B.12].

1.3. **Theorem.** *If $\{f_i^*\colon \mathcal{T} \to \mathcal{S}_i\}_{i\in I}$ is a jointly nil-conservative family of geometric functors, then the induced map*[1]

$$\varphi\colon \coprod_{i\in I} \mathrm{Spc}(\mathcal{S}_i^c) \to \mathrm{Spc}(\mathcal{T}^c) \tag{1.4}$$

is surjective.

TB is supported by the European Research Council (ERC) under Horizon Europe (grant No. 101042990). NC is partially supported by Spanish State Research Agency project PID2020-116481GB-I00, the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M), and the CERCA Programme/Generalitat de Catalunya. DH is supported by grant number TMS2020TMT02 from the Trond Mohn Foundation. BS is supported by NSF grant DMS-1903429.

[1] Throughout this paper, coproducts are taken in the category of topological spaces (as opposed to the category of spectral spaces).

1.5. *Remark.* If the family is finite, then this result can be deduced from the criterion [Bal18, Theorem 1.3] by first proving that the geometric functor $\prod_i f_i^*$ detects tensor nilpotence of morphisms with dualizable source, as in [BCH$^+$23, Section 2.3]. For an infinite family, such an argument cannot work directly, because $\mathrm{Spc}(\prod_{i\in I}\mathcal{S}_i^c) \neq \bigsqcup_{i\in I}\mathrm{Spc}(\mathcal{S}_i^c)$ whenever infinitely many of the $\mathcal{S}_i$ are non-trivial. Indeed, the spectrum $\mathrm{Spc}(\prod_{i\in I}\mathcal{S}_i^c)$ is a spectral space and in particular quasi-compact, while an infinite coproduct of non-empty spaces cannot be quasi-compact. Here we use implicitly that $(\prod_{i\in I}\mathcal{S}_i)^c \simeq \prod_{i\in I}\mathcal{S}_i^c$; see for example the proof of [Lur09, Proposition 5.5.7.6] combined with [Lur09, Proposition 5.5.7.8] for the corresponding ∞-categorical statement.

1.6. *Example.* If $\{k_i\}_{i\in I}$ is a family of fields, then the Zariski spectrum $\mathrm{Spec}(\prod_{i\in I}k_i)$ is homeomorphic to the Stone–Čech compactification of I.

1.7. *Remark.* Using the Balmer–Favi support [BF11] and the techniques of [BCHS23], it is possible to prove Theorem 1.3 for arbitrary indexing sets I under the additional hypothesis that $\mathrm{Spc}(\mathcal{T}^c)$ is weakly noetherian. However, since the construction of a surjective map as in (1.4) is often the first step in understanding $\mathrm{Spc}(\mathcal{T}^c)$, making any assumption on its topology is not desirable. Consequently, our proof relies on a suitable support theory for big objects which exists unconditionally without any point-set assumptions on $\mathrm{Spc}(\mathcal{T}^c)$. Such a theory is provided by the homological residue fields developed in [BKS19, Bal20b, Bal20a], from which we will draw freely. Indeed, we will derive Theorem 1.3 as a corollary of the following more complete statement:

1.8. **Theorem.** *A family $\{f_i^*\colon \mathcal{T}\to\mathcal{S}_i\}_{i\in I}$ of geometric functors is jointly nil-conservative if and only if the induced map on homological spectra*

$$\varphi^h\colon \bigsqcup_{i\in I}\mathrm{Spc}^h(\mathcal{S}_i^c)\to\mathrm{Spc}^h(\mathcal{T}^c) \tag{1.9}$$

is surjective.

