COSUPPORT IN TENSOR TRIANGULAR GEOMETRY

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Abstract. We develop a theory of cosupport and costratification in tensor triangular geometry. We study the geometric relationship between support and cosupport, provide a conceptual foundation for cosupport as categorically dual to support, and discover surprising relations between the theory of costratification and the theory of stratification. We prove that many categories in algebra, topology and geometry are costratified by developing and applying descent techniques. An overarching theme is that cosupport is relevant for diverse questions in tensor triangular geometry and that a full understanding of a category requires knowledge of both its support and its cosupport.

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1. Introduction

One approach for comparing the objects of a given category is via a support theory. The prototypical example is given by the support of a module in commutative algebra, which inspired similar constructions in related areas such as algebraic geometry, representation theory, and chromatic homotopy theory, to name just a few. The common perspective is to view the objects of the category as "bundles" over a geometric base space and record the points at which an object does not vanish. These theories of support have been remarkably successful in organizing the objects being studied, especially in derived and homotopical contexts [BCR97, Tho97, HS98].

Tensor triangular geometry [Bal10] puts these developments in a unified framework: It regards a tensor-triangulated category $\mathcal{T}$ as a bundle over a certain space, its Balmer spectrum, and constructs the universal theory of support for the compact objects of $\mathcal{T}$. This universal theory of support classifies the compact objects up to how they build each other using the naturally available categorical structure. The theory was extended beyond compact objects to "infinite-dimensional" contexts in [Nee92a, HPS97, BF11, BIK08, Ste13, BHS23]. The theory of support is more subtle for non-compact objects but in desirable situations extends the classification of compact objects to all objects. This is characterized by a property of the category called stratification.

The goal of this paper is to systematically develop a dual theory of cosupport in tensor triangular geometry as well as the accompanying notion of costratification. Conceptually, this may be motivated from three complementary points of view:

- (Geometric) In algebraic geometry, the support of a quasi-coherent sheaf is captured by its local cohomology. The cosupport corresponds to local homology. Grothendieck's local duality expresses the relation between the two notions through an adjunction, which can be formalized and provides a definition of cosupport in more general settings. This is how cosupport has traditionally been introduced into the literature.
- (Constructive) Theories of support provide an approach to understanding infinite-dimensional objects in terms of how they build each other using the naturally available "finite" structure together with infinite coproducts. This amounts to studying the localizing ideals of the category. While the idea of generating objects using coproducts is instilled in us from birth, it is not the only choice: We can also gain insight by considering how objects build each other using other constructions, such as infinite products. This leads to studying the colocalizing coideals of the category, and is the organizational framework that cosupport provides.
- (Categorical) The cosupport of a compactly generated category $\mathcal{T}$ can be understood as the support associated to the opposite category $\mathcal{T}^{\mathrm{op}}$. This provides a very conceptual understanding of cosupport which has been missing from the literature. It also highlights part of the subtlety of the theory as $\mathcal{T}^{\mathrm{op}}$ fails to be compactly generated in all but trivial cases.

Superficially, this seems to suggest that cosupport and its properties are merely a formal variant of the already established theory of support. However, this conclusion would be false, as we will demonstrate throughout this work. Moreover, it turns out that cosupport appears naturally even in situations where one is only interested in support, and only the combination of both provides a full picture.

Our construction of cosupport takes place in the context of a rigidly-compactly generated tensor-triangulated ("tt") category $\mathcal{T}$ whose Balmer spectrum of compact objects $\mathrm{Spc}(\mathcal{T}^c)$ is weakly noetherian (a mild point-set topological assumption introduced in [BHS23] which simultaneously generalizes noetherian and profinite). It takes the form of an assignment

$$\mathrm{Cosupp}\colon \big\{\text{objects of } \mathcal{T}\big\} \longrightarrow \big\{\text{subsets of } \mathrm{Spc}(\mathcal{T}^c)\big\}$$

satisfying a number of compatibilities with respect to the tensor-triangulated structure of $\mathcal{T}$. We briefly and informally summarize the main features of our theory of cosupport as follows:

- Cosupport illuminates key properties of support. For instance, the local-to-global principle for support is equivalent to the detection property for cosupport, and stratification can be characterized in terms of the combined behaviour of support and cosupport. The behaviour of cosupport also reflects the topology of the Balmer spectrum and thus provides insight into the structure of compact objects. More generally, we study the geometric relationship between support and cosupport and how they are intertwined through intrinsic dualities in $\mathcal{T}$.
- Cosupport affords an accompanying notion of costratification, which attempts to parameterize colocalizing coideals of $\mathcal{T}$ in terms of subsets of the Balmer spectrum. The basic properties of this theory are dual to those of stratification as developed in [BHS23]. For example, we prove that our theory of cosupport is the universal choice for classifying colocalizing coideals. In particular, this implies that if $\mathcal{T}$ is costratified in the sense of Benson–Iyengar–Krause [BIK12] then it is also costratified in our sense, but there are many classes of examples for which the converse fails.
- We clarify the relation between stratification and costratification, discovering a surprising asymmetry between the corresponding notions of (co)detection and (co)local-to-global principle. Nevertheless, while costratification is an *a priori* stronger property than stratification, we verify that all known examples of stratified categories are also costratified. These results are obtained as applications of general descent techniques. Our methods provide streamlined proofs of all known classifications of colocalizing coideals in the literature, and also establish new ones which were not previously accessible.
- We unify support and cosupport by showing that they both arise as particular instances of a more general notion of support defined at a level of generality which encompasses both $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$. This leads to a deeper conceptual understanding of cosupport as simply the support of the opposite category.

Content and summary of main results. We now proceed to give a more detailed outline of the main results of the paper. These can be loosely organized into six interconnected themes, as follows. Note however that the story being told here does not faithfully reflect the linear structure of the document, for which we instead refer to the end of the introduction.

Hypothesis. Throughout the introduction, $(\mathcal{T}, \otimes, \mathbb{1})$ will denote a rigidly-compactly generated tt-category with weakly noetherian spectrum $\mathrm{Spc}(\mathcal{T}^c)$. We will denote the internal hom of $\mathcal{T}$ by $\mathsf{hom}(-,-)$.

Theme I. Cosupport and costratification. The common starting point for the definition of support and cosupport in our tt-geometric setting is the existence of a suitable supply of idempotents $g_{\mathcal{P}} \in \mathcal{T}$ which can be used to isolate attention at each point $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$. The construction of these idempotents relies on our topological assumption that $\mathrm{Spc}(\mathcal{T}^c)$ is weakly noetherian. We then define the support and cosupport of an object $t \in \mathcal{T}$ (Definition 4.23) as the following subsets of $\mathrm{Spc}(\mathcal{T}^c)$:

$$\mathrm{Supp}(t) = \left\{\, \mathcal{P} \;\middle|\; g_{\mathcal{P}} \otimes t \neq 0 \,\right\} \quad \text{and} \quad \mathrm{Cosupp}(t) = \left\{\, \mathcal{P} \;\middle|\; \mathsf{hom}(g_{\mathcal{P}}, t) \neq 0 \,\right\}.$$

This definition of support is due to Balmer–Favi [BF11] and was studied in [BHS23]. The definition of cosupport is inspired by the constructions of [HS99], [Nee11] and especially [BIK12], where a similar definition is considered in the context of a triangulated category equipped with an auxiliary action of a commutative noetherian ring. We first extract the elementary properties the function $\mathrm{Cosupp}(t)$ satisfies and thereby formulate the axiomatic notion of a cosupport theory (Definition 3.1 and Proposition 4.25). This is summarized as follows:

Theorem. *Cosupport satisfies the conditions of an axiomatic cosupport theory:*

(a) $\mathrm{Cosupp}(0) = \varnothing$ *and* $\mathrm{Cosupp}(\mathcal{T}) = \mathrm{Spc}(\mathcal{T}^c)$*;*
(b) $\mathrm{Cosupp}(\Sigma t) = \mathrm{Cosupp}(t)$ *for every* $t \in \mathcal{T}$*;*
(c) $\mathrm{Cosupp}(c) \subseteq \mathrm{Cosupp}(a) \cup \mathrm{Cosupp}(b)$ *for any exact triangle* $a \to b \to c$ *in* $\mathcal{T}$*;*
(d) $\mathrm{Cosupp}(\prod_{i \in I} t_i) = \bigcup_{i \in I} \mathrm{Cosupp}(t_i)$ *for any set of objects* t_i *in* $\mathcal{T}$*;*
(e) $\mathrm{Cosupp}(\mathsf{hom}(s,t)) \subseteq \mathrm{Cosupp}(t)$ *for all* $s, t \in \mathcal{T}$*.*

Although basic, formulating the axioms correctly is subtle due to the asymmetric interaction between cosupport and support discussed in Theme II below, which is hinted at by the unusual form of axiom (e). The key principle which guides our choice of axioms is the observation that, while the collection of objects supported on a given set forms a localizing ideal, the collection of objects cosupported on a given set forms a colocalizing coideal (Definition 2.2). The above axioms (excluding $\mathrm{Cosupp}(\mathcal{T}) = \mathrm{Spc}(\mathcal{T}^c)$) are in fact equivalent to the statement that the collection of objects cosupported on a given set form a colocalizing coideal. Cosupport is thus intimately related to the study of colocalizing coideals in the same way that support is intimately related to the study of localizing ideals.

In [BHS23] we defined the category $\mathcal{T}$ to be stratified if the map

$$\mathrm{Supp}\colon \big\{\text{localizing ideals of } \mathcal{T}\big\} \longrightarrow \big\{\text{subsets of } \mathrm{Spc}(\mathcal{T}^c)\big\}$$

induced by the Balmer–Favi support is a bijection. Although one could consider the analogous statement for any support theory on $\mathcal{T}$, we proved that the Balmer–Favi notion of support provides the universal choice of support theory for the purposes of stratifying $\mathcal{T}$. This justifies defining stratification as a property of the category as above, rather than as a notion relative to a choice of auxiliary support theory.

Similarly, we say that $\mathcal{T}$ is costratified (Definition 7.1) if the map

$$\mathrm{Cosupp}\colon \big\{\text{colocalizing coideals of } \mathcal{T}\big\} \longrightarrow \big\{\text{subsets of } \mathrm{Spc}(\mathcal{T}^c)\big\}$$

induced by our tensor triangular cosupport theory is bijective. This intrinsic definition of costratification as a property of $\mathcal{T}$ is justified by a corresponding universality result for our cosupport theory:

Theorem (Informal version)**.** *Cosupport is the universal choice among costratifying cosupport theories for* $\mathcal{T}$ *which is compatible with the usual classification of compact* *objects in $\mathcal{T}$. In particular, if $\mathcal{T}$ is costratified in the sense of* [BIK12]*, then it is also costratified in our sense.*

More precise statements are in Corollary 11.10 and Corollary 11.12. We merely remark in passing that this universality result for cosupport is more subtle than the corresponding result for support, since the restriction of Cosupp to the compact objects does not in general coincide with Balmer's universal support for compact objects. Examples abound of tt-categories which are costratified in our sense but not in the sense of [BIK12], and so—in our tensor triangular setting—our theory of costratification is strictly more general than that of [BIK12].

In order to discuss our results further, we need additional preparation. Recall that the spectrum of $\mathcal{T}$ controls its global geometric structure through a divide and conquer approach: First decompose $\mathcal{T}$ into pieces $\Gamma_{\mathcal{P}}\mathcal{T}$ that capture those objects with support concentrated at a single point $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$ and secondly study these individual pieces (or "stalks"). Intuitively, the local-to-global principle stipulates that $\mathcal{T}$ can be reconstructed from its local pieces. In other words, each object of $\mathcal{T}$ can be built from its stalks. In particular, it implies the detection property: $\operatorname{Supp}(t) = \varnothing$ if and only if $t = 0$. If $\mathcal{T}$ satisfies the local-to-global principle, there is then a condition on the local pieces $\Gamma_{\mathcal{P}}\mathcal{T}$ which characterizes when the category $\mathcal{T}$ is stratified, namely that the stalks $\Gamma_{\mathcal{P}}\mathcal{T}$ are minimal as localizing ideals. This is the beginning to the theory of stratification developed systematically in [BHS23].

Now we can consider the analogue of these notions based on cosupport: We define the colocalizing coideal $\Lambda^{\mathcal{P}}\mathcal{T}$ of $\mathcal{T}$ consisting of all objects which are cosupported at a single point $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$ and we have a corresponding colocal-to-global principle (Definition 6.1) which morally states that every object can be built from these "costalks". This in turn implies the codetection property (Definition 5.5) which states that $\operatorname{Cosupp}(t) = \varnothing$ if and only if $t = 0$. In complete analogy with the theory of stratification we have (Theorem 7.7):

Theorem. *The following conditions are equivalent:*

(a) $\mathcal{T}$ is costratified;
(b) $\mathcal{T}$ satisfies the colocal-to-global principle, and the colocalizing coideal $\Lambda^{\mathcal{P}}\mathcal{T}$ is minimal for each $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$.

In this way, the theory of costratification has the same basic features as the theory of stratification developed in [BHS23].

Theme II. Asymmetry between stratification and costratification. Although stratification and costratification have analogous characterizations, as described above, there is a remarkable asymmetry in the relationship between the two properties. Surprisingly, the local-to-global principle is equivalent to the codetection property and these are both equivalent to the colocal-to-global principle (Theorem 6.4):

Theorem. *The following conditions are equivalent:*

(a) $\mathcal{T}$ satisfies the local-to-global principle;
(b) $\mathcal{T}$ satisfies the colocal-to-global principle;
(c) $\mathcal{T}$ satisfies the codetection property.

These conditions imply the detection property, but the converse does not hold in general.

Moreover, costratification implies stratification (Theorem 7.19 and Corollary 7.20):

Theorem. *If $\mathcal{T}$ is costratified, then it is also stratified. Moreover, in this case the map sending a localizing ideal $\mathcal{L}$ to its right orthogonal $\mathcal{L}^{\perp} = \{\, t \in \mathcal{T} \mid \mathsf{hom}(\mathcal{L}, t) = 0 \,\}$ induces a bijection*

$$\{\textit{localizing ideals of } \mathcal{T}\} \xrightarrow{\sim} \{\textit{colocalizing coideals of } \mathcal{T}\}$$

with inverse given by the left orthogonal.

We thus have a hierarchy of properties that a tt-category can possess, summarized by the following diagram:

$$\begin{tikzcd} & \boxed{\mathcal{T}\text{ is costratified}} \arrow[d, Rightarrow, shift left] & \\ & \boxed{\mathcal{T}\text{ is stratified}} \arrow[u, Rightarrow, dotted, shift left, "?"] \arrow[d, Rightarrow, shift left] & \\ \text{Codetection holds for }\mathcal{T} \arrow[r, Leftrightarrow] & \text{The local-to-global principle holds for }\mathcal{T} \arrow[u, Rightarrow, shift left, "/" marking] \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, shift left] & \text{The colocal-to-global principle holds for }\mathcal{T} \\ & \boxed{\text{Detection holds for }\mathcal{T}} \arrow[u, Rightarrow, shift left, "/" marking] & \end{tikzcd}$$

It remains an open question whether stratification implies costratification in general. This is related to questions concerning the existence of arbitrary Bousfield localizations and is thereby related to set-theoretic concerns such as Vopěnka's principle (see Remark 2.14 and Remarks 7.22–7.23). In light of this discrepancy, costratification remains an *a priori* deeper property than stratification. From a more practical point of view, the proofs of the known instances of costratification—such as [Nee11] or [BIK12]—are significantly more involved than their stratification counterparts. We will return to this topic later in Theme VI.

We now turn to another asymmetry between support and cosupport. A basic property of support is that

$$(\dagger) \qquad \operatorname{Supp}(s \otimes t) \subseteq \operatorname{Supp}(s) \cap \operatorname{Supp}(t)$$

for any $s, t \in \mathcal{T}$. The "half-$\otimes$ formula" states that this inclusion is an equality when the object s is compact. Moreover, this is promoted to a "full-$\otimes$ formula" (that is, ($\dagger$) is an equality for all objects) when the category is stratified. In contrast, the behaviour for cosupport is as follows:

Theorem. *The following statements hold:*

(a) *For any $s, t \in \mathcal{T}$,* $\operatorname{Cosupp}(\mathsf{hom}(s,t)) \subseteq \operatorname{Supp}(s) \cap \operatorname{Cosupp}(t)$.
(b) *For any $x \in \mathcal{T}^c$ and $t \in \mathcal{T}$,* $\operatorname{Cosupp}(\mathsf{hom}(x,t)) = \operatorname{Supp}(x) \cap \operatorname{Cosupp}(t)$.
(c) *Assume the local-to-global principle holds for $\mathcal{T}$. The following conditions are equivalent:*
   (i) *$\mathcal{T}$ is stratified;*
   (ii) *For all $s, t \in \mathcal{T}$, we have* $\operatorname{Cosupp}(\mathsf{hom}(s,t)) = \operatorname{Supp}(s) \cap \operatorname{Cosupp}(t)$.

This is established in Lemma 4.29, Proposition 4.35, and Theorem 7.15. Note how the inclusion in part (a), which is the analogue of property (†), involves both cosupport *and* support. Part (b) of the theorem provides the "half-hom formula". Part (c) establishes that the half-hom formula promotes to a "full-hom formula" if *and only if* the category is stratified (provided the local-to-global principle holds). This fundamental relationship between stratification and the behaviour of *cosupport* is a discovery due to Benson–Iyengar–Krause [BIK12]. One might have expected *a priori* that the full-hom formula would instead be more closely related to *co*stratification.

Theme III. The geometric relationship between support and cosupport. The above motivates the search for a systematic geometric description of the relation between support and cosupport as subsets of the spectrum. This turns out to be a subtle question, but there are several conceptual results we can prove. For example, we prove (Corollary 8.4):

Theorem. *If* $\mathcal{T}$ *has noetherian spectrum, then* $\min \operatorname{Supp}(t) = \min \operatorname{Cosupp}(t)$ *for any* $t \in \mathcal{T}$. *In general, if the spectrum is not noetherian, this identity can fail.*

In the situation of the theorem, one might wonder whether it is possible for the support and cosupport functions to coincide. We prove (Corollary 8.14):

Theorem. *If* $\mathcal{T}$ *has noetherian spectrum, then the following are equivalent:*

(a) $\operatorname{Supp}(t) = \operatorname{Cosupp}(t)$ *for all* $t \in \mathcal{T}$.
(b) $\operatorname{Spc}(\mathcal{T}^c)$ *is a finite discrete space.*

From a more general perspective, the question of whether support coincides with cosupport is related to the vanishing of the Tate construction and thereby reflects the topology of the Balmer spectrum. For example, we have (Corollary 8.12):

Theorem. *Assume that the codetection property holds. Let* $Y \subseteq \operatorname{Spc}(\mathcal{T}^c)$ *be a Thomason subset and let* $e_Y \in \mathcal{T}$ *be the associated left idempotent with* $\operatorname{Supp}(e_Y) = Y$. *Then* $\operatorname{Cosupp}(e_Y) \subseteq Y$ *if and only if there is a decomposition* $\operatorname{Spc}(\mathcal{T}^c) = Y \sqcup Y^c$ *as a disjoint union of closed sets.*

More refined statements could be made, but the significant point is that understanding cosupport is relevant even for questions purely about the structure of the category $\mathcal{T}^c$ of compact objects.

It would be desirable to find a process for computing the cosupport of a given object in terms of its support or vice versa. This turns out to be too optimistic: We provide explicit counterexamples showing that in general the support of an object does not determine its cosupport (Example 8.19) and, vice versa, the cosupport of an object does not determine its support (Example 8.20). There are, however, more refined ways to relate support and cosupport; for example, by considering pairs of objects related by some notion of duality. To this end, we undertake a general study of dualities in tt-categories (Definition 12.1) and analyze how support and cosupport transform under them. For example, Brown–Comenetz duality provides a way to construct objects with prescribed cosupport (Proposition 12.4):

Theorem. *For any* $t \in \mathcal{T}$, $\operatorname{Cosupp}(t^*) = \operatorname{Supp}(t)$, *where* t^* *denotes the Brown–Comenetz dual of* t.

Our results on dualities can be applied to a wide variety of examples. The following theorem (Proposition 12.15) provides an illustrative example of the type of result which can be obtained. To understand the statement, we say that an object $t \in \mathcal{T}$ has small cosupport if $\mathrm{Cosupp}(t) \subseteq \mathrm{Supp}(t)$. Examples include compact objects, and also dualizing complexes in algebraic geometry.

Theorem. *Let $\mathcal{T}$ be stratified and suppose $\kappa \in \mathcal{T}$ dualizes the subcategory $\mathcal{T}_0 \subseteq \mathcal{T}$. If the objects of $\mathcal{T}_0$ have small cosupport, then for any $t \in \mathcal{T}_0$ we have*

$$\mathrm{Cosupp}(t) = \mathrm{Supp}(t) \cap \mathrm{Cosupp}(\kappa).$$

Applied to Spanier–Whitehead duality, the statement of the theorem reduces to the half-hom formula. Applied to the derived category of a commutative noetherian ring which admits a dualizing complex, it extends the half-hom formula to bounded complexes of coherent sheaves. These results clarify a number of results in the literature concerning the relation between cosupport and completion.

Theme IV. Cosupport is support. A significant contribution of this paper is a unification of support and cosupport which shows that they are both manifestations of the same construction. This unification is not obvious. Attempts to find a deeper connection between support and cosupport are clouded by the fact that colocalizing coideals are not obviously categorically dual to localizing ideals. The key insight that leads to their unification is the realization that $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$ share the same rigid tt-category $\mathcal{T}^d \cong (\mathcal{T}^{\mathrm{op}})^d$ of dualizable objects and that localizing $\mathcal{T}^d$-submodules of $\mathcal{T}$ are precisely the localizing ideals of $\mathcal{T}$, while the localizing $\mathcal{T}^d$-submodules of $\mathcal{T}^{\mathrm{op}}$ are precisely the colocalizing coideals of $\mathcal{T}$. Technicalities arise because the opposite category $\mathcal{T}^{\mathrm{op}}$ is never compactly generated, but a framework which covers both $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$ is provided by the notion of a perfectly generated non-closed tt-category (Terminology 10.1). Summarizing results from Section 10, we have:

Theorem. *There is a theory of support for perfectly generated non-closed tt-categories $\mathcal{T}$ whose spectrum $\mathrm{Spc}(\mathcal{T}^d)$ of dualizable objects is weakly noetherian, and a corresponding notion of stratification for classifying the localizing $\mathcal{T}^d$-submodules of $\mathcal{T}$.*

Example 10.31, Example 10.32, and Theorem 10.35 then provide:

Theorem. *Let $\mathcal{T}$ be a rigidly-compactly generated tt-category.*

- *(a) Applied to $\mathcal{T}$ the above theory reduces to the theory of support and stratification for localizing ideals of $\mathcal{T}$ developed in* [BHS23].
- *(b) Applied to $\mathcal{T}^{\mathrm{op}}$ the above theory reduces to the theory of cosupport and costratification for colocalizing coideals of $\mathcal{T}$ developed in this paper.*
- *(c) In particular,*

$$\mathrm{Cosupp}_{\mathcal{T}}(t) = \mathrm{Supp}_{\mathcal{T}^{\mathrm{op}}}(t)$$

for all $t \in \mathcal{T}$, the colocalizing coideals of $\mathcal{T}$ are precisely the localizing $\mathcal{T}^d$-submodules of $\mathcal{T}^{\mathrm{op}}$, and $\mathcal{T}$ is costratified precisely when $\mathcal{T}^{\mathrm{op}}$ is stratified.

This provides sound conceptual foundations for the construction of cosupport.

Theme V. Base change of support and cosupport. Another major theme in this paper is the study of the behaviour of support and cosupport in a relative setting, that is under base change along functors between tt-categories. Such results are of fundamental importance in the subject, as they allow us to reduce problems in a category of interest to simpler categories. This is vital in our applications to classification problems discussed below. It also emphasizes the value of basing our theory of cosupport on the Balmer spectrum, which readily affords a geometric perspective for relating tt-categories.

The basic setup is a geometric functor $f^*\colon \mathcal{T} \to \mathcal{S}$, that is, a tt-functor which preserves coproducts. The functor f^* induces a continuous map on spectra

$$\varphi \coloneqq \mathrm{Spc}(f^*)\colon\ \mathrm{Spc}(\mathcal{S}^c) \to \mathrm{Spc}(\mathcal{T}^c).$$

Our assumptions also guarantee the existence of two layers of right adjoints:

$$f^* \dashv f_* \dashv f^!.$$

We obtain a variety of results concerning the image and preimage under φ of support and cosupport. An interesting feature is the prominent role that the double right adjoint $f^!$ plays concerning base change for cosupport. For example, one highlight (Corollary 14.19) establishes the Avrunin–Scott identities [AS82] in a general tensor triangular context:

Theorem. *If $f^*\colon \mathcal{T} \to \mathcal{S}$ a geometric functor with $\mathcal{T}$ stratified, then for any $t \in \mathcal{T}$:*

$$\mathrm{Supp}_{\mathcal{S}}(f^*(t)) = \varphi^{-1}(\mathrm{Supp}_{\mathcal{T}}(t)) \quad \textit{and} \quad \mathrm{Cosupp}_{\mathcal{S}}(f^!(t)) = \varphi^{-1}(\mathrm{Cosupp}_{\mathcal{T}}(t)).$$

These identities play a vital role in the study of descent properties for stratification, a topic discussed in more detail below.

Another result (Corollary 13.15) gives an unconditional description of the image:

Theorem. *If $f^*\colon \mathcal{T} \to \mathcal{S}$ is a geometric functor, then* $\operatorname{im}\varphi = \mathrm{Supp}_{\mathcal{T}}(f_*(\mathbb{1}_{\mathcal{S}}))$.

A number of more refined statements are obtained. For example, we prove that if f^* satisfies Grothendieck–Neeman duality (in the sense of [BDS16]) then the induced map on spectra φ is a closed map (Remark 13.26). We also exhibit a close relationship between the surjectivity of φ and the conservativity of the functors f^* and $f^!$; see, e.g., Proposition 13.33 and Corollary 14.24.

These are just the highlights of our base change results; they already demonstrate that cosupport arises naturally in the study of any geometric functor $f^*\colon \mathcal{T} \to \mathcal{S}$, providing insight into the behaviour of the double right adjoint $f^!\colon \mathcal{T} \to \mathcal{S}$.

Theme VI. Descent and applications. Another theme is the development of general techniques which allow us to establish costratification for numerous categories of interest. These techniques will provide streamlined proofs for all known classifications of colocalizing coideals in the literature and also apply to new classes of examples that were not previously accessible. To provide proper context for our results, we briefly review a blueprint for proving stratification, and then explain how to bootstrap this process to establish costratification.

Suppose $\mathcal{T}$ is a tt-category that we wish to prove is stratified. This proceeds naturally in three steps:

Step 1 Construct geometric functors $f_i^*\colon \mathcal{T} \to \mathcal{S}_i$ to simpler tt-categories $\mathcal{S}_i$ such that the images of the maps $\varphi_i \coloneqq \mathrm{Spc}(f_i^*)$ jointly cover $\mathrm{Spc}(\mathcal{T}^c)$.

Step 2 Prove stratification for the categories $\mathcal{S}_i$.

Step 3 Descend stratification along the functors f_i^*.

Progress usually hinges on improvements in Step 3, i.e., finding more general criteria for descending stratification. To this end, we add to the existing toolbox for establishing stratification in the form of *quasi-finite descent* (Theorem 17.16) and *nil-descent* (Theorem 17.20).

Now the question presents itself of whether a similar approach works for costratification. We show that this is indeed the case in the strongest possible way, provided we already have stratification. In other words, we prove that whenever we can descend stratification, we can also descend costratification. The key insight is the following 'bootstrap theorem':

Theorem. *Suppose $f^*\colon \mathcal{T} \to \mathcal{S}$ is a conservative geometric functor such that $\mathcal{S}$ is costratified. If $\mathcal{T}$ is stratified, then it is also costratified.*

This is Corollary 17.4 and is one of our main results. In order to run the descent strategy and apply the theorem, we also need a sufficient supply of costratified categories. One source is provided by our next result which combines Corollary 18.9 and Theorem 18.15:

Theorem. *The following tt-categories $\mathcal{T}$ are costratified (and hence also stratified):*

(a) $\mathcal{T}$ is pure-semisimple (e.g., a tt-field in the sense of [BKS19]). In this case, $\mathrm{Spc}(\mathcal{T}^c)$ is a finite discrete space.

(b) $\mathcal{T}$ is affine weakly regular in the sense of [DS16]. In this case, there is a canonical homeomorphism $\mathrm{Spc}(\mathcal{T}^c) \cong \mathrm{Spec}^h(\mathrm{End}^*_{\mathcal{T}}(\mathbb{1}))$.

Our methods are flexible and have wide applicability to diverse classes of examples in algebraic geometry, representation theory and homotopy theory. The next theorem collects a sample of our applications; further examples can be found in the main body of the paper.

Theorem.

(a) Let X be a quasi-compact and quasi-separated scheme which is topologically weakly noetherian. The derived category $\mathrm{D}_{\mathrm{qc}}(X)$ is stratified if and only if it is costratified. In particular, it follows that $\mathrm{D}_{\mathrm{qc}}(X)$ is costratified for any noetherian scheme X. (See Theorem 19.5.)

(b) For X a p-good connected space with noetherian mod p cohomology, the category of modules over the cochain algebra $\mathrm{Mod}(C^*(X;\mathbb{F}_p))$ is stratified if and only if it is costratified if and only if X satisfies Chouinard's condition. In particular, these conditions hold for connected noetherian H-spaces. (See Theorem 20.27.)

(c) The category of E_n-local spectra is costratified. (See Theorem 20.21.)

(d) Let G be a finite group and let $\mathbb{E}_G \in \mathrm{CAlg}(\mathrm{Sp}_G)$ be a commutative equivariant ring spectrum such that the non-equivariant derived categories $\mathrm{D}(\Phi^H\mathbb{E}_G)$ are costratified with noetherian spectrum for each $H \le G$, where $\Phi^H\mathbb{E}_G$ denotes the geometric fixed points. Then $\mathrm{D}(\mathbb{E}_G)$ is costratified. (See Theorem 20.31.)

(e) As a special case of (d), the category of derived Mackey functors is costratified for any finite group G. (See Corollary 20.36.)

(f) As a special case of (d) and [BCH+23a], the category of equivariant modules over Borel-equivariant Morava E-theory is costratified. (See Theorem 20.41.)

(g) The category of rational G-spectra is costratified for compact Lie groups G. (See Theorem 20.48.)

(h) The derived category of permutation modules $\mathrm{DPerm}(G,k)$ is costratified for any finite group G and field k. (See Theorem 19.13.)

In each of these cases, we get a classification of localizing ideals and colocalizing coideals in terms of the (known) underlying set of the corresponding Balmer spectrum.

For each of these examples, stratification was already known, and we give precise attributions and references in the main text.

Another new class of costratified categories, which uses the techniques of this paper but whose proof lies outside its scope, are stable module categories $\mathrm{StMod}(G, R)$ of finite groups with coefficients in noetherian commutative rings; see [BBI+23]. In the present paper, we include a streamlined proof of the classical $R = k$ case, handling elementary abelian groups via Galois ascent (Proposition 20.12) à la Mathew; see Theorem 19.10 and Theorem 20.14.

In summary, we have exhausted the list of all stratified tt-categories we are aware of and have shown that each of them is also costratified. This in particular includes the case of $X = S^3\langle 3\rangle$ in part (b) of the previous theorem, which was not accessible to previous technology; see Example 20.28.

Outline of the document. The paper consists of four parts.

Part I begins with Section 2 where we state our terminological conventions, introduce notation, and recall fundamental facts about colocalizing coideals and Bousfield localization. We define the notion of a cosupport theory in Section 3. In Section 4, we define the tensor triangular cosupport of a rigidly-compactly generated category and investigate its elementary properties. This study continues in Section 5 where we define the stalk and costalk at each point and introduce the detection and codetection properties. We then proceed to a study of the local-to-global principle and the colocal-to-global principle in Section 6 where we show that they are equivalent and in fact also equivalent to the codetection property. In Section 7, we define costratification, establish the conditions which characterize when a category is costratified, and also prove that costratification implies stratification. We conclude this part with Section 8 which is a study of the geometric behaviour of cosupport and its relation to support.

In Part II we switch from the world of rigidly-compactly generated tt-categories to a more general setting. We recall the notion of a perfectly generated triangulated category in Section 9 and recall how the opposite category of a compactly generated category is perfectly generated. In Section 10, we set up a theory of support for perfectly generated tt-categories, and show that it provides a unification of support and cosupport. In Section 11, we prove a universality result for our general theory of support, obtaining a universality result for cosupport as a special case. Finally, in Section 12 we study how support and cosupport are related under intrinsic dualities, such as Spanier–Whitehead duality and Brown–Comenetz duality.

In Part III we study base change and descent results for (co)support and (co)stratification. In particular, we study the image of a geometric functor, including critera for it to be surjective, in Section 13, and base change results for (co)support in Section 14. In Section 15, we apply these results to study descent for the local-to-global principle. In Section 16, we obtain local cogenerators for our costalk categories, which is an important technical ingredient for the results which follow. This culminates in Section 17 where we establish our key 'bootstrap' theorem which allows us to descend costratification whenever we can descend stratification. We also establish descent techniques for stratification to power the theorem.

In Part IV we turn to applications and examples. We discuss abstract tensor triangular examples in Section 18, algebraic examples in Section 19, and homotopical examples in Section 20. We conclude the paper in Section 21 with a list of open questions, which we hope will stimulate further research.

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Part I. Cosupport and costratification

2. Colocalizing coideals

2.1. *Terminology.* We follow the notation and terminology from [BHS23]. For the majority of the paper $\mathcal{T}$ will denote a rigidly-compactly generated tensor-triangulated category (with exceptions in Sections 9–11). We will denote the tensor by $-\otimes-$ and the unit object by $\mathbb{1}$. We will denote the internal hom by $\mathsf{hom}(-,-)$ and the abelian group of morphisms by $\mathrm{Hom}_{\mathcal{T}}(-,-)$ or $\mathcal{T}(-,-)$. We also write $t^\vee := \mathsf{hom}(t,\mathbb{1})$ for the dual of an object.

2.2. *Definition.* A *localizing subcategory* of $\mathcal{T}$ is a thick subcategory which is closed under coproducts. A *localizing ideal* is a localizing subcategory $\mathcal{L}$ which is also a tensor-ideal: $\mathcal{L}\otimes\mathcal{T}\subseteq\mathcal{L}$ where $\mathcal{L}\otimes\mathcal{T}=\left\{\, s\otimes t \,\middle|\, s\in\mathcal{L}, t\in\mathcal{T}\,\right\}$. A *colocalizing subcategory* of $\mathcal{T}$ is a thick subcategory which is closed under products. A *colocalizing coideal* is a colocalizing subcategory $\mathcal{C}$ with the property that $\mathsf{hom}(\mathcal{T},\mathcal{C})\subseteq\mathcal{C}$ where $\mathsf{hom}(\mathcal{T},\mathcal{C})=\left\{\,\mathsf{hom}(t,s)\,\middle|\, s\in\mathcal{C},\, t\in\mathcal{T}\,\right\}$.

2.3. *Notation.* We write $\mathrm{Loc}\langle\mathcal{E}\rangle$, $\mathrm{Locid}\langle\mathcal{E}\rangle$, $\mathrm{Coloc}\langle\mathcal{E}\rangle$, and $\mathrm{Colocid}\langle\mathcal{E}\rangle$ for the localizing subcategory, localizing ideal, colocalizing subcategory, and colocalizing coideal generated by a collection of objects $\mathcal{E}\subseteq\mathcal{T}$.

2.4. *Remark.* If $\mathcal{T}=\mathrm{Loc}\langle\mathcal{E}\rangle$ for a collection of objects $\mathcal{E}\subseteq\mathcal{T}$, then a colocalizing subcategory $\mathcal{C}$ is a coideal if and only if $\mathsf{hom}(\mathcal{E},\mathcal{C})\subseteq\mathcal{C}$. Thus, if $\mathcal{T}$ is monogenic (meaning $\mathcal{T}=\mathrm{Loc}\langle\mathbb{1}\rangle$) then every colocalizing subcategory is automatically a coideal.

2.5. *Remark.* For any object $t\in\mathcal{T}$, consider the three functors

$$t\otimes-\colon\mathcal{T}\to\mathcal{T},\quad \mathsf{hom}(t,-)\colon\mathcal{T}\to\mathcal{T},\quad\text{and}\quad \mathsf{hom}(-,t)\colon\mathcal{T}^{\mathrm{op}}\to\mathcal{T}.$$

Localizing ideals pull back under the first functor to localizing ideals; colocalizing coideals pull back under the second functor to colocalizing coideals; and colocalizing coideals pull back under the third functor to localizing ideals. It follows that for any collection of objects $\mathcal{E}\subseteq\mathcal{T}$, we have

$$t\otimes\mathrm{Locid}\langle\mathcal{E}\rangle\subseteq\mathrm{Locid}\langle t\otimes\mathcal{E}\rangle, \tag{2.6}$$

$$\mathsf{hom}(t,\mathrm{Colocid}\langle\mathcal{E}\rangle)\subseteq\mathrm{Colocid}\langle\mathsf{hom}(t,\mathcal{E})\rangle,\ \text{and} \tag{2.7}$$

$$\mathsf{hom}(\mathrm{Locid}\langle\mathcal{E}\rangle,t)\subseteq\mathrm{Colocid}\langle\mathsf{hom}(\mathcal{E},t)\rangle. \tag{2.8}$$

2.9. *Definition* (Orthogonal subcategories). If $\mathcal{E}\subseteq\mathcal{T}$ is a collection of objects, we define the *right orthogonal* of $\mathcal{E}$ to be the full subcategory

$$\mathcal{E}^\perp := \left\{\, t\in\mathcal{T}\,\middle|\,\mathsf{hom}(s,t)=0 \text{ for all } s\in\mathcal{E}\,\right\}.$$

Note that $\mathcal{E}^\perp$ is a colocalizing coideal of $\mathcal{T}$. Moreover, it follows from (2.8) that $\mathcal{E}^\perp=\mathrm{Locid}\langle\mathcal{E}\rangle^\perp$. Similarly, the *left orthogonal* is defined by

$${}^\perp\mathcal{E} := \left\{\, t\in\mathcal{T}\,\middle|\,\mathsf{hom}(t,s)=0 \text{ for all } s\in\mathcal{E}\,\right\}.$$

It is a localizing ideal of $\mathcal{T}$ and it follows from (2.7) that ${}^\perp\mathcal{E}={}^\perp\,\mathrm{Colocid}\langle\mathcal{E}\rangle$.

2.10. *Remark.* If $\mathcal{E}\subseteq\mathcal{T}$ is an ideal then

$$\mathcal{E}^\perp = \left\{\, t\in\mathcal{T}\,\middle|\,\mathcal{T}(s,t)=0 \text{ for all } s\in\mathcal{E}\,\right\}.$$

Similarly, if $\mathcal{E}\subseteq\mathcal{T}$ is a coideal then

$${}^\perp\mathcal{E} = \left\{\, t\in\mathcal{T}\,\middle|\,\mathcal{T}(t,s)=0 \text{ for all } s\in\mathcal{E}\,\right\}.$$

2.11. *Remark.* A localizing ideal $\mathcal{L}$ is *strictly localizing* if the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ has a right adjoint. This is the case if and only if $\mathcal{L}$ is the kernel of a Bousfield localization on $\mathcal{T}$. In this situation, the subcategory of local objects is $\mathcal{L}^{\perp}$. Significantly, if $\mathcal{L}$ is strictly localizing, then $\mathcal{L} = {}^{\perp}(\mathcal{L}^{\perp})$, see for example [Kra10, Proposition 4.9.1]. Similarly, a colocalizing coideal $\mathcal{C}$ is *strictly colocalizing* if the inclusion $\mathcal{C} \hookrightarrow \mathcal{T}$ has a left adjoint. This is the case if and only if $\mathcal{C}$ is the image of a Bousfield localization. Moreover, in this case, $\mathcal{C} = ({}^{\perp}\mathcal{C})^{\perp}$. We always have a function $\mathcal{L} \mapsto \mathcal{L}^{\perp}$ between the strictly localizing ideals of $\mathcal{T}$ and the strictly colocalizing coideals of $\mathcal{T}$. This assignment is a bijection modulo the set-theoretic question of whether there is only a set of such Bousfield localizations.