Proof. ($\Rightarrow$): Let $\mathcal{B}\in\mathrm{Spc}^h(\mathcal{T}^c)$ be a homological prime and consider the associated weak ring $E_{\mathcal{B}}\neq 0$; see [BKS19, Section 3]. By assumption, there exists some $i\in I$ such that $f_i^*(E_{\mathcal{B}})\neq 0$. For simplicity, write $f^*\coloneqq f_i^*$ and f_* for its right adjoint. By the unit-counit identity and the projection formula [BDS16, (2.16)], we deduce

$$f_*(\mathbb{1})\otimes E_{\mathcal{B}}\simeq f_*f^*(E_{\mathcal{B}})\neq 0.$$

Note that as a right adjoint to a tt-functor, f_* is lax symmetric monoidal, hence $f_*(\mathbb{1})$ is a weak ring in $\mathcal{T}$. Since the homological support coincides with the naive homological support for weak rings [Bal20a, Theorem 4.7], this implies that $\mathcal{B}\in\mathrm{Supp}^h(f_*(\mathbb{1}))$. By [Bal20a, Theorem 5.12], we conclude that

$$\mathcal{B}\in\mathrm{Supp}^h(f_*(\mathbb{1}))=\mathrm{im}(\mathrm{Spc}^h(f^*)),$$

thereby verifying that (1.9) is surjective.

($\Leftarrow$): If $R\in\mathcal{T}$ is a nonzero weak ring, then $\mathrm{Supp}^h(R)\neq\varnothing$ by [Bal20a, Thm. 1.8]. Hence, if (1.9) is surjective then there exists an $i\in I$ such that

$$\mathrm{im}(\mathrm{Spc}^h(f_i^*))\cap\mathrm{Supp}^h(R)\neq\varnothing.$$

By [Bal20a, Theorem 1.2(d) and Theorem 1.9], this implies $\mathrm{Supp}^h(f_{i*}f_i^*(R)) = \mathrm{Supp}^h(f_{i*}(\mathbb{1})\otimes R)=\mathrm{Supp}^h(f_{i*}(\mathbb{1}))\cap\mathrm{Supp}^h(R)\neq\varnothing$ so that $f_i^*(R)\neq 0$. ∎

Proof of Theorem 1.3. In order to deduce Theorem 1.3 from Theorem 1.8, we employ the naturality of the homological comparison map ϕ from [Bal20a, Theorem 5.10], resulting in a commutative square:

$$\begin{array}{ccc} \bigsqcup_{i\in I}\operatorname{Spc}^h(\mathcal{S}_i^c) & \xrightarrow{\varphi^h} & \operatorname{Spc}^h(\mathcal{T}^c) \\ {\scriptstyle \sqcup\phi_{\mathcal{S}_i}}\downarrow & & \downarrow{\scriptstyle \phi_{\mathcal{T}}} \\ \bigsqcup_{i\in I}\operatorname{Spc}(\mathcal{S}_i^c) & \xrightarrow[\varphi]{} & \operatorname{Spc}(\mathcal{T}^c). \end{array}$$

By [Bal20b, Corollary 3.9], the vertical maps are surjective, and so is the top horizontal map by Theorem 1.8. It follows that φ is also surjective. □

1.10. *Remark.* It is an open question whether the converse to Theorem 1.3 holds, that is, whether the surjectivity of φ in (1.4) implies that the family $\{f_i^*\}_{i\in I}$ is jointly nil-conservative. It is known that the family need not be jointly conservative (see [BCHS23, Example 14.26]). In light of Theorem 1.8, the converse of Theorem 1.3 would follow from Balmer's "Nerves of Steel" Conjecture that the homological and tensor triangular spectra always coincide; see [BHS21a].