2.12. *Remark.* The question of whether Bousfield localizations always exist has an interesting history, starting with [Bou79]. Of particular note for our purposes is the following:

2.13. **Theorem** (Neeman)**.** *If $\mathcal{T}$ is a well generated triangulated category then every set-generated localizing subcategory is strictly localizing.*

Proof. The key point is that a set-generated localizing subcategory of a well generated triangulated category is itself well generated, hence the inclusion has a right adjoint by Brown representability; see [Nee01, Remark 1.16 and Proposition 1.21] or [Kra10, Proposition 4.9.1 and Theorem 7.2.1]. □

2.14. *Remark.* There is no analogous result for colocalizing subcategories (morally, because the proof techniques do not apply to $\mathcal{T}^{\mathrm{op}}$). This is the heart of the issue for why there could be "more" colocalizing subcategories and why one might expect a classification of colocalizing coideals to be "harder" than a classification of localizing ideals. For example, if there is only a set of localizing ideals (e.g., if $\mathcal{T}$ is stratified in the sense of [BHS23]) then every localizing ideal is set-generated by [KS19, Lemma 3.3] and hence is strictly localizing by Theorem 2.13. Hence in this case the assignment $\mathcal{L} \mapsto \mathcal{L}^{\perp}$ provides an injection from the set of localizing ideals into the collection of colocalizing coideals. A priori, there might be more colocalizing coideals. The difference would disappear if we knew that all colocalizing coideals were strictly colocalizing, but therein lie set-theoretic dragons. For example, Casacuberta–Gutiérrez–Rosický [CGR14] prove that if a large cardinal axiom known as Vopěnka's principle holds (see [AR94, Chapter 6]) then every colocalizing subcategory of $\mathcal{T}$ is strictly colocalizing provided $\mathcal{T} = \mathrm{Ho}(\mathcal{M})$ is the homotopy category of a stable combinatorial model category. It remains an open question whether this is true for arbitrary $\mathcal{T}$ and without assuming axioms beyond ZFC.

2.15. *Remark.* The modified version of [KS19, Lemma 3.3.1] provided by [BHS23, Proposition 3.5] actually establishes that if there is a set of *set-generated* localizing ideals then all localizing ideals are set-generated. With this in hand, a variant of the above argument runs as follows: If there is a set of colocalizing coideals then there is a set of strictly colocalizing coideals, hence a set of strictly localizing ideals, hence a set of set-generated localizing ideals (by invoking Theorem 2.13), and hence all localizing ideals are set-generated (and strictly localizing). In summary: If there is a set of colocalizing coideals then there is a set of localizing ideals and they correspond to the strictly colocalizing coideals of $\mathcal{T}$.

3. Cosupport theories

Our goal is to classify colocalizing coideals using a suitable notion of "cosupport" for the objects of $\mathcal{T}$. First we axiomatize the properties such a cosupport theory should satisfy.

3.1. *Definition.* Let X be a topological space and let $\mathfrak{C}\colon \mathcal{T} \to \mathcal{P}(X)$ be a function, where $\mathcal{P}(X)$ denotes the power set of (the underlying set of) X. This function extends to collections $\mathcal{E}$ of objects in $\mathcal{T}$ by setting $\mathfrak{C}(\mathcal{E}) = \bigcup_{t\in\mathcal{E}} \mathfrak{C}(t)$. The pair $(X, \mathfrak{C})$ is called a *cosupport theory* if it satisfies the following conditions:

(a) $\mathfrak{C}(0) = \varnothing$ and $\mathfrak{C}(\mathcal{T}) = X$;
(b) $\mathfrak{C}(\Sigma t) = \mathfrak{C}(t)$ for every $t \in \mathcal{T}$;
(c) $\mathfrak{C}(c) \subseteq \mathfrak{C}(a) \cup \mathfrak{C}(b)$ for any exact triangle $a \to b \to c \to \Sigma a$ in $\mathcal{T}$;
(d) $\mathfrak{C}(\prod_{i\in I} t_i) = \bigcup_{i\in I} \mathfrak{C}(t_i)$ for any set of objects t_i in $\mathcal{T}$;
(e) $\mathfrak{C}(\mathsf{hom}(s,t)) \subseteq \mathfrak{C}(t)$ for all $s, t \in \mathcal{T}$.

We also refer to $\mathfrak{C}$ as a *cosupport function.*

3.2. *Remark.* These axioms (excluding $\mathfrak{C}(\mathcal{T}) = X$) are equivalent to the statement that for any subset $Y \subseteq X$, the subcategory $\{\, t \in \mathcal{T} \mid \mathfrak{C}(t) \subseteq Y \,\}$ is a colocalizing coideal of $\mathcal{T}$. In particular, $\mathfrak{C}(\mathrm{Colocid}\langle\mathcal{E}\rangle) = \mathfrak{C}(\mathcal{E})$ for any collection of objects $\mathcal{E} \subseteq \mathcal{T}$. For example, if $\mathrm{Colocid}\langle t_1\rangle = \mathrm{Colocid}\langle t_2\rangle$ then $\mathfrak{C}(t_1) = \mathfrak{C}(t_2)$.

3.3. *Remark.* Note that we have not yet made use of the topology on X. For now, it could therefore be dropped from the definition. We also remark in passing that our axiomatization is similar but not exactly the same as the one given by [Ver22, Definition 3.2]. Further discussion on this point will be given in Remark 11.14.

3.4. *Example.* Let $\mathrm{D}(R)$ be the derived category of a commutative noetherian ring R. Neeman [Nee11] has given a classification of the colocalizing coideals of $\mathrm{D}(R)$ using the assignment

$$\{\text{colocalizing coideals of } \mathrm{D}(R)\} \xrightarrow{B(-)} \{\text{subsets of } \mathrm{Spec}(R)\}$$

where $B(\mathcal{C}) \coloneqq \{\, \mathfrak{p} \in \mathrm{Spec}(R) \mid \kappa(\mathfrak{p}) \in \mathcal{C} \,\}$ and $\kappa(\mathfrak{p})$ is the residue field of R associated to the prime ideal $\mathfrak{p}$. We claim that the function $B(t) \coloneqq B(\mathrm{Colocid}\langle t\rangle)$ defined on objects $t \in \mathrm{D}(R)$ is a cosupport theory in the sense of Definition 3.1. Indeed, $B(0) = \{\, \mathfrak{p} \in \mathrm{Spec}(R) \mid \kappa(\mathfrak{p}) \in (0) \,\} = \varnothing$, while $\kappa(\mathfrak{p}) \in B(\kappa(\mathfrak{p}))$ by definition so that $\bigcup_{\mathfrak{p}\in\mathrm{Spec}(R)} B(\kappa(\mathfrak{p})) = \mathrm{Spec}(R)$. Condition (b) is clear, while (c) follows because if $a \to b \to c \to \Sigma a$ is an exact triangle, then c is in the colocalizing subcategory generated by a and b. Finally, (e) is a consequence of the inclusion $\mathrm{Colocid}\langle \mathsf{hom}(s,t)\rangle \subseteq \mathrm{Colocid}\langle t\rangle$. Note that this cosupport theory is defined for any commutative ring R, although Neeman's theorem requires R noetherian.

3.5. *Example.* An alternative approach is given in [SWW17], where the cosupport of $t \in \mathrm{D}(R)$ is defined by

$$\text{co-supp}_R(t) = \{\, \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathsf{hom}(k(\mathfrak{p}), t) \neq 0 \,\}.$$

Most of the axioms for cosupport are verified in [SWW17, Prop. 4.7–4.9]. Axiom (b) is not verified, but is clear from the definition. The axiom $\text{co-supp}_R(\mathrm{D}(R)) = \mathrm{Spec}(R)$ follows, for example, from the observation that $\mathsf{hom}(k(\mathfrak{p}), k(\mathfrak{p})) \neq 0$. The only remaining axiom to check is (e). To this end, suppose $\mathsf{hom}(k(\mathfrak{p}), \mathsf{hom}(s,t)) \neq 0$. Then $\mathsf{hom}(k(\mathfrak{p}) \otimes s, t) \neq 0$. But $k(\mathfrak{p}) \otimes s$ is a coproduct of suspensions of $k(\mathfrak{p})$, so that $\mathsf{hom}(k(\mathfrak{p}), t) \neq 0$, as well.

3.6. *Example.* Let $\mathcal{S}_{E(n)}$ denote the category of $E(n)$-local spectra; see [HS99] and [BHS23, Section 10]. For $t \in \mathcal{S}_{E(n)}$, Hovey and Strickland [HS99, Section 6] consider the chromatic cosupport, defined by

$$\text{co-supp}(t) \coloneqq \left\{\, m \in \{0, \ldots, n\} \;\middle|\; \mathsf{hom}(K(m), t) \neq 0 \,\right\},$$

where $K(m)$ is the m-th Morava K-theory. Arguments similar to those used for Example 3.5 establish that this defines a theory of cosupport on $\mathcal{S}_{E(n)}$.

3.7. *Example.* If a rigidly-compactly generated tt-category $\mathcal{T}$ is equipped with a central action by a graded commutative noetherian ring R, then Benson–Iyengar–Krause [BIK12] provide a cosupport theory $(\operatorname{cosupp}_R(\mathcal{T}), \operatorname{cosupp}_R)$ whose space of cosupports $\operatorname{cosupp}_R(\mathcal{T}) \subseteq \operatorname{Spec}^h(R)$ lies in the homogeneous spectrum of the acting ring. The cosupport axioms are established in Sections 4, 8 and 9 of their paper. For example, we could take the derived category $\mathrm{D}(R)$ of a noetherian commutative ring acted upon by R itself, or we could take the stable module category $\operatorname{StMod}(kG)$ of a finite group G over a field k acted upon by the group cohomology ring $H^*(G;k)$. In the latter example the space of cosupports is $\operatorname{Proj}(H^*(G,k)) \subsetneq \operatorname{Spec}^h(H^*(G,k))$.

4. Tensor triangular cosupport

In this section we introduce the main cosupport theory of interest to us, which is related to the Balmer–Favi support (a.k.a. small support) introduced in [BF11] and studied in depth in [BHS23]. Throughout $\mathcal{T}$ will denote a rigidly-compactly generated tensor-triangulated category whose spectrum $\operatorname{Spc}(\mathcal{T}^c)$ is weakly noetherian. We will briefly recall what the latter topological condition means, before proceeding with the definition of cosupport. Further discussion is found in [BHS23, Section 2]. First we recall some details about smashing and finite localizations and their idempotent triangles.

4.1. *Remark.* Let $\mathcal{T}$ be a rigidly-compactly generated tt-category. Recall from [BF11, Theorem 2.13] that a *smashing ideal* is a strictly localizing ideal $\mathcal{L}$ which satisfies the following equivalent conditions:

(a) $\mathcal{L}^\perp$ is a localizing subcategory of $\mathcal{T}$;
(b) $\mathcal{L}^\perp$ is a localizing ideal of $\mathcal{T}$;
(c) $\mathcal{L}^\perp$ is an ideal of $\mathcal{T}$.

Associated to a smashing ideal is an idempotent exact triangle

$$e \to \mathbb{1} \to f \to \Sigma e$$

and we have the following diagram of adjunctions

$$\begin{tikzcd}
\mathcal{L} = e \otimes \mathcal{T} \arrow[rr, bend left=10, "\mathsf{hom}(e,-)"] \arrow[rr, phantom, "\cong"] \arrow[dr, hook, bend right=30, "\mathrm{incl}"'] \arrow[dr, bend left=15, "\mathsf{hom}(e,-)"] & & \mathsf{hom}(e, \mathcal{T}) = \mathcal{L}^{\perp\perp} \arrow[ll, bend left=10, "e\otimes-"] \arrow[dl, hook, bend left=30, "\mathrm{incl}"] \arrow[dl, hook, bend right=15, "e\otimes-"'] \\
& \mathcal{T} \arrow[ul, "e\otimes-"] \arrow[ur, "\mathsf{hom}(e,-)"] \arrow[d, two heads, shift right=4, "f\otimes-"'] \arrow[d, two heads, shift left=4, "\mathsf{hom}(f,-)"] & \\
& \mathcal{L}^\perp = f \otimes \mathcal{T} = \mathsf{hom}(f, \mathcal{T}) \arrow[u, hook, "\mathrm{incl}"'] &
\end{tikzcd} \tag{4.2}$$

in which each of the six "vertical" sequences $\bullet \hookrightarrow \mathcal{T} \twoheadrightarrow \bullet$ is a Bousfield localization. See [BF11, Theorem 3.5], [BS17, Remark 5.3], [BHV18, Section 2] and [HPS97, Section 3.3] for further discussion.

4.3. *Example* (Finite localizations). Let $\mathcal{T}^c_Y \coloneqq \{ x \in \mathcal{T}^c \mid \mathrm{supp}(x) \subseteq Y \}$ denote the thick ideal of compact objects corresponding to a Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$. The localizing ideal $\mathcal{T}_Y \coloneqq \mathrm{Locid}\langle \mathcal{T}^c_Y \rangle = \mathrm{Loc}\langle \mathcal{T}^c_Y \rangle$ is a smashing ideal; see [BHS23, Remark 1.23]. We write

$$e_Y \to \mathbb{1} \to f_Y \to \Sigma e_Y$$

for the corresponding idempotent triangle.

4.4. *Example.* If $\mathcal{T} = \mathrm{D}(R)$ is the derived category of a commutative ring R and $I \subseteq R$ is a finitely generated ideal, we can take $Y \coloneqq V(I)$, the set of prime ideals of R containing I. In this case, $e_Y \otimes \mathcal{T}$ is the category of I-torsion complexes and $\mathsf{hom}(e_Y, \mathcal{T})$ is the category of I-adically complete complexes; see [Gre01, Section 5] and [Ste18, Example 2.24]. The equivalence $e_Y \otimes \mathcal{T} \cong \mathsf{hom}(e_Y, \mathcal{T})$ first arose in the work of Matlis [Mat78] and Greenlees–May [GM92]; cf. [PSY14]. The functors $e_Y \otimes -$ and $\mathsf{hom}(e_Y, -)$ can be interpreted in terms of local cohomology and local homology, respectively; see, e.g., [ATJLL97, DG02, BHV18].

4.5. **Lemma.** *For each Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$,*

$$\mathrm{Coloc}\langle \mathcal{T}^c_Y \otimes \mathcal{T} \rangle = \mathsf{hom}(e_Y, \mathcal{T}) = (\mathcal{T}_Y)^{\perp\perp}.$$

Proof. This is proved in [HPS97, Theorem 3.3.5(e)]. ☐

4.6. *Definition.* A subset $W \subseteq \mathrm{Spc}(\mathcal{T}^c)$ is said to be *weakly visible* if it can be written as the intersection of a Thomason subset and the complement of a Thomason subset: $W = Y_1 \cap Y_2^c$. We can then define an idempotent

$$g_W \coloneqq e_{Y_1} \otimes f_{Y_2}.$$

This object of $\mathcal{T}$ only depends, up to isomorphism, on the subset W; see [BF11, Corollary 7.5]. We say that a point $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$ is weakly visible if the singleton subset $\{\mathcal{P}\}$ is weakly visible, and we define

$$g_{\mathcal{P}} \coloneqq g_{\{\mathcal{P}\}}.$$

That is, $g_{\mathcal{P}} = e_{Y_1} \otimes f_{Y_2}$ for any choice of Thomason subsets $Y_1, Y_2 \subseteq \mathrm{Spc}(\mathcal{T}^c)$ such that $\{\mathcal{P}\} = Y_1 \cap Y_2^c$.

4.7. *Remark.* For a weakly visible point $\mathcal{P}$, we can always take $Y_2 = \mathrm{gen}(\mathcal{P})^c$, where $\mathrm{gen}(\mathcal{P}) = \{ \mathcal{Q} \mid \mathcal{P} \subseteq \mathcal{Q} \}$ and $Y_1 = \mathrm{supp}(a)$ for some $a \in \mathcal{T}^c$; see [BHS23, Remark 2.8].

4.8. *Remark.* The intersection $W_1 \cap W_2$ of two weakly visible subsets is again weakly visible and $g_{W_1} \otimes g_{W_2} = g_{W_1 \cap W_2}$. Moreover, $g_W = 0$ if and only if $W = \varnothing$. These facts are proved in [BHS23, Lemma 1.27].

4.9. *Example.* If Y is a Thomason subset then both Y and Y^c are weakly visible subsets. We have $g_Y = e_Y$ and $g_{Y^c} = f_Y$. Remark 4.8 thus specializes to give $e_{Y_1} \otimes e_{Y_2} = e_{Y_1 \cap Y_2}$ and $f_{Y_1} \otimes f_{Y_2} = f_{Y_1 \cup Y_2}$ for Thomason subsets Y_1 and Y_2; see also [BF11, Theorem 5.18] and [BHS23, Lemma 1.27].

4.10. *Definition.* A spectral space X is *weakly noetherian* if every point is weakly visible. This is the topological condition we will require of $\mathrm{Spc}(\mathcal{T}^c)$ in order to construct our cosupport theory.

4.11. *Remark.* A spectral space X is weakly noetherian if and only if its Hochster dual X^* has the property that every point is locally closed. The latter is a separation axiom between T_0 and T_1 called T_D and shows up in work on the analogue of these constructions for the smashing spectrum; see [BS21, Ver21] and [DST19, Section 4.5].

4.12. *Remark.* We define the specialization order on X by $x \le y$ if and only if x is a specialization of y, that is, $x \in \overline{\{y\}}$. Be warned that this is opposite to how the specialization order is defined in [DST19]. According to our convention, closed points are minimal for the specialization order.

4.13. **Proposition.** *Weakly noetherian spectral spaces satisfy the descending chain condition (DCC) on irreducible closed sets.*

Proof. It suffices to prove the result for the Balmer spectrum $X = \mathrm{Spc}(\mathcal{K})$ of an essentially small tensor-triangulated category $\mathcal{K}$ since every spectral space arises in this way.[1] We claim that if X is weakly noetherian then $\mathcal{K}$ satisfies the descending chain condition on prime ideals. Indeed, if $\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \mathcal{P}_3 \supseteq \cdots$ is a descending chain of prime ideals, then consider the prime ideal $\mathcal{P} \coloneqq \bigcap_{n=1}^{\infty} \mathcal{P}_n$. If $\mathcal{P}$ is weakly visible then $\{\mathcal{P}\} = \mathrm{supp}(a) \cap \mathrm{gen}(\mathcal{P})$ for some $a \in \mathcal{K}$ (Remark 4.7). Then $a \notin \mathcal{P}$ implies $a \notin \mathcal{P}_n$ for some n, so that $\mathcal{P}_n \in \mathrm{supp}(a)$. Since $\mathcal{P}_n \in \mathrm{gen}(\mathcal{P})$ we conclude that $\mathcal{P} = \mathcal{P}_n$. □

4.14. *Remark.* There is also a purely point-set topological proof of the previous proposition. The result is equivalent to the statement that any weakly noetherian spectral space X has the descending chain condition for the specialization order on X (Remark 4.12). To establish this, let $Y = (y_1 \ge y_2 \ge \cdots)$ be a descending specialization chain in X. Let $\overline{Y}^{\mathrm{con}}$ denote the closure of Y in the constructible topology. By [DST19, Theorem 4.2.6], Y has an infimum $y_\infty \in X$ which is contained in $\overline{Y}^{\mathrm{con}}$. Since X is weakly noetherian, $\{y_\infty\} = Z \cap \mathrm{gen}(y_\infty)$ for some Thomason closed subset $Z \subseteq X$. To establish that $y_\infty = y_n$ for some $n \ge 1$, it suffices to prove that $Y \cap Z \neq \varnothing$ since $Y \subseteq \mathrm{gen}(y_\infty)$. If $Y \cap Z = \varnothing$ then $Y \subseteq Z^c$, but Z^c is closed in the constructible topology. Thus it would follow that $\overline{Y}^{\mathrm{con}} \subseteq Z^c$ which is a contradiction since $y_\infty \in \overline{Y}^{\mathrm{con}} \cap Z$.

4.15. *Example.* Let X denote the Hochster dual of the Zariski spectrum $\mathrm{Spec}(\mathbb{Z})$. Specialization chains in X are of length at most one, hence X satisfies the DCC on irreducible closed sets. However, the point of X corresponding to the generic point of $\mathrm{Spec}(\mathbb{Z})$ is contained in every Thomason subset of X. Thus, this point is not weakly visible. This example shows that the converse to Proposition 4.13 is false.

4.16. *Notation.* Let X be a spectral space. For any subset $V \subseteq X$, we write $\min V$ for the collection of points in V which are minimal for the specialization order ($x \le y$ iff $x \in \overline{\{y\}}$) among the points of V:

$$\min V \coloneqq \left\{\, x \in V \,\middle|\, \overline{\{x\}} \cap V = \{x\} \,\right\}.$$

For example, $\min X$ is the set of closed points of X, and $V \cap \min X \subseteq \min V$.

4.17. *Remark.* Proposition 4.13 establishes that if X is weakly noetherian then

$$V \neq \varnothing \Longrightarrow \min V \neq \varnothing.$$

[1] Every spectral space arises as the Zariski spectrum of a commutative ring by Hochster's theorem [Hoc69] and it follows from Thomason's theorem [Tho97] that the Zariski spectrum of a commutative ring coincides with the Balmer spectrum of its derived category of perfect complexes.

4.18. *Definition.* We say that a point $\mathcal{P}$ is *visible* if $\overline{\{\mathcal{P}\}}$ is Thomason; in other words, it is a weakly visible point for which we can take $Y_1 = \overline{\{\mathcal{P}\}}$. Recall that a spectral space is noetherian if and only if every point is visible; see [BF11, Corollary 7.14]. Also note that the closed point of a local category is weakly visible if and only if it is visible; see [BHS23, Remark 2.9].

4.19. *Notation.* We write $\operatorname{vis} X$ for the set of visible points in X. We will also abuse notation slightly and write, for example, $\operatorname{vis} \mathcal{T}$ for $\operatorname{vis}(\operatorname{Spc}(\mathcal{T}^c))$.

4.20. *Notation.* For a weakly visible point $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$, we have the functor

$$\Gamma_{\mathcal{P}} := - \otimes g_{\mathcal{P}} : \mathcal{T} \to \mathcal{T}$$

and its right adjoint

$$\Lambda^{\mathcal{P}} := \mathsf{hom}(g_{\mathcal{P}}, -) : \mathcal{T} \to \mathcal{T}.$$

More generally, we have functors

$$\Gamma_W := - \otimes g_W \dashv \mathsf{hom}(g_W, -) =: \Lambda^W$$

for any weakly visible subset $W \subseteq \operatorname{Spc}(\mathcal{T}^c)$.

4.21. *Remark.* Given $t_1, t_2 \in \mathcal{T}$, adjunction provides the following useful formulas:

$$\Lambda^{\mathcal{P}} \mathsf{hom}(t_1, t_2) \simeq \mathsf{hom}(\Gamma_{\mathcal{P}} t_1, t_2) \simeq \mathsf{hom}(t_1, \Lambda^{\mathcal{P}} t_2). \tag{4.22}$$

4.23. *Definition* (The Balmer–Favi support and cosupport). Let $\mathcal{T}$ be a rigidly-compactly generated tt-category whose spectrum $\operatorname{Spc}(\mathcal{T}^c)$ is weakly noetherian. The *support* of an object $t \in \mathcal{T}$ is defined as

$$\begin{aligned}\operatorname{Supp}(t) &:= \left\{ \mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c) \,\middle|\, \Gamma_{\mathcal{P}} t \neq 0 \right\} \\ &= \left\{ \mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c) \,\middle|\, g_{\mathcal{P}} \otimes t \neq 0 \right\}.\end{aligned}$$

The *cosupport* of an object $t \in \mathcal{T}$ is defined as

$$\begin{aligned}\operatorname{Cosupp}(t) &:= \left\{ \mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c) \,\middle|\, \Lambda^{\mathcal{P}} t \neq 0 \right\} \\ &= \left\{ \mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c) \,\middle|\, \mathsf{hom}(g_{\mathcal{P}}, t) \neq 0 \right\}.\end{aligned}$$

4.24. *Remark.* As discussed in [BHS23, Remark 2.12], the function Supp defines a support theory on $\mathcal{T}$ in the sense of [BHS23, Definition 7.1]. Our present goal is to study the properties of the function Cosupp and the relationship between the two theories.

4.25. **Proposition.** *The pair* $(\operatorname{Spc}(\mathcal{T}^c), \operatorname{Cosupp})$ *defines a cosupport theory.*

Proof. For any subset $Y \subseteq \operatorname{Spc}(\mathcal{T}^c)$, observe that

$$\left\{ t \in \mathcal{T} \,\middle|\, \operatorname{Cosupp}(t) \subseteq Y \right\} = \bigcap_{\mathcal{P} \in Y^c} \ker(\Lambda^{\mathcal{P}}(-))$$

is a colocalizing subcategory since each $\Lambda^{\mathcal{P}} = \mathsf{hom}(g_{\mathcal{P}}, -)$ is a product preserving exact functor. Moreover, it is a coideal by the formula (4.22). This establishes, by Remark 3.2, that Cosupp satisfies all properties of a cosupport theory except $\operatorname{Cosupp}(\mathcal{T}) = \operatorname{Spc}(\mathcal{T}^c)$. For this just note that since the object $g_{\mathcal{P}}$ is nonzero (Remark 4.8), the internal hom $\mathsf{hom}(g_{\mathcal{P}}, g_{\mathcal{P}})$ is nonzero. Thus, $\mathcal{P} \in \operatorname{Cosupp}(g_{\mathcal{P}})$ and hence $\operatorname{Spc}(\mathcal{T}^c) = \bigcup_{\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)} \operatorname{Cosupp}(g_{\mathcal{P}})$. □

4.26. **Lemma.** *Let* $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$. *Then*

$$\{\mathcal{P}\} \subseteq \operatorname{Cosupp}(g_{\mathcal{P}}) \subseteq \operatorname{gen}(\mathcal{P}).$$

Proof. The first inclusion $\{\mathcal{P}\} \subseteq \mathrm{Cosupp}(g_{\mathcal{P}})$ is established in the proof of Proposition 4.25. For the second inclusion we need to show that if $\mathcal{P} \notin \overline{\{\mathcal{Q}\}}$ then $\mathsf{hom}(g_{\mathcal{Q}}, g_{\mathcal{P}}) = 0$. Write $\{\mathcal{Q}\} = U_1 \cap U_2^c$ and $\{\mathcal{P}\} = V_1 \cap V_2^c$ for Thomason subsets U_1, U_2, V_1, V_2. Then we have equalities

$$\begin{aligned} \mathsf{hom}(g_{\mathcal{Q}}, g_{\mathcal{P}}) &= \mathsf{hom}(e_{U_1} \otimes f_{U_2}, e_{V_1} \otimes f_{V_2}) \\ &= \mathsf{hom}(e_{U_1} \otimes f_{U_2} \otimes f_{V_2}, e_{V_1} \otimes f_{V_2}) \\ &= \mathsf{hom}(e_{U_1} \otimes f_{U_2} \otimes f_{V_2}, e_{U_1} \otimes e_{V_1} \otimes f_{V_2}) \\ &= \mathsf{hom}(e_{U_1} \otimes f_{U_2 \cup V_2}, e_{U_1 \cap V_1} \otimes f_{V_2}), \end{aligned}$$

where the last step uses Example 4.9. It follows that $\mathsf{hom}(g_{\mathcal{Q}}, g_{\mathcal{P}})$ vanishes if either of the following two conditions hold:

$$U_1 \subseteq U_2 \cup V_2 \quad \text{or} \quad U_1 \cap V_1 \subseteq V_2.$$

By Remark 4.7 we can always take $U_2 := \mathrm{gen}(\mathcal{Q})^c$ and $V_2 := \mathrm{gen}(\mathcal{P})^c$. If $\mathcal{P} \notin \overline{\{\mathcal{Q}\}}$ then $\mathcal{Q} \in V_2$ hence $U_1 \cap U_2^c \subseteq V_2$ which means $U_1 \subseteq U_2 \cup V_2$ and the proof is complete. ☐

4.27. *Remark.* In general, $\mathrm{Cosupp}(g_{\mathcal{P}}) \neq \{\mathcal{P}\}$; see Example 8.16.

4.28. *Remark.* The geometric behavior of cosupport is slightly counterintuitive. For example, it is often the case that $\mathrm{Cosupp}(\mathbb{1}) \subsetneq \mathrm{Spc}(\mathcal{T}^c)$ is a proper subset; see Example 8.7. We will study the geometry of cosupport, as well as its subtle geometric relationship with support, in Section 8. At present we focus on general properties.

4.29. **Lemma.** *For $t_1, t_2 \in \mathcal{T}$, there is an inclusion*

$$\mathrm{Cosupp}(\mathsf{hom}(t_1, t_2)) \subseteq \mathrm{Supp}(t_1) \cap \mathrm{Cosupp}(t_2). \tag{4.30}$$

Proof. If $\mathcal{P} \in \mathrm{Cosupp}(\mathsf{hom}(t_1, t_2))$ then $\Lambda^{\mathcal{P}}\mathsf{hom}(t_1, t_2) \neq 0$ by definition, and hence $\mathsf{hom}(\Gamma_{\mathcal{P}} t_1, t_2) \neq 0$ and $\mathsf{hom}(t_1, \Lambda^{\mathcal{P}} t_2) \neq 0$ by Remark 4.21. This implies that $\Gamma_{\mathcal{P}} t_1 \neq 0$ and $\Lambda^{\mathcal{P}} t_2 \neq 0$, and the result follows. ☐

4.31. *Remark.* The inclusion (4.30) is our most fundamental relationship between support and cosupport. It is an equality (for all objects t_1 and t_2) if and only if $\mathcal{T}$ is stratified; see Theorem 7.15. This demonstrates the significance of cosupport even if one is only interested in localizing ideals. In general, (4.30) is an equality when $t_1 = e_Y$ or $t_1 = f_Y$ for $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ a Thomason subset:

4.32. **Lemma.** *Let $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ be a Thomason subset and let $t \in \mathcal{T}$. Then:*

- *(a)* $\mathrm{Cosupp}(\mathsf{hom}(e_Y, t)) = Y \cap \mathrm{Cosupp}(t)$*;*
- *(b)* $\mathrm{Cosupp}(\mathsf{hom}(f_Y, t)) = Y^c \cap \mathrm{Cosupp}(t)$*;*
- *(c)* $\mathrm{Cosupp}(\Lambda^{\mathcal{P}} t) = \{\mathcal{P}\} \cap \mathrm{Cosupp}(t)$ *for any* $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$.

Proof. First note that for any Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ and $\mathcal{P} \in Y$ we have $g_{\mathcal{P}} \otimes e_Y = g_{\mathcal{P}}$ by Remark 4.8.

Now, by Lemma 4.29 and [BHS23, Lemma 2.13] we have

$$\mathrm{Cosupp}(\mathsf{hom}(e_Y, t)) \subseteq Y \cap \mathrm{Cosupp}(t).$$

To establish the reverse inclusion, let $\mathcal{P} \in Y \cap \mathrm{Cosupp}(t)$. By the previous paragraph, we have

$$\mathsf{hom}(g_{\mathcal{P}}, \mathsf{hom}(e_Y, t)) \simeq \mathsf{hom}(g_{\mathcal{P}}, t) \neq 0$$

so that $\mathcal{P} \in \mathrm{Cosupp}(\mathsf{hom}(e_Y, t))$ as well. This establishes the equality in part (a). The equality in part (b) can be proved similarly. The final statement is a consequence of (a) and (b) by writing $\{\mathcal{P}\} = Y_1 \cap Y_2^c$:

$$\begin{aligned}\mathrm{Cosupp}(\Lambda^{\mathcal{P}} t) &= \mathrm{Cosupp}(\mathsf{hom}(e_{Y_1} \otimes f_{Y_2}, t)) \\ &= \mathrm{Cosupp}(\mathsf{hom}(e_{Y_1}, \mathsf{hom}(f_{Y_2}, t))) \\ &= Y_1 \cap \mathrm{Cosupp}(\mathsf{hom}(f_{Y_2}, t)) \\ &= \{\mathcal{P}\} \cap \mathrm{Cosupp}(t). \qquad \square\end{aligned}$$

4.33. *Remark.* We next establish a cosupport-theoretic analogue of the "half $\otimes$-theorem" of [BF11, Theorem 7.22]; see also [BHS23, Lemma 2.18]. This requires the following lemma:

4.34. **Lemma.** *Let $s_1, s_2, t \in \mathcal{T}$.*

(a) If $s_1 \in \mathrm{Locid}\langle s_2 \rangle$, then $\mathrm{Cosupp}\,\mathsf{hom}(s_1, t) \subseteq \mathrm{Cosupp}\,\mathsf{hom}(s_2, t)$.
(b) If $s_1 \in \mathrm{Colocid}\langle s_2 \rangle$, then $\mathrm{Cosupp}\,\mathsf{hom}(t, s_1) \subseteq \mathrm{Cosupp}\,\mathsf{hom}(t, s_2)$.

Proof. Bearing in mind Remark 3.2, part (a) follows from (2.8) while part (b) follows from (2.7). $\square$

4.35. **Proposition** (Half-hom Theorem)**.** *If $t \in \mathcal{T}$ and $x \in \mathcal{T}^c$, then*

$$\mathrm{Cosupp}\,\mathsf{hom}(x, t) = \mathrm{supp}(x) \cap \mathrm{Cosupp}(t).$$

Proof. The support $\mathrm{supp}(x) \subseteq \mathrm{Spc}(\mathcal{T}^c)$ of the compact object $x \in \mathcal{T}^c$ is a Thomason subset and we have an equality $\mathrm{Locid}\langle x \rangle = \mathrm{Locid}\langle e_{\mathrm{supp}(x)} \rangle$ of compactly generated localizing ideals. By Lemma 4.34, we obtain

$$\mathrm{Cosupp}\,\mathsf{hom}(x, t) = \mathrm{Cosupp}\,\mathsf{hom}(e_{\mathrm{supp}(x)}, t) = \mathrm{supp}(x) \cap \mathrm{Cosupp}(t),$$

where the last equality uses Lemma 4.32. $\square$

4.36. *Example.* For any compact $x \in \mathcal{T}^c$, we have

$$\mathrm{Cosupp}(x) = \mathrm{supp}(x) \cap \mathrm{Cosupp}(\mathbb{1}).$$

Indeed, since x and its dual $x^\vee = \mathsf{hom}(x, \mathbb{1})$ generate the same thick ideal of compact objects, they have the same cosupport. Thus $\mathrm{Cosupp}(x) = \mathrm{Cosupp}(\mathsf{hom}(x, \mathbb{1})) = \mathrm{supp}(x) \cap \mathrm{Cosupp}(\mathbb{1})$ by Proposition 4.35. It follows that, although

$$\mathrm{Cosupp}(\mathcal{T}) = \mathrm{Spc}(\mathcal{T}^c)$$

(as established in Proposition 4.25), we have

$$\mathrm{Cosupp}(\mathcal{T}^c) = \mathrm{Cosupp}(\mathbb{1}).$$

It is possible for the latter to be a proper subset of $\mathrm{Spc}(\mathcal{T}^c)$; see Example 8.7.

5. The stalk and costalk

5.1. *Definition.* We let $\Gamma_{\mathcal{P}}\mathcal{T}$ and $\Lambda^{\mathcal{P}}\mathcal{T}$ denote the essential images of the functors $\Gamma_{\mathcal{P}} \colon \mathcal{T} \to \mathcal{T}$ and $\Lambda^{\mathcal{P}} \colon \mathcal{T} \to \mathcal{T}$, respectively (Notation 4.20). We call $\Gamma_{\mathcal{P}}\mathcal{T}$ the *stalk* of $\mathcal{T}$ at $\mathcal{P}$ and call $\Lambda^{\mathcal{P}}\mathcal{T}$ the *costalk* of $\mathcal{T}$ at $\mathcal{P}$.

5.2. **Proposition.** *Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$.*

(a) The full subcategory $\Gamma_{\mathcal{P}}\mathcal{T}$ is a localizing ideal and

$$\Gamma_{\mathcal{P}}\mathcal{T} = \left\{\, t \in \mathcal{T} \,\middle|\, t \simeq \Gamma_{\mathcal{P}} t \,\right\}.$$

(b) The full subcategory $\Lambda^{\mathcal{P}}\mathcal{T}$ is a colocalizing coideal and

$$\Lambda^{\mathcal{P}}\mathcal{T} = \left\{ t \in \mathcal{T} \,\middle|\, t \simeq \Lambda^{\mathcal{P}} t \right\}.$$

Proof. We prove part (b); the proof of (a) is similar. Since $\mathcal{P}$ is weakly visible we can write $\{\mathcal{P}\} = Y_1 \cap Y_2^c$ with Y_1 and Y_2 Thomason subsets. Consider the exact triangle

$$e_{Y_1 \cap Y_2} \to e_{Y_1} \to g_{\mathcal{P}} \to \Sigma e_{Y_1 \cap Y_2}$$

obtained from $e_{Y_2} \to \mathbb{1} \to f_{Y_2} \to \Sigma e_{Y_2}$ by tensoring with e_{Y_1}. We first establish that $t \in \Lambda^{\mathcal{P}}\mathcal{T}$ if and only if the canonical maps

$$\mathsf{hom}(g_{\mathcal{P}}, t) \to \mathsf{hom}(e_{Y_1}, t) \leftarrow t \tag{5.3}$$

induced from $e_{Y_1} \to g_{\mathcal{P}}$ and $e_{Y_1} \to \mathbb{1}$ are isomorphisms. Indeed, if $t \in \Lambda^{\mathcal{P}}\mathcal{T}$ then $t \simeq \mathsf{hom}(g_{\mathcal{P}}, t')$ for some $t' \in \mathcal{T}$. Since $\mathcal{P} \notin Y_1 \cap Y_2$ we have $e_{Y_1 \cap Y_2} \otimes g_{\mathcal{P}} = 0$ and hence $\mathsf{hom}(e_{Y_1 \cap Y_2}, t) = \mathsf{hom}(e_{Y_1 \cap Y_2} \otimes g_{\mathcal{P}}, t') = 0$. Also, $\mathsf{hom}(f_{Y_1}, t) = \mathsf{hom}(f_{Y_1} \otimes g_{\mathcal{P}}, t') = 0$ since $\mathcal{P} \in Y_1$ implies $f_{Y_1} \otimes g_{\mathcal{P}} = 0$. (Here we have repeatedly invoked Remark 4.8.)

We have thus established that $t \in \Lambda^{\mathcal{P}}\mathcal{T}$ if and only if the canonical maps in (5.3) are isomorphisms if and only if there exists an isomorphism $t \simeq \mathsf{hom}(g_{\mathcal{P}}, t) = \Lambda^{\mathcal{P}} t$. Then, it follows from the latter characterization that $\Lambda^{\mathcal{P}}\mathcal{T}$ is closed under exact triangles and hence is a triangulated subcategory of $\mathcal{T}$. Now, $\Lambda^{\mathcal{P}}\mathcal{T}$ is closed under products since $\Lambda^{\mathcal{P}} = \mathsf{hom}(g_{\mathcal{P}}, -)$ is a right adjoint. Moreover, it is a coideal: if $t_1 \in \mathcal{T}$ and $t_2 \in \Lambda^{\mathcal{P}}\mathcal{T}$ then

$$\mathsf{hom}(t_1, t_2) \simeq \mathsf{hom}(t_1, \mathsf{hom}(g_{\mathcal{P}}, t_2)) \simeq \mathsf{hom}(g_{\mathcal{P}}, \mathsf{hom}(t_1, t_2)) = \Lambda^{\mathcal{P}} \mathsf{hom}(t_1, t_2)$$

by adjunction. □

5.4. *Remark.* It follows from Lemma 4.32 and [BHS23, Lemma 2.13] that

$$\operatorname{Supp}(\Gamma_{\mathcal{P}}\mathcal{T}) \subseteq \{\mathcal{P}\} \quad \text{and} \quad \operatorname{Cosupp}(\Lambda^{\mathcal{P}}\mathcal{T}) \subseteq \{\mathcal{P}\}.$$

This leads to the following:

5.5. *Definition.* We say that $\mathcal{T}$ satisfies

(a) the *detection property* if $\operatorname{Supp}(t) = \varnothing$ implies $t = 0$ for all $t \in \mathcal{T}$;
(b) the *codetection property* if $\operatorname{Cosupp}(t) = \varnothing$ implies $t = 0$ for all $t \in \mathcal{T}$.