1.11. *Remark.* While Theorem 1.3 is in general not enough to determine the topology on $\operatorname{Spc}(\mathcal{T}^c)$ even when φ is a bijection (see for instance [BHS21b, Remark 15.12]), there are cases in which it can be used to compute the topology. Recall that Balmer [Bal10] constructs a natural *comparison map*

$$\rho_{\mathcal{T}}\colon \operatorname{Spc}(\mathcal{T}^c) \to \operatorname{Spec}^h(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}))$$

between the tensor triangular spectrum and the Zariski spectrum of the graded endomorphism ring of the unit object. If $\mathcal{T}$ is *noetherian* in the sense that $\operatorname{End}^*_{\mathcal{T}}(C)$ is noetherian as an $\operatorname{End}^*_{\mathcal{T}}(\mathbb{1})$-module for each $C\in\mathcal{T}^c$, then $\rho_{\mathcal{T}}$ is a homeomorphism if and only if it is a bijection [Lau21, Corollary 2.8]. The following result provides a 'fiberwise' criterion for Balmer's comparison map to be a homeomorphism:

1.12. **Corollary.** *Let $\mathcal{T}$ be a noetherian rigidly-compactly generated tt-category and consider a family of geometric tt-functors $\{f_i^*\colon \mathcal{T}\to\mathcal{S}_i\}_{i\in I}$ satisfying the following properties:*

(a) the family $\{f_i^*\}_{i\in I}$ is jointly nil-conservative;
(b) $\rho_{\mathcal{S}_i}$ is a bijection for all $i\in I$;
(c) the induced map on Zariski spectra

$$\bigsqcup_{i\in I}\operatorname{Spec}^h(\operatorname{End}^*_{\mathcal{S}_i}(\mathbb{1})) \to \operatorname{Spec}^h(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1}))$$

is a bijection.

Then $\rho_{\mathcal{T}}$ is a homeomorphism.

Proof. Naturality of the comparison map yields a commutative diagram

$$\begin{array}{ccc} \bigsqcup_{i\in I}\operatorname{Spc}(\mathcal{S}_i^c) & \xrightarrow{\quad\varphi\quad} & \operatorname{Spc}(\mathcal{T}^c) \\ {\scriptstyle \sqcup\rho_{\mathcal{S}_i}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathcal{T}}} \\ \bigsqcup_{i\in I}\operatorname{Spec}^h(\operatorname{End}^*_{\mathcal{S}_i}(\mathbb{1})) & \longrightarrow & \operatorname{Spec}^h(\operatorname{End}^*_{\mathcal{T}}(\mathbb{1})). \end{array}$$

On the one hand, by assumption, both the left vertical and the bottom horizontal maps are bijections, so φ has to be injective. On the other hand, Theorem 1.3 implies that φ is also surjective, hence bijective. It follows that $\rho_{\mathcal{T}}$ is a bijection and thus a homeomorphism because $\mathcal{T}$ is noetherian. □

1.13. *Remark.* Corollary 1.12 offers an alternative perspective on the Hopkins–Neeman theorem [Hop87, Nee92] for noetherian commutative rings:

1.14. *Example.* Let $\mathrm{D}(R)$ be the derived category of a noetherian commutative ring R. For any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, consider the residue field $\kappa(\mathfrak{p})$, constructed as the quotient field of $R/\mathfrak{p}$, and write $f_{\mathfrak{p}}^*\colon \mathrm{D}(R) \to \mathrm{D}(\kappa(\mathfrak{p}))$ for the associated base-change functor. We claim that the family $\{f_{\mathfrak{p}}^*\}_{\mathfrak{p}\in\mathrm{Spec}(R)}$ satisfies the assumptions of Corollary 1.12. Indeed, (b) and (c) are immediate: since $\kappa(\mathfrak{p})$ is a field, $\rho_{\mathrm{D}(\kappa(\mathfrak{p}))}$ is a bijection (between singletons), while (c) holds by construction. Finally, the family $\{f_{\mathfrak{p}}^*\}$ is jointly conservative by [Nee92, Lemma 2.12],[1] which verifies (a). Therefore, the comparison map

$$\rho_{\mathrm{D}(R)}\colon \mathrm{Spc}(\mathrm{D}(R)) \xrightarrow{\sim} \mathrm{Spec}(R)$$

is a homeomorphism. □

1.15. *Remark.* The extension to arbitrary commutative rings follows by absolute noetherian approximation as in Thomason's work [Tho97]; cf. [Lau21, Lemma 2.12].