5.6. **Lemma.** *Let $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$.*

(a) If $\mathcal{T}$ has the detection property then

$$\Gamma_{\mathcal{P}}\mathcal{T} = \left\{ t \in \mathcal{T} \,\middle|\, \operatorname{Supp}(t) \subseteq \{\mathcal{P}\} \right\}.$$

(b) If $\mathcal{T}$ has the codetection property then

$$\Lambda^{\mathcal{P}}\mathcal{T} = \left\{ t \in \mathcal{T} \,\middle|\, \operatorname{Cosupp}(t) \subseteq \{\mathcal{P}\} \right\}.$$

Proof. The inclusion $\subseteq$ in both (a) and (b) follows from Remark 5.4. We now establish the reverse inclusion of part (b); the proof of the reverse inclusion of part (a) is similar. We can write $\{\mathcal{P}\} = Y_1 \cap \operatorname{gen}(\mathcal{P})$ with $Y_1 \subseteq \operatorname{Spc}(\mathcal{T}^c)$ Thomason, so that $g_{\mathcal{P}} = e_{Y_1} \otimes f_{\operatorname{gen}(\mathcal{P})^c}$. If $\operatorname{Cosupp}(t) \subseteq \{\mathcal{P}\}$ then $\mathsf{hom}(e_{\operatorname{gen}(\mathcal{P})^c}, t)$ and $\mathsf{hom}(f_{Y_1}, t)$ have empty cosupport by Lemma 4.32. It follows that $\mathsf{hom}(e_{\operatorname{gen}(\mathcal{P})^c}, t) = 0$ and $\mathsf{hom}(f_{Y_1}, t) = 0$ since $\mathcal{T}$ has the codetection property. Hence $t \simeq \mathsf{hom}(f_{\operatorname{gen}(\mathcal{P})^c}, t)$ and $t \simeq \mathsf{hom}(e_{\operatorname{gen}(\mathcal{P})^c}, t)$. Together, this implies $t \simeq \mathsf{hom}(g_{\mathcal{P}}, t) \in \Lambda^{\mathcal{P}}\mathcal{T}$ and we are done. □

5.7. **Proposition.** *If $\mathcal{T}$ satisfies the codetection property then it also satisfies the detection property.*

Proof. Let $t \in \mathcal{T}$ and suppose $\mathrm{Supp}(t) = \varnothing$. Then

$$\mathrm{Cosupp}(\mathsf{hom}(t,t)) \subseteq \mathrm{Supp}(t) \cap \mathrm{Cosupp}(t)$$

by Lemma 4.29 which implies that $\mathrm{Cosupp}(\mathsf{hom}(t,t)) = \varnothing$. The codetection property then implies $\mathsf{hom}(t,t) = 0$. Hence $\mathcal{T}(t,t) = \mathcal{T}(\mathbb{1}, \mathsf{hom}(t,t)) = 0$, so that $t = 0$. □

5.8. *Example.* In contrast, the following example shows that the detection property does not imply the codetection property. Let R be an absolutely flat ring and let $\mathcal{T} = \mathrm{D}(R)$. Since $\mathrm{Spc}(\mathcal{T}^c) \cong \mathrm{Spec}(R)$ has dimension zero, $\{\mathfrak{p}\} = \mathrm{Spec}(R) \cap \mathrm{gen}(\mathfrak{p})$. Hence $g_{\mathfrak{p}} \cong f_{\mathrm{gen}(\mathfrak{p})^c} \cong R_{\mathfrak{p}} \cong k(\mathfrak{p})$ is the residue field of the prime ideal $\mathfrak{p}$; see [Ste14a, Lemma 4.2 and Equation (4.1)] and [BHS23, Example 1.36]. Hence, we have

$$\mathrm{Cosupp}(t) = \left\{\, \mathfrak{p} \in \mathrm{Spec}(R) \,\middle|\, \mathsf{hom}(k(\mathfrak{p}), t) \neq 0 \,\right\}$$

for any $t \in \mathrm{D}(R)$. The detection property holds for any absolutely flat ring R by [Ste14a, Lemma 4.1]. On the other hand, the codetection property does not hold if R is not semi-artinian. This follows from the argument in the proof of [Ste14a, Theorem 4.8]: If R is not semi-artinian then there exists a superdecomposable (pure-)injective R-module $E \neq 0$ by [Trl96, Lemma 1.2]. For each $\mathfrak{p} \in \mathrm{Spec}(R)$, we must have $\mathrm{Hom}_R(k(\mathfrak{p}), E) = 0$ for otherwise E would contain the (injective) simple module $k(\mathfrak{p})$ as a direct summand, contradicting the superdecomposability of E. It follows that $\mathrm{Cosupp}(E) = \varnothing$ even though $E \neq 0$. An explicit example of an absolutely flat ring which is not semi-artinian is an infinite product of fields, such as $R = \prod_{\mathbb{N}} \mathbb{F}_p$. Theorem 6.4 below will provide further insight into this example and the relationship between the detection and codetection properties.

5.9. **Lemma.** *Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$.*

(a) *If $t \in \Gamma_{\mathcal{P}}\mathcal{T}$, then $\mathrm{Supp}(t) \subseteq \mathrm{Cosupp}(t)$.*
(b) *If $t \in \Lambda^{\mathcal{P}}\mathcal{T}$, then $\mathrm{Cosupp}(t) \subseteq \mathrm{Supp}(t)$.*

Proof. If $t \in \Gamma_{\mathcal{P}}\mathcal{T}$ then $t \simeq \Gamma_{\mathcal{P}} t$ by Proposition 5.2 and hence $\mathrm{Supp}(\Gamma_{\mathcal{P}} t) \subseteq \{\mathcal{P}\}$ by Remark 5.4. The claim is vacuously true if $t = 0$; otherwise

$$0 \neq \mathcal{T}(t,t) \simeq \mathcal{T}(\Gamma_{\mathcal{P}} t, t) \simeq \mathcal{T}(t, \Lambda^{\mathcal{P}} t)$$

shows that $\Lambda^{\mathcal{P}} t \neq 0$, i.e., $\mathcal{P} \in \mathrm{Cosupp}(t)$. This establishes (a). The proof of (b) uses a similar argument. If $t \in \Lambda^{\mathcal{P}}\mathcal{T}$ then $t \simeq \Lambda^{\mathcal{P}} t$ by Proposition 5.2 and hence $\mathrm{Cosupp}(\Lambda^{\mathcal{P}} t) \subseteq \{\mathcal{P}\}$ by Remark 5.4. If $t = 0$, the inclusion holds. If not, the same displayed equation applies to show that $\Gamma_{\mathcal{P}} t \neq 0$. □

5.10. **Corollary.** *For any $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$, we have $\mathrm{Cosupp}(\Lambda^{\mathcal{P}}\Gamma_{\mathcal{P}}\mathbb{1}) = \{\mathcal{P}\}$. Moreover, for every subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$, there exists an object $t \in \mathcal{T}$ with $\mathrm{Cosupp}(t) = Y$.*

Proof. The first statement is a consequence of Lem. 5.9(a) and Lem. 4.32(c), together with the observation that $\Gamma_{\mathcal{P}}\mathbb{1} = g_{\mathcal{P}} \neq 0$ (Remark 4.8). Then, given $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ the object $t \coloneqq \prod_{\mathcal{P} \in Y} \Lambda^{\mathcal{P}}\Gamma_{\mathcal{P}}\mathbb{1}$ has cosupport equal to Y. □

5.11. *Remark.* Let $Y_1, Y_2 \subseteq \mathrm{Spc}(\mathcal{T}^c)$ be Thomason subsets. Recall from (4.2) that for any smashing localization $e \to \mathbb{1} \to f \to \Sigma e$ we have an equivalence $e \otimes \mathcal{T} \cong \mathsf{hom}(e, \mathcal{T})$ and an equality $f \otimes \mathcal{T} = \mathsf{hom}(f, \mathcal{T})$. In particular, it follows that $\mathsf{hom}(f, f \otimes t) \simeq f \otimes t$ and $f \otimes \mathsf{hom}(f,t) \simeq \mathsf{hom}(f,t)$ for any $t \in \mathcal{T}$. From this, and the fact that idempotent functors are fully faithful on their essential image, we deduce that we have a diagram of adjunctions

$$\begin{tikzcd}
\mathsf{hom}(e_{Y_1}, \mathcal{T}) \arrow[r, shift left] \arrow[r, phantom, "\cong"] \arrow[d, shift left, "\mathsf{hom}(f_{Y_2},-)"] \arrow[d, phantom, "\dashv"] & e_{Y_1} \otimes \mathcal{T} \arrow[l, hook, shift left] \arrow[d, shift right, "-\otimes f_{Y_2}"'] \arrow[d, phantom, "\dashv"] \\
\mathsf{hom}(f_{Y_2}, \mathsf{hom}(e_{Y_1}, \mathcal{T}),) \arrow[u, hook, shift left, "-\otimes f_{Y_2}"] & f_{Y_2} \otimes e_{Y_1} \otimes \mathcal{T} \arrow[u, hook, shift right, "\mathsf{hom}(f_{Y_2},-)"']
\end{tikzcd}$$

where the hooked arrows are fully faithful. In particular, writing $W \coloneqq Y_1 \cap Y_2^c$, the $\Gamma_W : \mathcal{T} \rightleftarrows \mathcal{T} : \Lambda^W$ adjunction restricts to an adjoint equivalence

$$\begin{tikzcd}
\Lambda^W \mathcal{T} \arrow[r, bend left] \arrow[r, phantom, "\cong"] & \Gamma_W \mathcal{T}. \arrow[l, hook, bend left]
\end{tikzcd}$$

This type of equivalence has been previously observed in [BIK12, Prop. 5.1]. For $Y_2 = \varnothing$ it specializes to the Matlis–Greenlees–May style equivalence mentioned in Example 4.4; cf. [DG02, Theorem 2.1]. In particular, we obtain an equivalence

$$\Lambda^{\mathcal{P}} \mathcal{T} \cong \Gamma_{\mathcal{P}} \mathcal{T}$$

between the costalk and stalk for any weakly visible point $\mathcal{P}$. This equivalence provides another perspective on Lemma 5.9.

5.12. *Example.* Consider a smashing ideal $\mathcal{L}$ with idempotent triangle

$$e \to \mathbb{1} \to f \to \Sigma e.$$

Then $\mathcal{L} = e \otimes \mathcal{T}$ and it follows from Remark 4.1 that its right orthogonal $\mathcal{L}^{\perp} = f \otimes \mathcal{T} = \mathsf{hom}(f, \mathcal{T})$ is both a localizing ideal and colocalizing coideal, and $\mathcal{L}^{\perp\perp} = \mathsf{hom}(e, \mathcal{T})$ is a colocalizing coideal. Using these descriptions we can readily check that

$$\mathrm{Supp}(e) = \mathrm{Supp}(\mathcal{L}) = \mathrm{Cosupp}(\mathcal{L}^{\perp\perp})$$

and

$$\mathrm{Supp}(f) = \mathrm{Supp}(\mathcal{L}^{\perp}) = \mathrm{Cosupp}(\mathcal{L}^{\perp}).$$

For example, $\mathrm{Cosupp}(\mathcal{L}^{\perp}) = \mathrm{Cosupp}(\mathsf{hom}(f, \mathcal{T})) \subseteq \mathrm{Supp}(f)$ by Lemma 4.29. Conversely, if $\mathcal{P} \in \mathrm{Supp}(f)$ then $\mathcal{P} \in \mathrm{Cosupp}(\mathcal{L}^{\perp})$ since $g_{\mathcal{P}} \otimes f \neq 0$ implies

$$\mathsf{hom}(g_{\mathcal{P}}, \mathsf{hom}(f, g_{\mathcal{P}} \otimes f)) = \mathsf{hom}(g_{\mathcal{P}} \otimes f, g_{\mathcal{P}} \otimes f) \neq 0.$$

A similar argument gives $\mathrm{Cosupp}(\mathcal{L}^{\perp\perp}) = \mathrm{Supp}(e)$.

6. The local-to-global principle

6.1. *Definition* (Local-to-global principle). Let $\mathcal{T}$ be a rigidly-compactly generated tt-category with $\mathrm{Spc}(\mathcal{T}^c)$ weakly noetherian. Recall from [BHS23, Definition 3.8] that $\mathcal{T}$ satisfies the *local-to-global principle for localizing ideals* (or simply the *local-to-global principle*) if for every object $t \in \mathcal{T}$, we have

$$\mathrm{Locid}\langle t \rangle = \mathrm{Locid}\langle \Gamma_{\mathcal{P}} t \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c) \rangle. \tag{6.2}$$

Note that this implies the detection property (Definition 5.5) since the right-hand side of (6.2) is the same as $\mathrm{Locid}\langle \Gamma_{\mathcal{P}} t \mid \mathcal{P} \in \mathrm{Supp}(t) \rangle$. Similarly, we say that $\mathcal{T}$ satisfies the *local-to-global principle for colocalizing coideals* (or simply the *colocal-to-global principle*) if for every object $t \in \mathcal{T}$, we have

$$\mathrm{Colocid}\langle t \rangle = \mathrm{Colocid}\langle \Lambda^{\mathcal{P}} t \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c) \rangle. \tag{6.3}$$

Note that this implies the codetection property since the right-hand side of (6.3) is the same as $\mathrm{Colocid}\langle \Lambda^{\mathcal{P}} t \mid \mathcal{P} \in \mathrm{Cosupp}(t)\rangle$.

6.4. **Theorem.** *Let $\mathcal{T}$ be a rigidly-compactly generated tt-category with weakly noetherian spectrum. The following are equivalent:*

(a) $\mathcal{T}$ satisfies the codetection property;
(b) $\mathcal{T}$ satisfies the local-to-global principle for localizing ideals;
(c) $\mathcal{T}$ satisfies the local-to-global principle for colocalizing coideals.

Proof. Let $\mathcal{L} := \mathrm{Locid}\langle g_{\mathcal{P}} \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)\rangle$. An object $t \in \mathcal{T}$ satisfies $\mathrm{Cosupp}(t) = \varnothing$ if and only if $t \in \mathcal{L}^{\perp}$. Thus, the codetection property is equivalent to $\mathcal{L}^{\perp} = 0$. On the other hand, the local-to-global principle for localizing ideals is equivalent to $\mathcal{L} = \mathcal{T}$ since if $\mathbb{1} \in \mathcal{L}$ then for any $t \in \mathcal{T}$ we have

$$t = t \otimes \mathbb{1} \in t \otimes \mathcal{L} \subseteq \mathrm{Locid}\langle t \otimes g_{\mathcal{P}} \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)\rangle$$

by (2.6). Clearly $\mathcal{L} = \mathcal{T}$ implies $\mathcal{L}^{\perp} = 0$. On the other hand, $\mathcal{L}$ is strictly localizing by Theorem 2.13, hence we have $\mathcal{L} = {}^{\perp}(\mathcal{L}^{\perp})$ (Remark 2.11) so that $\mathcal{L}^{\perp} = 0$ implies $\mathcal{L} = \mathcal{T}$. Thus, (a) and (b) are equivalent. Condition (c) evidently implies (a) so it remains to show that (b) implies (c). This is also immediate since if $\mathbb{1} \in \mathcal{L}$ then for any $t \in \mathcal{T}$ we have

$$t = \mathsf{hom}(\mathbb{1}, t) \in \mathsf{hom}(\mathcal{L}, t) \subseteq \mathrm{Colocid}\langle \mathsf{hom}(g_{\mathcal{P}}, t) \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)\rangle$$

by (2.8). □

6.5. **Corollary.** *If $\mathrm{Spc}(\mathcal{T}^c)$ is noetherian, then $\mathcal{T}$ satisfies the local-to-global principle for colocalizing coideals.*

Proof. The local-to-global principle for localizing ideals holds by [BHS23, Theorem 3.21] and hence the result follows from Theorem 6.4. □

6.6. *Remark.* Although the proof of Theorem 6.4 is not difficult in hindsight, we regard the statement as quite surprising. The local-to-global and colocal-to-global principles are equivalent, and in fact are equivalent to the codetection property. In particular, this shows that the theory of cosupport is highly relevant even for the task of classifying localizing ideals via a theory of support. The theorem also provides another way to see that the codetection property does not always hold. Indeed, [Ste14a, Theorem 4.8] establishes that if R is an absolutely flat ring which is not semi-artinian then $\mathrm{D}(R)$ does not satisfy the local-to-global principle; cf. Example 5.8. This example also shows that the detection property does not imply the codetection property. It remains a tantalizing possibility that the detection property is *always*[2] satisfied. We do not know of any counterexamples.

[2] When the spectrum is not weakly noetherian, the definition of support requires modification, as the example of p-local spectra shows. For some remarks in this direction, see [BHS21, Remark 5.14]. A general approach to extending the Balmer–Favi support to points which are not weakly visible is considered by William Sanders [San17] and has been further developed in recent work of Changhan Zou [Zou23].

7. Costratification

7.1. *Definition* (Costratification). Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category with $\operatorname{Spc}(\mathcal{T}^c)$ weakly noetherian. We say that $\mathcal{T}$ is *costratified* if the cosupport theory $(\operatorname{Spc}(\mathcal{T}^c), \operatorname{Cosupp})$ (Definition 4.23) induces a bijection

$$\big\{\text{colocalizing coideals of } \mathcal{T}\big\} \xrightarrow{\operatorname{Cosupp}} \big\{\text{subsets of } \operatorname{Spc}(\mathcal{T}^c)\big\}. \tag{7.2}$$

The inclusion-preserving function (7.2) is always surjective by Corollary 5.10, so costratification amounts to its injectivity.

7.3. *Remark.* For a subset $Y \subseteq \operatorname{Spc}(\mathcal{T})$, we use the notation

$$\operatorname{Supp}^{-1}(Y) := \big\{\, t \in \mathcal{T} \,\big|\, \operatorname{Supp}(t) \subseteq Y \,\big\}$$

for the localizing ideal of objects supported on Y, and

$$\operatorname{Cosupp}^{-1}(Y) := \big\{\, t \in \mathcal{T} \,\big|\, \operatorname{Cosupp}(t) \subseteq Y \,\big\}$$

for the colocalizing coideal of objects cosupported on Y. If $\mathcal{T}$ is costratified, then the inverse of (7.2) is necessarily given by $Y \mapsto \operatorname{Cosupp}^{-1}(Y)$. Indeed, we always have the inclusion $\mathcal{C} \subseteq \operatorname{Cosupp}^{-1}(\operatorname{Cosupp}(\mathcal{C}))$ and costratification is equivalent to these inclusions being equalities.

7.4. *Remark.* If a colocalizing coideal is generated by a set of objects $\mathcal{E}$ then it is also generated by a single object: $\operatorname{Colocid}\langle\mathcal{E}\rangle = \operatorname{Colocid}\langle\prod_{t\in\mathcal{E}} t\rangle$. We refer to such colocalizing coideals as the *set-generated* colocalizing coideals.

7.5. **Proposition.** *If the class of set-generated colocalizing coideals of $\mathcal{T}$ forms a set, then every colocalizing coideal of $\mathcal{T}$ is generated by a set.*

Proof. The argument in [KS19, Lemma 3.3.1] goes through verbatim with "localizing subcategory" and "Loc" replaced by "colocalizing coideal" and "Colocid"; cf. [BHS23, Proposition 3.5]. □

7.6. *Remark.* It follows from Corollary 5.10 and Lemma 4.32 that the colocalizing coideal $\Lambda^{\mathcal{P}}\mathcal{T}$ is nonzero and in fact has cosupport equal to the single point $\{\mathcal{P}\}$. Thus, a necessary condition for costratification is that $\Lambda^{\mathcal{P}}\mathcal{T}$ contains no nontrivial proper colocalizing coideals, i.e., that it is a *minimal* colocalizing coideal.

7.7. **Theorem.** *Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category with weakly noetherian spectrum. The following are equivalent:*

- *(a) The local-to-global principle for colocalizing coideals holds for $\mathcal{T}$, and for all $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$, $\Lambda^{\mathcal{P}}\mathcal{T}$ is a minimal colocalizing coideal of $\mathcal{T}$.*
- *(b) For all $t \in \mathcal{T}$, $\operatorname{Colocid}\langle t\rangle = \operatorname{Colocid}\langle \Lambda^{\mathcal{P}}\mathcal{T} \mid \mathcal{P} \in \operatorname{Cosupp}(t)\rangle$.*
- *(c) The function*

$$\big\{\textit{colocalizing coideals of } \mathcal{T}\big\} \xrightarrow{\operatorname{Cosupp}} \big\{\textit{subsets of } \operatorname{Spc}(\mathcal{T}^c)\big\}$$

is injective (and hence a bijection by Corollary 5.10); that is, $\mathcal{T}$ is costratified.

Proof. $(a) \Rightarrow (b)$: By the local-to-global principle for colocalizing coideals we have

$$\operatorname{Colocid}\langle t\rangle = \operatorname{Colocid}\langle \Lambda^{\mathcal{P}} t \mid \mathcal{P} \in \operatorname{Cosupp}(t)\rangle.$$

Since $\Lambda^{\mathcal{P}}\mathcal{T}$ is a minimal colocalizing coideal, for any $t \in \mathcal{T}$ we have equalities $\operatorname{Colocid}\langle\Lambda^{\mathcal{P}} t\rangle = \Lambda^{\mathcal{P}}\mathcal{T}$ for any $\mathcal{P} \in \operatorname{Cosupp}(t)$. Therefore,

$$\operatorname{Colocid}\langle t\rangle = \operatorname{Colocid}\langle \Lambda^{\mathcal{P}}\mathcal{T} \mid \mathcal{P} \in \operatorname{Cosupp}(t)\rangle.$$

$(b) \Rightarrow (c)$: We will actually show that the map

$$\big\{\text{set-generated colocalizing coideals of } \mathcal{T}\big\} \xrightarrow{\text{Cosupp}} \big\{\text{subsets of } \operatorname{Spc}(\mathcal{T}^c)\big\} \tag{7.8}$$

is injective. It will follow that there is only a set of set-generated colocalizing coideals. Hence by Proposition 7.5, every colocalizing coideal is set-generated, so we will have established (c). Note that the map is always surjective by Corollary 5.10. Recall that every set-generated colocalizing coideal of $\mathcal{T}$ is generated by a single object (Remark 7.4) and that $\operatorname{Cosupp}(\operatorname{Colocid}\langle t\rangle) = \operatorname{Cosupp}(t)$ (Remark 3.2). Thus the injectivity of (7.8) is equivalent to

$$\forall t_1, t_2 \in \mathcal{T}, \operatorname{Cosupp}(t_1) = \operatorname{Cosupp}(t_2) \Longrightarrow \operatorname{Colocid}\langle t_1\rangle = \operatorname{Colocid}\langle t_2\rangle.$$

If $\operatorname{Cosupp}(t_1) = \operatorname{Cosupp}(t_2)$ then (b) implies that $\operatorname{Colocid}\langle t_1\rangle = \operatorname{Colocid}\langle t_2\rangle$, so we are done.

$(c) \Rightarrow (a)$: Suppose the function Cosupp is injective. Applied to the zero coideal, we obtain the codetection property and hence the local-to-global principle by Theorem 6.4. Suppose then that $0 \neq \mathcal{C} \subseteq \Lambda^{\mathcal{P}}\mathcal{T}$ is a colocalizing coideal. Then

$$\varnothing \neq \operatorname{Cosupp}(\mathcal{C}) \subseteq \operatorname{Cosupp}(\Lambda^{\mathcal{P}}\mathcal{T}) = \{\mathcal{P}\}$$

using Lemma 4.32. Hence $\operatorname{Cosupp}(\mathcal{C}) = \operatorname{Cosupp}(\Lambda^{\mathcal{P}}\mathcal{T})$ and so $\mathcal{C} = \Lambda^{\mathcal{P}}\mathcal{T}$. This establishes the minimality of $\Lambda^{\mathcal{P}}\mathcal{T}$. □

7.9. *Terminology.* We say that $\mathcal{T}$ satisfies (or has) "cominimality at $\mathcal{P}$" if $\Lambda^{\mathcal{P}}\mathcal{T}$ is a minimal colocalizing coideal of $\mathcal{T}$ as in Remark 7.6 and part (a) of Theorem 7.7.

7.10. *Remark.* Recall from [BHS23] that $\mathcal{T}$ is said to be *stratified* if the Balmer–Favi support theory $(\operatorname{Spc}(\mathcal{T}^c), \operatorname{Supp})$ provides a bijection

$$\big\{\text{localizing ideals of } \mathcal{T}\big\} \xrightarrow{\text{Supp}} \big\{\text{subsets of } \operatorname{Spc}(\mathcal{T}^c)\big\}.$$

Theorem 7.7 should be compared with [BHS23, Theorem 4.1] which provides a directly analogous characterization of stratification. Note that the statement of part (b) in Theorem 7.7 is slightly different than the formulation of part (b) in [BHS23, Theorem 4.1]. Morally this is because cosupport is not controlled by the unit $\mathbb{1}$ in the same way that support is.

7.11. *Remark.* Our next goal is to prove that costratification implies stratification. To this end, we will utilize the following crucial result which is a minor modification of [BIK11a, Lemma 3.9].

7.12. **Lemma** (Benson–Iyengar–Krause)**.** *Let $\mathcal{T}$ be a rigidly-compactly generated tt-category and let $\mathcal{L}$ be a nonzero localizing ideal of $\mathcal{T}$. The following statements are equivalent:*

(a) The localizing ideal $\mathcal{L}$ is minimal.
(b) $\mathsf{hom}(t_1, t_2) \neq 0$ for any two nonzero objects $t_1, t_2 \in \mathcal{L}$.

Proof. $(a) \Rightarrow (b)$: If $t_1 \neq 0$ then minimality implies $\mathcal{L} = \operatorname{Locid}\langle t_1\rangle$. Hence if $\mathsf{hom}(t_1, t_2) = 0$ then $\mathsf{hom}(\mathcal{L}, t_2) = 0$. In particular $\mathsf{hom}(t_2, t_2) = 0$ so that $t_2 = 0$.

$(b) \Rightarrow (a)$: Let $0 \neq t_1 \in \mathcal{L}$. Then $\mathcal{L}_1 \coloneqq \operatorname{Locid}\langle t_1\rangle = \operatorname{Loc}\langle t_1 \otimes \mathcal{T}^c\rangle$ is a set-generated localizing subcategory and hence is strictly localizing by Theorem 2.13. By Remark 2.11, $\mathcal{L}_1 \subseteq \mathcal{L}$ is the kernel of a Bousfield localization on $\mathcal{T}$. For any $t_2 \in \mathcal{L}$ consider the associated Bousfield localization triangle $\Gamma t_2 \to t_2 \to \mathrm{L}t_2 \to \Sigma\Gamma t_2$. The first two objects are in $\mathcal{L}$, thus so is $\mathrm{L}t_2$. However, (b) implies that no nonzero object of $\mathcal{L}$ can be contained in $(\mathcal{L}_1)^{\perp}$; that is, $(\mathcal{L}_1)^{\perp} \cap \mathcal{L} = \{0\}$. Hence $\mathrm{L}t_2 = 0$, since $\Gamma t_2 \in \mathcal{L}_1$, so that $\Gamma t_2 \simeq t_2$ is contained in $\mathcal{L}_1$. This proves that $\mathcal{L}_1 = \mathcal{L}$. ☐

7.13. *Remark.* We highlight that the proof of Lemma 7.12 crucially depends on the existence of Bousfield localizations for arbitrary set-generated localizing ideals. In light of Remark 2.14, one cannot establish minimality of colocalizing coideals by a similar argument.

7.14. *Remark.* Lemma 7.12 admits the following relative version: Let $\mathcal{L}$ be a localizing ideal of $\mathcal{T}$ and let $\mathcal{J} \subsetneq \mathcal{L}$ be a set-generated localizing subcategory. Then $\mathcal{L}$ is minimal among those localizing ideals which properly contain $\mathcal{J}$ if and only if $\mathsf{hom}(t_1, t_2) \neq 0$ for any two objects $t_1, t_2 \in \mathcal{L} \setminus \mathcal{J}$. In the proof one needs to consider $\mathcal{L}_1 := \mathrm{Locid}\langle \mathcal{J} \cup \{t_1\}\rangle$.

7.15. **Theorem.** *Let $\mathcal{T}$ be a rigidly-compactly generated tt-category whose spectrum is weakly noetherian. Assume that the local-to-global principle for localizing ideals holds. The following conditions are equivalent:*

(a) *$\mathcal{T}$ is stratified.*
(b) $\mathrm{Cosupp}\,\mathsf{hom}(t_1, t_2) = \mathrm{Supp}(t_1) \cap \mathrm{Cosupp}(t_2)$ *for all* $t_1, t_2 \in \mathcal{T}$.
(c) $\mathsf{hom}(t_1, t_2) = 0$ *implies* $\mathrm{Supp}(t_1) \cap \mathrm{Cosupp}(t_2) = \varnothing$ *for all* $t_1, t_2 \in \mathcal{T}$.

Proof. Using Lemma 5.9 we can follow the proof of Theorem 9.5 of [BIK12], which we spell out for the reader.

$(a) \Rightarrow (b)$: One inclusion $(\subseteq)$ is Lemma 4.29. For the other inclusion, let $\mathcal{P} \in \mathrm{Supp}(t_1) \cap \mathrm{Cosupp}(t_2)$. In particular, $\Gamma_{\mathcal{P}} t_1 \neq 0$. Since $\Gamma_{\mathcal{P}}\mathcal{T}$ is a nontrivial minimal localizing ideal, $g_{\mathcal{P}} \in \mathrm{Locid}\langle \Gamma_{\mathcal{P}} t_1\rangle$. Using [BIK12, Lemma 8.4] and the assumption $\mathcal{P} \in \mathrm{Cosupp}(t_2)$ we see that

$$0 \neq \Lambda^{\mathcal{P}} t_2 = \mathsf{hom}(g_{\mathcal{P}}, t_2) \in \mathrm{Colocid}\langle \mathsf{hom}(\Gamma_{\mathcal{P}} t_1, t_2)\rangle.$$

Therefore $0 \neq \mathsf{hom}(\Gamma_{\mathcal{P}} t_1, t_2) \simeq \mathsf{hom}(g_{\mathcal{P}}, \mathsf{hom}(t_1, t_2))$ and as a consequence we have that $\mathcal{P} \in \mathrm{Cosupp}\,\mathsf{hom}(t_1, t_2)$.

$(b) \Rightarrow (c)$: This follows from $\mathrm{Cosupp}(0) = \varnothing$.

$(c) \Rightarrow (a)$: The local-to-global principle for localizing ideals holds by hypothesis. We show that the criteria of Lemma 7.12 are satisfied for $\Gamma_{\mathcal{P}}\mathcal{T}$. Let $t_1, t_2 \in \Gamma_{\mathcal{P}}\mathcal{T}$ be nonzero objects. Then $\mathcal{P} \in \mathrm{Supp}(t_1)$ and $\mathcal{P} \in \mathrm{Supp}(t_2)$. By Lemma 5.9, we also have $\mathcal{P} \in \mathrm{Cosupp}(t_2)$. Hence (c) implies $\mathsf{hom}(t_1, t_2) \neq 0$. ☐

7.16. *Remark.* The appearance of cosupport in the above characterization of *stratification* should be regarded as significant: it is a fundamental characterization of stratification which utilizes the notion of cosupport. This motivates the study of cosupport even for questions purely about localizing ideals.

7.17. *Remark.* Recall that stratification implies that the Balmer–Favi support satisfies the full tensor product property: $\mathrm{Supp}(t_1 \otimes t_2) = \mathrm{Supp}(t_1) \cap \mathrm{Supp}(t_2)$ for all $t_1, t_2 \in \mathcal{T}$; see [BHS23, Theorem 8.2]. It would be natural to guess that costratification would promote the half-hom theorem (Proposition 4.35) to a full hom-theorem: $\mathrm{Cosupp}(\mathsf{hom}(t_1, t_2)) = \mathrm{Supp}(t_1) \cap \mathrm{Cosupp}(t_2)$. Perhaps surprisingly, Theorem 7.15 shows that this full hom-theorem is actually equivalent to *stratification*.

7.18. **Corollary.** *If $\mathcal{T}$ is stratified then:*

(a) $\mathcal{L}^{\perp} = \mathrm{Cosupp}^{-1}(\mathrm{Supp}(\mathcal{L})^c)$ *for every localizing ideal* $\mathcal{L}$.
(b) ${}^{\perp}\mathcal{C} = \mathrm{Supp}^{-1}(\mathrm{Cosupp}(\mathcal{C})^c)$ *for every colocalizing coideal* $\mathcal{C}$.

Proof. Stratification implies the local-to-global principle by [BHS23, Theorem 4.1] and hence also the codetection property by Theorem 6.4. Hence we can use Theorem 7.15 to observe that

$$\begin{aligned}\mathcal{L}^{\perp} &= \left\{ t \in \mathcal{T} \,\middle|\, \mathsf{hom}(s,t) = 0 \text{ for all } s \in \mathcal{L} \right\} \\ &= \left\{ t \in \mathcal{T} \,\middle|\, \operatorname{Cosupp} \mathsf{hom}(s,t) = \varnothing \text{ for all } s \in \mathcal{L} \right\} \\ &= \left\{ t \in \mathcal{T} \,\middle|\, \operatorname{Supp}(s) \cap \operatorname{Cosupp}(t) = \varnothing \text{ for all } s \in \mathcal{L} \right\} \\ &= \operatorname{Cosupp}^{-1}(\operatorname{Supp}(\mathcal{L})^c)\end{aligned}$$

and similarly

$$\begin{aligned}{}^{\perp}\mathcal{C} &= \left\{ s \in \mathcal{T} \,\middle|\, \mathsf{hom}(s,t) = 0 \text{ for all } t \in \mathcal{C} \right\} \\ &= \left\{ s \in \mathcal{T} \,\middle|\, \operatorname{Cosupp} \mathsf{hom}(s,t) = \varnothing \text{ for all } t \in \mathcal{C} \right\} \\ &= \left\{ s \in \mathcal{T} \,\middle|\, \operatorname{Supp}(s) \cap \operatorname{Cosupp}(t) = \varnothing \text{ for all } t \in \mathcal{C} \right\} \\ &= \operatorname{Supp}^{-1}(\operatorname{Cosupp}(\mathcal{C})^c).\end{aligned}$$

□

7.19. **Theorem.** *If $\mathcal{T}$ is costratified, then it is also stratified.*

Proof. Since the colocal-to-global principle is equivalent to the local-to-global principle by Theorem 6.4, it suffices by [BHS23, Theorem 4.1] and Theorem 7.7 to establish that $\Gamma_{\mathcal{P}}\mathcal{T}$ is minimal for all $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$. For this we follow the proof of [BIK12, Theorem 9.7] by invoking Lemma 7.12. To this end, we fix two nonzero objects $t_1, t_2 \in \Gamma_{\mathcal{P}}\mathcal{T}$ and show that $\mathsf{hom}(t_1, t_2) \neq 0$. Since $t_1 \in \Gamma_{\mathcal{P}}\mathcal{T}$, we have that $\Gamma_{\mathcal{P}} t_1 \simeq t_1$ by Proposition 5.2. Then by Remark 4.21, $\mathsf{hom}(t_1, t_2) \simeq \mathsf{hom}(t_1, \Lambda^{\mathcal{P}} t_2)$. Since $t_2 \in \Gamma_{\mathcal{P}}\mathcal{T}$, by Lemma 5.9 we have that $\mathcal{P} \in \operatorname{Cosupp}(t_2)$, that is, $\Lambda^{\mathcal{P}} t_2 \neq 0$. Because $\mathcal{T}$ is costratified, the cominimality of $\Lambda^{\mathcal{P}}\mathcal{T}$ implies that $\Lambda^{\mathcal{P}} t_1 \in \operatorname{Colocid}\langle \Lambda^{\mathcal{P}} t_2 \rangle$. To conclude, we observe that because $t_1 \neq 0$, we have $0 \neq \mathsf{hom}(t_1, t_1) \simeq \mathsf{hom}(t_1, \Lambda^{\mathcal{P}} t_1)$, which by (2.7) is only possible if $0 \neq \mathsf{hom}(t_1, \Lambda^{\mathcal{P}} t_2) \simeq \mathsf{hom}(t_1, t_2)$. □

7.20. **Corollary.** *If $\mathcal{T}$ is costratified, then the map sending a subcategory $\mathcal{L}$ to $\mathcal{L}^{\perp}$ induces a bijection*

$$\{\textit{localizing ideals of } \mathcal{T}\} \xrightarrow{\sim} \{\textit{colocalizing coideals of } \mathcal{T}\}$$

The inverse map sends $\mathcal{C}$ to ${}^{\perp}\mathcal{C}$.

Proof. Costratification and stratification (Theorem 7.19) provide bijections of both sets with the set of all subsets of $\operatorname{Spc}(\mathcal{T}^c)$. The map which sends a subset to its complement then induces a bijection from the set of localizing ideals to the set of colocalizing ideals which, by Corollary 7.18, is given by $\mathcal{L} \mapsto \mathcal{L}^{\perp}$ with inverse $\mathcal{C} \mapsto {}^{\perp}\mathcal{C}$. □

7.21. **Proposition.** *If $\mathcal{T}$ is stratified but not costratified, then $\mathcal{T}$ has a colocalizing coideal which is not strictly colocalizing.*

Proof. A strictly colocalizing coideal must be of the form $\mathcal{L}^{\perp}$ for a localizing ideal $\mathcal{L}$ (see Remark 2.11). If $\mathcal{T}$ is stratified then for any $Y \subseteq \operatorname{Spc}(\mathcal{T}^c)$, we have

$$(\operatorname{Supp}^{-1}(Y))^{\perp} = (\operatorname{Locid}\langle g_{\mathcal{P}} \mid \mathcal{P} \in Y \rangle)^{\perp} = \left\{ g_{\mathcal{P}} \,\middle|\, \mathcal{P} \in Y \right\}^{\perp} = \operatorname{Cosupp}^{-1}(Y^c).$$

Now, since $\mathcal{T}$ is not costratified, there is some colocalizing coideal $\mathcal{C}$ such that $\mathcal{C} \neq \operatorname{Cosupp}^{-1}(\operatorname{Cosupp}(\mathcal{C}))$. But if $\mathcal{C}$ was strictly colocalizing then

$$\mathcal{C} = \mathcal{L}^{\perp} = (\operatorname{Supp}^{-1}(\operatorname{Supp}(\mathcal{L})))^{\perp} = \operatorname{Cosupp}^{-1}(\operatorname{Supp}(\mathcal{L})^c)$$

and $\mathrm{Supp}(\mathcal{L})^c = \mathrm{Supp}({}^{\perp}\mathcal{C})^c = \mathrm{Cosupp}(\mathcal{C})$ by Corollary 7.18(b), so that $\mathcal{C} = \mathrm{Cosupp}^{-1}(\mathrm{Cosupp}(\mathcal{C}))$. ∎

7.22. *Remark.* We do not know whether stratification implies costratification. This seems to be a difficult question. For example, suppose we could find an example $\mathcal{T}$ which is stratified but not costratified. By Proposition 7.21, this category would have a colocalizing coideal which is not strictly colocalizing. Assuming the counterexample arises from a combinatorial model, this would prove (Remark 2.14) that Vopěnka's principle is inconsistent with ZFC. Whether that is likely or unlikely is beyond our expertise.

7.23. *Remark.* On the other hand, if Vopěnka's principle is taken as an axiom then stratification implies costratification for any example that arises from a stable combinatorial model. Indeed, in this case every colocalizing coideal $\mathcal{C}$ of $\mathcal{T} = \mathrm{Ho}(\mathcal{M})$ is strictly colocalizing (Remark 2.14). Hence $\mathcal{C} = ({}^{\perp}\mathcal{C})^{\perp}$. If $\mathcal{T}$ is stratified then $({}^{\perp}\mathcal{C})^{\perp} = \mathrm{Cosupp}^{-1}(\mathrm{Cosupp}(\mathcal{C}))$ by Corollary 7.18 and costratification follows.

7.24. *Remark.* We can summarize the situation with the diagram displayed on page 6. It assembles the implications of Proposition 5.7, Theorem 6.4, Theorem 7.19, along with Example 5.8 and [BHS23, Example 4.6].

8. The geometry of cosupport

We have already seen some relations between support and cosupport in the previous sections. We now dig deeper into the somewhat mysterious relationship between the support and cosupport of a given object. Throughout this section $\mathcal{T}$ will denote a rigidly-compactly generated tt-category whose spectrum $\mathrm{Spc}(\mathcal{T}^c)$ is weakly noetherian.

8.1. **Proposition.** *Assume that codetection holds. Let $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ be a Thomason subset. Then for any $t \in \mathcal{T}$ we have:*

$$\mathrm{Supp}(t) \cap Y = \varnothing \text{ if and only if } \mathrm{Cosupp}(t) \cap Y = \varnothing.$$

Proof. The codetection property implies the detection property by Proposition 5.7. Then observe that

$$\begin{aligned}
\mathrm{Cosupp}(t) \cap Y = \varnothing &\iff \mathrm{Cosupp}(\mathsf{hom}(e_Y, t)) = \varnothing && \text{(Lemma 4.32)}\\
&\iff \mathsf{hom}(e_Y, t) = 0 && \text{(codetection)}\\
&\iff \mathsf{hom}(x, t) = 0 \text{ for all } x \in \mathcal{T}^c_Y\\
&\iff x \otimes t = 0 \text{ for all } x \in \mathcal{T}^c_Y\\
&\iff e_Y \otimes t = 0\\
&\iff \mathrm{Supp}(e_Y \otimes t) = \varnothing && \text{(detection)}\\
&\iff \mathrm{Supp}(t) \cap Y = \varnothing
\end{aligned}$$

which establishes the claim. ∎

8.2. *Remark.* The next theorem (and its corollary) expresses a fundamental relation between support and cosupport, generalizing [BIK12, Theorem 4.13]. The statement uses Notation 4.19 and Notation 4.16.