1.16. *Example.* Let G be a finite group and let R be a noetherian commutative ring equipped with trivial G-action. We write $\mathrm{Rep}(G,R)$ for the tt-category of R-linear derived representations of G introduced in [Bar21]. This category is noetherian and rigidly-compactly generated with subcategory of compact objects given by $\mathrm{D}^b(\mathrm{mod}(G,R))$, the bounded derived category of $R[G]$-modules whose underlying complex of R-modules is perfect. If k is a field, then $\mathrm{Rep}(G,k)$ coincides with the homotopy category of unbounded chain complexes of injective $k[G]$-modules studied in [BK08]; for an extension of this homological model to coefficients in R, see [BBI$^+$23].

For any prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, there is a geometric fiber-functor

$$F_{\mathfrak{p}}^*\colon \mathrm{Rep}(G,R) \to \mathrm{Rep}(G,\kappa(\mathfrak{p})),$$

which is induced by base-change along the canonical map $R \to \kappa(\mathfrak{p})$. We claim that the family $\{F_{\mathfrak{p}}^*\}_{\mathfrak{p}\in\mathrm{Spec}(R)}$ satisfies the conditions of Corollary 1.12. Indeed, the joint conservativity of the family is the content of [BBI$^+$23, Proposition 3.23], while $\rho_{\mathrm{D}^b(k[G])}$ is a homeomorphism by [BCR97] for any field k. It remains to verify condition (c). To this end, note that the map on Zariski spectra induced by $F_{\mathfrak{p}}^*$ identifies with the composite

$$\mathrm{Spec}^h(H^*(G,\kappa(\mathfrak{p}))) \xrightarrow{\sim} \mathrm{Spec}^h(H^*(G,R)\otimes_R \kappa(\mathfrak{p})) \to \mathrm{Spec}^h(H^*(G,R)).$$

The first map is a homeomorphism by [Lau21, Corollary 8.29]; see also [BIKP22, Corollary 5.6]. Varying the second map over $\mathrm{Spec}(R)$ assembles into a bijection

$$\bigsqcup_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Spec}^h(H^*(G,R)\otimes_R \kappa(\mathfrak{p})) \xrightarrow{\sim} \mathrm{Spec}^h(H^*(G,R)),$$

which verifies (c) of Corollary 1.12 for $\{F_{\mathfrak{p}}^*\}_{\mathfrak{p}\in\mathrm{Spec}(R)}$. We conclude that $\rho_{\mathrm{D}^b(\mathrm{mod}(G,R))}$ is a homeomorphism. □

[1] Note that the proof of this lemma does not rely on the nilpotence theorem or the thick subcategory theorem for $\mathrm{D}(R)$.

1.17. *Remark.* Example 1.16 recovers the main theorem of [Lau21] for rings equipped with trivial G-action — modulo the straightforward reduction from the general case to the case where R is noetherian, as explained at the beginning of the proof of [Lau21, Theorem 11.1]. We remark that the key input to our proof is the joint conservativity of the functors $\{F_{\mathfrak{p}}^*\}$ on the 'big' categories $\mathrm{Rep}(G,R)$ and emphasize that the proof of this does not rely on the stratification of $\mathrm{Rep}(G,k)$.

1.18. *Remark.* The previous example extends to any finite flat group scheme over a noetherian commutative ring; cf. [BBI+23]. The key input is the recent generalization of the Friedlander–Suslin theorem [FS97] due to van der Kallen [vdK22].

Acknowledgements. We readily thank Paul Balmer for helpful conversations and Julia Pevtsova for raising the question of whether 'fiberwise phenomena' admit a general tt-geometric explanation. We would also like to thank the Max Planck Institute and the Hausdorff Research Institute for Mathematics for their hospitality in the context of the Trimester program *Spectral Methods in Algebra, Geometry, and Topology* funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2047/1 – 390685813.

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