8.3. **Theorem.** *Assume that codetection holds. Then for any $t \in \mathcal{T}$ we have*

$$\mathrm{vis}\,\mathcal{T} \cap \min \mathrm{Supp}(t) = \mathrm{vis}\,\mathcal{T} \cap \min \mathrm{Cosupp}(t).$$

Proof. Let $\mathcal{P}$ be a visible point. Then $\{\mathcal{P}\} = \overline{\{\mathcal{P}\}} \cap \mathrm{gen}(\mathcal{P})$ and $\overline{\{\mathcal{P}\}} \setminus \{\mathcal{P}\} = \overline{\{\mathcal{P}\}} \cap \mathrm{gen}(\mathcal{P})^c$ is the intersection of two Thomason subsets and hence is itself Thomason. Applying Proposition 8.1 to the Thomason $\overline{\{\mathcal{P}\}} \setminus \{\mathcal{P}\}$ yields

$$\mathrm{Supp}(t) \cap \overline{\{\mathcal{P}\}} \subseteq \{\mathcal{P}\} \iff \mathrm{Cosupp}(t) \cap \overline{\{\mathcal{P}\}} \subseteq \{\mathcal{P}\}.$$

It follows, by invoking Proposition 8.1 again for the Thomason $\overline{\{\mathcal{P}\}}$, that

$$\mathrm{Supp}(t) \cap \overline{\{\mathcal{P}\}} = \{\mathcal{P}\} \iff \mathrm{Cosupp}(t) \cap \overline{\{\mathcal{P}\}} = \{\mathcal{P}\}.$$

Recall that $\mathcal{P} \in \min V$ means, by definition, that $V \cap \overline{\{\mathcal{P}\}} = \{\mathcal{P}\}$ so we are done. ☐

8.4. **Corollary.** *If $\mathcal{T}$ has noetherian spectrum and $t \in \mathcal{T}$, then*

$$\min \mathrm{Supp}(t) = \min \mathrm{Cosupp}(t).$$

Proof. Since the spectrum is noetherian, the colocal-to-global principle holds (by Corollary 6.5) and hence the codetection property holds (by Theorem 6.4). Moreover, a spectral space is noetherian if and only if each of its points is visible, by [BF11, Proposition 7.13]; see also [BHS23, Remark 2.2]. Hence the result follows immediately from Theorem 8.3. ☐

8.5. *Remark.* Recall that a spectral space has Krull dimension zero if and only if its specialization order is trivial if and only if it is T_1 if and only if it is Hausdorff if and only if it is profinite (see, e.g., [BHS23, Remark 2.4]). In this case we have:

8.6. **Corollary.** *If the spectrum of $\mathcal{T}$ is zero-dimensional and $\mathcal{T}$ satisfies the codetection property then for any $t \in \mathcal{T}$ we have*

$$\mathrm{vis}\,\mathcal{T} \cap \mathrm{Supp}(t) = \mathrm{vis}\,\mathcal{T} \cap \mathrm{Cosupp}(t).$$

Proof. Since the specialization order is trivial, we have $\min W = W$ for any subset $W \subseteq \mathrm{Spc}(\mathcal{T}^c)$. Hence the statement follows from Theorem 8.3. ☐

8.7. *Example.* Let $\mathcal{T} = \mathrm{K}(\mathrm{Inj}\, kG)$ denote the homotopy category of complexes of injective kG-modules with G a finite group and k a field whose characteristic divides the order of G. In [BIK12, Example 11.1] the authors prove that $\mathrm{Cosupp}(\mathbb{1}) = \{\mathfrak{m}\}$ is a singleton, where $\mathfrak{m}$ is the unique maximal ideal in $H^*(G,k)$. This shows that the statement of Corollary 8.4 is, in the given generality, optimal.

8.8. *Example.* Consider $\mathbb{N}^+$, the one-point compactification of the discrete set $\mathbb{N}$ of natural numbers with ∞ denoting the accumulation point, and let $R = C(\mathbb{N}^+, k)$ be the commutative ring of locally constant functions on $\mathbb{N}^+$ with values in a field k. If $\mathrm{D}(R)$ denotes the derived category of R-modules, we have homeomorphisms

$$\mathrm{Spc}(\mathrm{D}(R)^c) \cong \mathrm{Spec}(R) \cong \mathbb{N}^+.$$

Note that this space is profinite, hence weakly noetherian and generically noetherian[3] but not noetherian. The canonical inclusion $\mathbb{N} \subset \mathbb{N}^+$ exhibits $\mathbb{N}$ as a Thomason subset and we can consider $\Lambda^{\mathbb{N}}\mathbb{1} = \mathsf{hom}(e_{\mathbb{N}}, \mathbb{1})$. We claim:

(a) $\infty \in \mathrm{Supp}\,\Lambda^{\mathbb{N}}\mathbb{1}$; and
(b) $\mathrm{Cosupp}\,\Lambda^{\mathbb{N}}\mathbb{1} \subseteq \mathbb{N}$.

[3] Recall that a spectral space X is said to be *generically noetherian* if $\mathrm{gen}(x)$ is noetherian for every $x \in X$; see [BHS23, Definition 9.5].

Since $\min W = W$ for any subset $W \subseteq \mathbb{N}^+$, we deduce that

$$\min \operatorname{Supp} t \neq \min \operatorname{Cosupp} t$$

for $t = \Lambda^{\mathbb{N}}\mathbb{1} \in \mathrm{D}(R)$. Thus Corollary 8.4 does not hold without the noetherian assumption, leading to the more precise statement Theorem 8.3. Moreover, since $\operatorname{vis} \mathrm{D}(R)^c = \mathbb{N}$, this example also shows that the restriction to visible points in Corollary 8.6 is necessary.

Claim (b) follows from Lemma 4.32, so it remains to prove Claim (a). To this end, note that since the space is Hausdorff, $\operatorname{gen}(\infty) = \{\infty\}$ and $g_\infty = f_{\operatorname{gen}(\infty)^c} = f_{\mathbb{N}}$. Thus, $\infty \notin \operatorname{Supp} \Lambda^{\mathbb{N}}\mathbb{1}$ would mean that $f_{\mathbb{N}} \otimes \Lambda^{\mathbb{N}}\mathbb{1} = 0$. This would imply that the homotopy pullback square

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d] & \Lambda^{\mathbb{N}}\mathbb{1} \arrow[d] \\ f_{\mathbb{N}} \arrow[r] & f_{\mathbb{N}} \otimes \Lambda^{\mathbb{N}}\mathbb{1} \end{tikzcd}$$

degenerates to an isomorphism $\mathbb{1} \simeq f_{\mathbb{N}} \oplus \Lambda^{\mathbb{N}}\mathbb{1}$. By naturality, this would force $\Lambda^{\mathbb{N}}\mathbb{1} \simeq e_{\mathbb{N}}$ and consequently a decomposition $\mathbb{N}^+ = \mathbb{N} \sqcup \{\infty\}$. This is a contradiction, so $\infty \in \operatorname{Supp} \Lambda^{\mathbb{N}}\mathbb{1}$ as claimed.

In fact, $\operatorname{Cosupp} \Lambda^{\mathbb{N}}\mathbb{1} = \mathbb{N}$, since for any $n \in \mathbb{N}$, one can use the delta function $\delta_n \in C(\mathbb{N}^+, k)$ to construct a nontrivial map $g_n \to \mathbb{1}$ in $\mathrm{D}(R)$, bearing in mind Example 5.8. It then follows from Theorem 8.3 and (a) that $\operatorname{Supp} \Lambda^{\mathbb{N}}\mathbb{1} = \mathbb{N}^+$.

8.9. *Remark.* The above examples show that $\operatorname{Supp}(t) \neq \operatorname{Cosupp}(t)$ in general. In fact, it turns out that the relationship between $\operatorname{Supp}(t)$ and $\operatorname{Cosupp}(t)$ has an interesting connection with the Tate construction.

8.10. **Proposition.** *Let $Y \subseteq \operatorname{Spc}(\mathcal{T}^c)$ be a Thomason subset and let*

$$e_Y \to \mathbb{1} \to f_Y \xrightarrow{\theta} \Sigma e_Y$$

be the associated finite (co)localization. The following are equivalent:

(a) The internal hom $\mathsf{hom}(f_Y, \Sigma e_Y)$ vanishes.
(b) The map $\theta\colon f_Y \to \Sigma e_Y$ vanishes.
(c) The Thomason Y is both open and closed and the spectrum decomposes

$$\operatorname{Spc}(\mathcal{T}^c) = Y \sqcup Y^c$$

as a disjoint union of closed subsets.
(d) The complement Y^c is Thomason.

Proof. This is established by [PSW22, Proposition 2.29]. □

8.11. **Lemma.** *Let $Y \subseteq \operatorname{Spc}(\mathcal{T}^c)$ be a Thomason subset. The following are equivalent:*

(a) $\operatorname{Cosupp}(e_Y) \subseteq Y$.
(b) $\operatorname{Cosupp}(\mathsf{hom}(f_Y, \Sigma e_Y)) = \varnothing$.

Proof. By Lemma 4.32, we have

$$\operatorname{Cosupp}(\mathsf{hom}(f_Y, \Sigma e_Y)) = Y^c \cap \operatorname{Cosupp}(e_Y)$$

and hence (a) is equivalent to (b). □

8.12. **Corollary.** *Assume that the codetection property holds and let* $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ *be a Thomason subset. Then* $\mathrm{Cosupp}(e_Y) \subseteq Y$ *if and only if there is a decomposition* $\mathrm{Spc}(\mathcal{T}^c) = Y \sqcup Y^c$ *as a disjoint union of closed subsets.*

Proof. Since we are assuming that the codetection property holds, Lemma 8.11 says that $\mathrm{Cosupp}(e_Y) \subseteq Y$ if and only if $\mathsf{hom}(f_Y, \Sigma e_Y) = 0$. The result then follows from Proposition 8.10. □

8.13. **Proposition.** *Let* $\mathcal{T}$ *be a rigidly-compactly generated tt-category with* $\mathrm{Spc}(\mathcal{T}^c)$ *weakly noetherian. The following are equivalent:*

(a) The codetection property holds and $\mathrm{Supp}(t) = \mathrm{Cosupp}(t)$ *for all* $t \in \mathcal{T}$.
(b) $\mathrm{Spc}(\mathcal{T}^c)$ *is a finite discrete space.*

Proof. $(a) \Rightarrow (b)$: Recall that $\mathrm{Supp}(e_Y) = Y$ for any Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$. Hence Corollary 8.12 implies that Y is both open and closed. Since $\mathrm{Spc}(\mathcal{T}^c)$ is assumed weakly noetherian, every point is the intersection of a Thomason and the complement of a Thomason. Hence every point is open and so the topology is discrete. Since $\mathrm{Spc}(\mathcal{T}^c)$ is quasi-compact, it is then finite and discrete. $(b) \Rightarrow (a)$: This is provided by Corollary 8.4, Theorem 6.4 and Corollary 6.5 since a finite discrete space is both noetherian and zero-dimensional. □

8.14. **Corollary.** *If* $\mathrm{Spc}(\mathcal{T}^c)$ *is noetherian then the following are equivalent:*

(a) $\mathrm{Supp}(t) = \mathrm{Cosupp}(t)$ *for all* $t \in \mathcal{T}$.
(b) $\mathrm{Spc}(\mathcal{T}^c)$ *is a finite discrete space.*

Proof. This follows from Proposition 8.13 and Corollary 6.5. □

8.15. *Example.* If $\mathrm{Spc}(\mathcal{T}^c) = *$ is a single point then Supp = Cosupp. Indeed, both just detect whether an object is nonzero.

8.16. *Example* (Two connected points). Let $\mathcal{T}$ be a rigidly-compactly generated tt-category whose spectrum

$$\mathrm{Spc}(\mathcal{T}^c) = \begin{array}{l} \bullet\ \ \mathfrak{m} \\ | \\ \bullet\ \ \eta \end{array}$$

consists of two connected points: a closed point $\mathfrak{m}$ and a generic point η. There is only one non-trivial Thomason subset, namely $\{\mathfrak{m}\}$, hence there is only one non-trivial finite localization

$$e_{\{\mathfrak{m}\}} \to \mathbb{1} \to f_{\{\mathfrak{m}\}} \to \Sigma e_{\{\mathfrak{m}\}}.$$

Moreover, $g(\mathfrak{m}) = e_{\{\mathfrak{m}\}}$ and $g(\eta) = f_{\{\mathfrak{m}\}}$. We can directly observe that the codetection property holds (i.e., without invoking Corollary 6.5). In fact we can directly observe that the local-to-global principle and the colocal-to-global principle hold in this example. From Lemma 4.26 we have that $\mathrm{Cosupp}(g(\eta)) = \{\eta\}$ is a single point. On the other hand, since the two points are connected, we know by Corollary 8.12 that $\mathrm{Cosupp}(g(\mathfrak{m})) = \{\mathfrak{m}, \eta\}$ is the whole space. The nature of $\mathrm{Cosupp}(\mathbb{1})$ is more subtle. We always have $\mathfrak{m} \in \mathrm{Cosupp}(\mathbb{1})$, either by a direct argument or by Corollary 8.4. On the other hand, both $\eta \in \mathrm{Cosupp}(\mathbb{1})$ and $\eta \notin \mathrm{Cosupp}(\mathbb{1})$ are possible, as the next sub-example demonstrates.

8.17. *Example.* Take $\mathcal{T} = \mathrm{D}(R)$ to be the derived category of a discrete valuation ring $(R, \mathfrak{m}, k)$. In this example, $\mathrm{Cosupp}(\mathbb{1}) \subseteq \{\mathfrak{m}\}$ if and only if R is complete. This follows from the triangle

$$\mathsf{hom}(f_{\{\mathfrak{m}\}}, \mathbb{1}) \to \mathbb{1} \to \mathsf{hom}(e_{\{\mathfrak{m}\}}, \mathbb{1})$$

and $g(\eta) = f_{\{\mathfrak{m}\}}$, once one recognizes that in this case $\mathsf{hom}(e_{\{\mathfrak{m}\}}, -)$ is the derived functor of $\mathfrak{m}$-adic completion; see Example 4.4 and the references therein.

8.18. *Remark.* If a category is stratified then its support function determines and is determined by its cosupport function. Indeed, for any $t \in \mathcal{T}$, we have $\mathrm{Supp}(t) = \mathrm{Cosupp}(\{t\}^{\perp})^c$ and $\mathrm{Cosupp}(t) = \mathrm{Supp}({}^{\perp}\{t\})^c$ by Corollary 7.18. However, the question remains whether the support of an individual object $\mathrm{Supp}(t)$ determines its cosupport $\mathrm{Cosupp}(t)$, or vice versa. The next two examples demonstrate that in general this is false.

8.19. *Example.* Let $Y \coloneqq \mathrm{supp}(x)$ for a compact object $x \in \mathcal{T}^c$. Then $\mathrm{Supp}(x) = \mathrm{Supp}(e_Y) = Y$. In particular, $\mathrm{Cosupp}(x) \subseteq Y$ by Example 4.36. On the other hand, we have seen in Corollary 8.12 that if $\mathrm{Cosupp}(e_Y) \subseteq Y$ then the spectrum disconnects into Y and its complement (provided the codetection property holds). Therefore there are many examples where we have two objects t_1 and t_2 with $\mathrm{Supp}(t_1) = \mathrm{Supp}(t_2)$ and yet $\mathrm{Cosupp}(t_1) \neq \mathrm{Cosupp}(t_2)$. For an explicit example, this occurs for any category $\mathcal{T}$ whose spectrum is two connected points (Example 8.16) and $t_1 = e_{\{\mathfrak{m}\}}$ and $t_2 = x$ any compact object with $\mathrm{supp}(x) = \{\mathfrak{m}\}$. Moreover, there are such examples which are costratified; e.g., the derived category $\mathcal{T} = \mathrm{D}(R)$ of a discrete valuation ring (see Proposition 19.1 below). We conclude that an object's support does not in general determine its cosupport, even in costratified examples.

8.20. *Example.* Let $\mathcal{T}$ be a local category where $\mathrm{Cosupp}(\mathbb{1}) = \{\mathfrak{m}\}$ just consists of the closed point. For example, we could take $\mathcal{T} = \mathrm{D}(R)$ for R a complete discrete valuation ring (Example 8.17) or $\mathcal{T} = \mathrm{K}(\mathrm{Inj}\, kG)$ (Example 8.7). Then for any nonzero compact object x, we have $\mathrm{Cosupp}(x) = \mathrm{supp}(x) \cap \mathrm{Cosupp}(\mathbb{1}) = \{\mathfrak{m}\}$ (Example 4.36). Thus, any two nonzero compact objects have the same cosupport: $\mathrm{Cosupp}(x) = \mathrm{Cosupp}(y)$. On the other hand, provided the spectrum is not a single point, there exist nonzero compact objects with differing support. We conclude that an object's cosupport does not in general determine its support, even in costratified examples.

8.21. *Remark.* In Section 12 we will establish a further important relationship between support and cosupport, namely that the support of an object coincides with the cosupport of its Brown–Comenetz dual; see Proposition 12.9. The discussion there will also elaborate on the relationship with completion.

Part II. Perfection and duality

The opposite category $\mathcal{T}^{\mathrm{op}}$ of a triangulated category $\mathcal{T}$ inherits a triangulated structure (with $\Sigma_{\mathcal{T}^{\mathrm{op}}} = \Sigma_{\mathcal{T}}^{-1}$) and the notion of a thick subcategory is self-dual. In fact, the thick subcategories of a triangulated category $\mathcal{T}$ coincide with the thick subcategories of its opposite triangulated category $\mathcal{T}^{\mathrm{op}}$. Moreover, the localizing subcategories of $\mathcal{T}$ are precisely the colocalizing subcategories of $\mathcal{T}^{\mathrm{op}}$ and vice versa. However, localizing ideals and colocalizing coideals are not obviously dual notions. As it stands, the notion of a colocalizing coideal is somewhat mysterious, besides its appearance in right orthogonals of localizing ideals. Similarly, definitions of cosupport in the literature — including the notion we have studied above — are not very conceptually motivated. It is not too far a stretch to say that one scratches one's head the first time one sees the definition of cosupport. Fortunately, in this part of the paper we will provide a conceptual understanding for it all, exhibit localizing ideals as dual to colocalizing coideals, support as dual to cosupport, and stratification as dual to costratification. In order to achieve this, we will work in a more general setting that encompasses both $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$. This leads us to perfection:

9. Perfect generation

The opposite category $\mathcal{T}^{\mathrm{op}}$ of a compactly generated triangulated category $\mathcal{T}$ is never compactly generated.[4] The more general notion of "perfectly generated" triangulated category is more flexible in this respect, encompassing both compactly generated triangulated categories and their opposites. This notion has a three-fold role to play in this work. Firstly, it will provide an adequate level of generality in which the Balmer–Favi approach to support can be constructed, which will facilitate a comparison between support and cosupport. Secondly, it will provide another way to construct objects with prescribed cosupport. Thirdly, it will be important for controlling descent of costratification.

9.1. *Definition* (Krause). Recall from [Kra10, Section 5] that a triangulated category $\mathcal{T}$ is *perfectly generated* if it has coproducts and there exists a set of objects $\mathcal{G} \subseteq \mathcal{T}$ satisfying the following two conditions:

(PG1) $\mathrm{Loc}\langle\mathcal{G}\rangle = \mathcal{T}$.

(PG2) Given any family $(x_i \to y_i)_{i\in I}$ of morphisms in $\mathcal{T}$ such that the induced map $\mathcal{T}(g, x_i) \to \mathcal{T}(g, y_i)$ is surjective for all $g \in \mathcal{G}$ and $i \in I$, the induced map

$$\mathcal{T}(g, \coprod_i x_i) \to \mathcal{T}(g, \coprod_i y_i)$$

is surjective.

Dually, we say that a triangulated category $\mathcal{T}$ is *perfectly cogenerated* if $\mathcal{T}^{\mathrm{op}}$ is perfectly generated.

9.2. *Remark.* As noted in [Kra10, Remark 5.1.2], in the presence of (PG2), the generation condition (PG1) is equivalent to the (*a priori* weaker) generation condition that an object $t \in \mathcal{T}$ vanishes if $\mathcal{T}(\Sigma^n g, t) = 0$ for all $g \in \mathcal{G}$ and $n \in \mathbb{Z}$.

[4] Boardman [Boa70] proved this for $\mathcal{T} = \mathrm{SH}$. The general result is due to Neeman [Nee01, Appendix E.1].

9.3. *Example.* A compactly generated triangulated category is perfectly generated. The (PG2) condition is automatic when $\mathcal{G}$ is a set of compact objects. More generally, a well generated triangulated category is perfectly generated; see [Kra01].

9.4. *Definition.* Let $\mathcal{T}$ be a compactly generated triangulated category. For any compact object $c \in \mathcal{T}^c$, the functor

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{T}(c,-),\mathbb{Q}/\mathbb{Z})\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab} \tag{9.5}$$

is homological and sends coproducts in $\mathcal{T}$ to products in Ab, hence by Brown representability is represented by an object $I_c \in \mathcal{T}$. Consequently, we have a natural isomorphism

$$\mathcal{T}(t, I_c) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{T}(c,t),\mathbb{Q}/\mathbb{Z}) \tag{9.6}$$

for any $t \in \mathcal{T}$ and $c \in \mathcal{T}^c$.

9.7. *Remark.* The significance of $\mathbb{Q}/\mathbb{Z}$ in the above construction is that it is an injective cogenerator of the category of abelian groups. Injectivity ensures that the functor (9.5) is homological, while cogeneration amounts to the fact that every nonzero abelian group admits a nonzero homomorphism to $\mathbb{Q}/\mathbb{Z}$. It follows that an object $t = 0$ if and only if $\mathcal{T}(c,t) = 0$ for all $c \in \mathcal{T}^c$ if and only if $\mathcal{T}(t, I_c) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{T}(c,t),\mathbb{Q}/\mathbb{Z}) = 0$ for all $c \in \mathcal{T}^c$. Moreover, we have:

9.8. **Lemma.** *For any morphism $x \to y$ in $\mathcal{T}$ and $c \in \mathcal{T}^c$, the induced map $\mathcal{T}(y, I_c) \to \mathcal{T}(x, I_c)$ is surjective if and only if the induced map $\mathcal{T}(c,x) \to \mathcal{T}(c,y)$ is injective.*

Proof. One implication follows from the injectivity of $\mathbb{Q}/\mathbb{Z}$: The induced map $\mathcal{T}(y, I_c) \to \mathcal{T}(x, I_c)$ is surjective if the induced map $\mathcal{T}(c,x) \to \mathcal{T}(c,y)$ is injective. The other implication uses the cogeneration property of $\mathbb{Q}/\mathbb{Z}$: If $\mathcal{T}(y, I_c) \to \mathcal{T}(x, I_c)$ is surjective then by the natural isomorphism (9.6) any homomorphism of abelian groups $\mathcal{T}(c,x) \to \mathbb{Q}/\mathbb{Z}$ factors through $\mathcal{T}(c,x) \to \mathcal{T}(c,y)$. This implies that $\mathcal{T}(c,x) \to \mathcal{T}(c,y)$ is injective because if $f \in \mathcal{T}(c,x)$ is nonzero we can extend a nonzero homomorphism of abelian groups $\langle f \rangle \to \mathbb{Q}/\mathbb{Z}$ to a homomorphism $\mathcal{T}(c,x) \to \mathbb{Q}/\mathbb{Z}$ which does not annihilate f and hence $\mathcal{T}(c,x) \to \mathcal{T}(c,y)$ cannot annihilate f. $\square$

9.9. **Proposition.** *If $\mathcal{T}$ is compactly generated then $\mathcal{T}^{\mathrm{op}}$ is perfectly generated. More precisely, if $\mathcal{G} \subseteq \mathcal{T}$ is a set of compact generators, then $\mathcal{T}^{\mathrm{op}}$ is perfectly generated by $I(\mathcal{G}) := \{\, I_c \mid c \in \mathcal{G} \,\}$. In particular, $\mathcal{T} = \mathrm{Coloc}\langle I(\mathcal{G})\rangle$.*

Proof. Suppose $t \in \mathcal{T}$ is such that $\mathcal{T}(t, I_c) = 0$ for all $c \in \mathcal{G}$. Since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator of Ab, this is equivalent to the statement that $\mathcal{T}(c,t) = 0$ for all $c \in \mathcal{G}$ (Remark 9.7). By assumption, $\mathcal{G}$ is a generating set for $\mathcal{T}$, hence $t = 0$. This shows that $I(\mathcal{G})$ is a set of cogenerators for $\mathcal{T}$ (Remark 9.2). In order to verify condition (PG2), let $f_i\colon x_i \to y_i$ be a collection of maps in $\mathcal{T}$ with $\mathcal{T}(f_i, I_c)$ surjective for all $c \in \mathcal{G}$. In light of Lemma 9.8, this translates to the statement that the maps $\mathcal{T}(c, f_i)$ are injective for all $c \in \mathcal{G}$. It follows that $\prod_i \mathcal{T}(c, f_i) = \mathcal{T}(c, \prod_i f_i)$ is injective in Ab for each $c \in \mathcal{G}$, which in turn is equivalent to the surjectivity of $\mathcal{T}(\prod_i f_i, I_c)$ for all $c \in \mathcal{G}$. This establishes that $I(\mathcal{G})$ is a set of perfect cogenerators for $\mathcal{T}$. $\square$

9.10. *Remark.* The Brown representability theorem holds for perfectly generated categories; see [Kra02, Kra10]. In particular, if $\mathcal{T}$ is perfectly generated then any coproduct-preserving exact functor $f^*\colon \mathcal{T} \to \mathcal{S}$ admits a right adjoint f_*.

9.11. **Proposition.** *Suppose the triangulated category $\mathcal{T}$ is perfectly generated by $\mathcal{G} \subseteq \mathcal{T}$. Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a coproduct-preserving exact functor whose right adjoint f_* preserves coproducts. Then the following are equivalent:*

(a) The category $\mathcal{S}$ is perfectly generated by $f^*(\mathcal{G})$.
(b) The right adjoint f_* is conservative.

Proof. One verifies directly that $f^*(\mathcal{G})$ satisfies (PG2). Here one uses the assumption that f_* preserves coproducts. Then using Remark 9.2, the equivalence of (a) and (b) just follows from observing that $f_*(s) = 0$ if and only if $\mathcal{T}(\Sigma^n\mathcal{G}, f_*s) = 0$ if and only if $\mathcal{S}(\Sigma^n f^*(\mathcal{G}), s) = 0$. ∎

9.12. *Remark.* With "perfectly generated" replaced by "compactly generated", Proposition 9.11 is well-known. In this case, the assumption that f_* preserves coproducts can be removed since it is implied by (and is in fact equivalent to) the condition in part (a) that f^* preserves the compactness of the generators; see [Nee96, Theorem 5.1].

10. Cosupport is support

By a tensor-triangulated category we usually mean a triangulated category equipped with a compatible *closed* symmetric monoidal structure as spelled out in [HPS97, Appendix A]. Indeed, the closed structure is used in a fundamental way in the definition of cosupport and is featured in the very definition of colocalizing coideal. Although the opposite category of a symmetric monoidal category inherits a symmetric monoidal structure, it does not inherit a *closed* symmetric monoidal structure. For this reason, we will need the following weaker notion:

10.1. *Terminology.* In this paper, by a *non-closed tensor-triangulated category* we mean a triangulated category $\mathcal{T}$ equipped with a symmetric monoidal structure such that $-\otimes t\colon \mathcal{T} \to \mathcal{T}$ is an exact functor for each $t \in \mathcal{T}$ (with the usual compatibility between the associated suspension isomorphisms; see [Bal10, Definition 3]). We will not assume that $\mathcal{T}$ has an internal hom, and even when we assume that $\mathcal{T}$ has coproducts, we do not assume that $-\otimes t\colon \mathcal{T} \to \mathcal{T}$ preserves them. However we will require that the full subcategory $\mathcal{T}^d \subseteq \mathcal{T}$ of dualizable objects is an essentially small triangulated subcategory and that $(-)^\vee\colon \mathcal{T}^d \to (\mathcal{T}^d)^{\mathrm{op}}$ preserves exact triangles.

10.2. *Remark.* The requirement that $\mathcal{T}^d$ is a triangulated subcategory amounts to the assumption that an extension of dualizable objects is again dualizable. If $\mathcal{T}$ is idempotent-complete (for example, if it has countable coproducts) then $\mathcal{T}^d$ is a thick subcategory.

10.3. *Remark.* If $\mathcal{T}$ is a non-closed tensor-triangulated category then $\mathcal{T}^d$ is a rigid essentially small tensor-triangulated subcategory of $\mathcal{T}$.

10.4. *Remark.* If $\mathcal{T}$ is a rigidly-compactly generated tensor-triangulated category in the usual sense then $\mathcal{T}^d = \mathcal{T}^c$. Nevertheless, we will often write $\mathcal{T}^d$ when emphasizing dualizability over compactness. The results which follow provide evidence that it is $\mathcal{T}^d$ which is more fundamental to tensor triangular geometry and that $\mathrm{Spc}(\mathcal{T}^d)$ is the correct definition of "the" Balmer spectrum of $\mathcal{T}$.

10.5. *Example.* Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category in the usual sense. Its opposite category $\mathcal{T}^{\mathrm{op}}$ is a non-closed tensor-triangulated category and $\mathcal{T}^d \cong (\mathcal{T}^{\mathrm{op}})^d$ are equivalent tensor-triangulated categories, hence have the same Balmer spectrum. In fact, since thick ideals of dualizable objects are closed under taking duals, $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$ have exactly the same thick ideals of dualizable objects, the same Balmer spectrum of dualizable objects, the same universal support for dualizable objects, and so on. For all intents and purposes, $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$ are just two extensions of "the same" rigid tensor-triangulated category that we will sometimes denote by $\mathcal{K}$.

10.6. *Remark.* The following definition is the key to reconciling localizing ideals and colocalizing coideals.

10.7. *Definition.* Let $\mathcal{T}$ be a non-closed tensor-triangulated category which has small coproducts. A *localizing* $\mathcal{T}^d$*-submodule* of $\mathcal{T}$ is a localizing subcategory $\mathcal{L} \subseteq \mathcal{T}$ such that $\mathcal{T}^d \otimes \mathcal{L} \subseteq \mathcal{L}$.

10.8. *Remark.* The localizing $\mathcal{T}^d$-submodule generated by a collection of objects $\mathcal{E} \subseteq \mathcal{T}$ coincides with $\mathrm{Loc}\langle \mathcal{E} \otimes \mathcal{T}^d \rangle$.

10.9. *Example.* Let $\mathcal{T}$ be a rigidly-compactly generated tt-category (in the usual sense). A localizing subcategory $\mathcal{L}$ is a localizing ideal if and only if $\mathcal{L} \otimes \mathcal{T} \subseteq \mathcal{L}$ if and only if $\mathcal{L} \otimes \mathcal{T}^d \subseteq \mathcal{L}$ if and only if $\mathcal{L}$ is a $\mathcal{T}^d$-submodule of $\mathcal{T}$. That is, the localizing ideals of $\mathcal{T}$ are precisely the localizing $\mathcal{T}^d$-submodules of $\mathcal{T}$. The key reason for this is that $\mathcal{T} = \mathrm{Loc}\langle \mathcal{T}^d \rangle$ is generated by the subcategory of dualizable objects.

10.10. *Example.* Let $\mathcal{T}$ be a rigidly-compactly generated tt-category (in the usual sense). A colocalizing subcategory $\mathcal{C}$ of $\mathcal{T}$ is a colocalizing coideal if and only if $\mathsf{hom}(\mathcal{T}, \mathcal{C}) \subseteq \mathcal{C}$ if and only if $\mathsf{hom}(\mathcal{T}^d, \mathcal{C}) \subseteq \mathcal{C}$ if and only if $\mathcal{T}^d \otimes \mathcal{C} \subseteq \mathcal{C}$ if and only if $\mathcal{C}$ is a localizing $\mathcal{T}^d$-submodule of $\mathcal{T}^{\mathrm{op}}$. That is, the colocalizing coideals of $\mathcal{T}$ are precisely the localizing $\mathcal{T}^d$-submodules of $\mathcal{T}^{\mathrm{op}}$. Thus, the task of classifying localizing ideals and colocalizing coideals is, in both cases, the question of classifying the localizing $\mathcal{T}^d$-submodules of $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$, respectively.

10.11. *Remark.* Our next goal is to explain how cosupport is related to support. Recall from Proposition 9.9 that if $\mathcal{T}$ is compactly generated then $\mathcal{T}^{\mathrm{op}}$ is perfectly generated.

10.12. **Lemma.** *Let $\mathcal{T}$ be a non-closed tensor-triangulated category and let $\mathcal{E} \subseteq \mathcal{T}^d$ be a set of dualizable objects. If $\mathcal{T}$ is perfectly generated by $\mathcal{G} \subseteq \mathcal{T}$ then $\mathrm{Loc}\langle \mathcal{E} \otimes \mathcal{T} \rangle$ is perfectly generated by $\mathcal{E} \otimes \mathcal{G}$.*

Proof. Let $\mathcal{L} := \mathrm{Loc}\langle \mathcal{E} \otimes \mathcal{G} \rangle$. Then $\{ t \in \mathcal{T} \mid \mathcal{E} \otimes t \in \mathcal{L} \}$ is a localizing subcategory since the objects in $\mathcal{E}$ are dualizable. It contains $\mathcal{G}$ and hence since $\mathcal{T} = \mathrm{Loc}\langle \mathcal{G} \rangle$ we conclude that $\mathcal{E} \otimes \mathcal{T} \subseteq \mathrm{Loc}\langle \mathcal{E} \otimes \mathcal{G} \rangle$. This establishes (PG1). Condition (PG2) can be readily checked from the definition. The key point here is that for a dualizable object $d \in \mathcal{T}^d$, the functor $d \otimes - \colon \mathcal{T} \to \mathcal{T}$ has a right adjoint $d^\vee \otimes - \colon \mathcal{T} \to \mathcal{T}$ which itself preserves coproducts; cf. Proposition 9.11. ∎

10.13. **Proposition.** *Let $\mathcal{T}$ be a perfectly generated non-closed tensor-triangulated category and write $\mathcal{K} := \mathcal{T}^d$. For every Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{K})$, the localizing subcategory $\mathrm{Loc}\langle \mathcal{K}_Y \otimes \mathcal{T} \rangle$ is strictly localizing, where $\mathcal{K}_Y := \{ a \in \mathcal{K} \mid \mathrm{supp}(a) \subseteq Y \}$. We write*

$$\Gamma_Y(t) \to t \to \mathrm{L}_Y(t) \to \Sigma\Gamma_Y(t) \tag{10.14}$$

for the associated Bousfield localization triangle.

Proof. By Lemma 10.12, the localizing subcategory $\mathrm{Loc}\langle \mathcal{K}_Y \otimes \mathcal{T}\rangle$ is perfectly generated. It follows from [Kra10, Proposition 5.2.1] that it is strictly localizing. $\square$

10.15. *Remark.* The subcategory $\Gamma_Y(\mathcal{T}) = \mathrm{Loc}\langle \mathcal{K}_Y \otimes \mathcal{T}\rangle$ of colocal objects is a localizing $\mathcal{T}^d$-submodule of $\mathcal{T}$. The proof is a standard thick subcategory argument which uses the fact that for a dualizable object $a \in \mathcal{T}^d$, the functor $a \otimes - \colon \mathcal{T} \to \mathcal{T}$ preserves coproducts. Note that we are not assuming that $t \otimes -$ preserve coproducts in general (for an arbitrary $t \in \mathcal{T}$). This is related to the fact that $\Gamma_Y(\mathcal{T})$ is not necessarily a localizing ideal of $\mathcal{T}$.

10.16. *Remark.* The subcategory of local objects $\mathrm{L}_Y(\mathcal{T}) = \mathrm{Loc}\langle \mathcal{K}_Y \otimes \mathcal{T}\rangle^{\perp}$ coincides with $\bigcap_{a\in\mathcal{K}_Y} \ker(a \otimes -)$ and hence is itself a localizing $\mathcal{T}^d$-submodule of $\mathcal{T}$. This implies that the Bousfield localization (10.14) is smashing in the sense that Γ_Y and L_Y preserve coproducts. It follows that

$$\Gamma_Y(a \otimes t) \simeq a \otimes \Gamma_Y(t) \quad \text{and} \quad \mathrm{L}_Y(a \otimes t) \simeq a \otimes \mathrm{L}_Y(t) \tag{10.17}$$

for any $a \in \mathcal{T}^d$ and $t \in \mathcal{T}$.

10.18. *Remark.* The localizing $\mathcal{T}^d$-submodule generated by the thick ideal $\mathcal{K}_Y$ is given by $\mathrm{Loc}\langle \mathcal{K}_Y\rangle = \mathrm{Loc}\langle \mathcal{K}_Y \otimes \mathcal{T}^d\rangle$; cf. Remark 10.8. The localizing $\mathcal{T}^d$-submodule $\Gamma_Y(\mathcal{T}) = \mathrm{Loc}\langle \mathcal{K}_Y \otimes \mathcal{T}\rangle$ is potentially larger.

10.19. **Lemma.** *The assignment $Y \mapsto \Gamma_Y\mathcal{T}$, regarded as a map from the lattice of Thomason subsets of $\mathrm{Spc}(\mathcal{T}^d)$ to the lattice of localizing $\mathcal{T}^d$-submodules of $\mathcal{T}$, preserves arbitrary joins and finite meets.*

Proof. For notational simplicity, write $\mathcal{K} \coloneqq \mathcal{T}^d$. It is immediate that the map preserves the greatest and least elements. We first prove that it preserves arbitrary joins. To this end, consider a union $\bigcup_i Y_i$ of Thomason subsets of $\mathrm{Spc}(\mathcal{K})$. The inclusion $\bigvee_i \Gamma_{Y_i}\mathcal{T} \subseteq \Gamma_{\bigcup_i Y_i}\mathcal{T}$ is trivial. To establish the other inclusion we need to check that $\mathcal{K}_{\bigcup_i Y_i} \otimes \mathcal{T} \subseteq \bigvee_i \Gamma_{Y_i}\mathcal{T}$. Observe that

$$\left\{\, a \in \mathcal{T}^d \;\middle|\; a \otimes \mathcal{T} \subseteq \bigvee\nolimits_i \Gamma_{Y_i}\mathcal{T} \,\right\}$$

is a thick ideal of $\mathcal{T}^d$. It contains $\mathcal{K}_{Y_i}$ for each i. Hence it contains $\mathrm{thickid}\langle \bigcup_i \mathcal{K}_{Y_i}\rangle$ which — by the classification of thick ideals of the rigid tt-category $\mathcal{K}$ — coincides with $\mathcal{K}_{\bigcup_i Y_i}$. This establishes the desired claim.

It remains to prove that the function preserves finite meets. That is,

$$\Gamma_{Y_1}\mathcal{T} \cap \Gamma_{Y_2}\mathcal{T} = \Gamma_{Y_1\cap Y_2}\mathcal{T}$$

for any pair of Thomason subsets Y_1, Y_2. The $\supseteq$ inclusion is immediate. For the $\subseteq$ inclusion we need to establish that

$$\mathrm{Loc}\langle \mathcal{K}_{Y_1} \otimes \mathcal{T}\rangle \cap \mathrm{Loc}\langle \mathcal{K}_{Y_2} \otimes \mathcal{T}\rangle \subseteq \mathrm{Loc}\langle \mathcal{K}_{Y_1\cap Y_2} \otimes \mathcal{T}\rangle. \tag{10.20}$$

First note that $\mathcal{K}_{Y_1\cap Y_2} = \mathcal{K}_{Y_1} \cap \mathcal{K}_{Y_2} = \mathrm{thick}\langle \mathcal{K}_{Y_1} \otimes \mathcal{K}_{Y_2}\rangle$ where the last equality uses the rigidity of $\mathcal{K}$. Then consider an object t contained in the left-hand side of (10.20). Since $t \in \mathrm{Loc}\langle \mathcal{K}_{Y_2} \otimes \mathcal{T}\rangle$, we have

$$\Gamma_{Y_1}(t) \in \Gamma_{Y_1}(\mathrm{Loc}\langle \mathcal{K}_{Y_2} \otimes \mathcal{T}\rangle) \subseteq \mathrm{Loc}\langle \Gamma_{Y_1}(\mathcal{K}_{Y_2} \otimes \mathcal{T})\rangle,$$

where the inclusion uses that Γ_{Y_1} preserves coproducts (Remark 10.16). But also $t \simeq \Gamma_{Y_1}(t)$ since $t \in \mathrm{Locid}\langle \mathcal{K}_{Y_1} \otimes \mathcal{T}\rangle$; hence

$$t \in \mathrm{Loc}\langle \Gamma_{Y_1}(\mathcal{K}_{Y_2} \otimes \mathcal{T})\rangle.$$

Now using (10.17) and the fact that tensoring with dualizable objects preserves coproducts, we have

$$\begin{aligned}\operatorname{Loc}\langle\Gamma_{Y_1}(\mathcal{K}_{Y_2}\otimes\mathcal{T})\rangle=\operatorname{Loc}\langle\mathcal{K}_{Y_2}\otimes\Gamma_{Y_1}\mathcal{T}\rangle&=\operatorname{Loc}\langle\mathcal{K}_{Y_2}\otimes\operatorname{Loc}\langle\mathcal{K}_{Y_1}\otimes\mathcal{T}\rangle\rangle\\&=\operatorname{Loc}\langle\mathcal{K}_{Y_2}\otimes\mathcal{K}_{Y_1}\otimes\mathcal{T}\rangle\\&=\operatorname{Loc}\langle\mathcal{K}_{Y_1\cap Y_2}\otimes\mathcal{T}\rangle\end{aligned}\tag{10.21}$$

which establishes (10.20). □

10.22. **Lemma.** *Let $Y_1, Y_2 \subseteq \operatorname{Spc}(\mathcal{T}^d)$ be two Thomason subsets. Then we have:*

(a) $\Gamma_{Y_1}\Gamma_{Y_2}=\Gamma_{Y_1\cap Y_2}=\Gamma_{Y_2}\Gamma_{Y_1}$;
(b) $\mathrm{L}_{Y_1}\mathrm{L}_{Y_2}=\mathrm{L}_{Y_1\cup Y_2}=\mathrm{L}_{Y_2}\mathrm{L}_{Y_1}$;
(c) $\Gamma_{Y_1}\mathrm{L}_{Y_2}=\mathrm{L}_{Y_2}\Gamma_{Y_1}$.

Proof. First note that if $Y\subseteq Y'$ then $\Gamma_Y\simeq\Gamma_Y\Gamma_{Y'}$. Hence, $\Gamma_{Y_1\cap Y_2}\simeq\Gamma_{Y_1\cap Y_2}\Gamma_{Y_2}\simeq\Gamma_{Y_1\cap Y_2}\Gamma_{Y_1}\Gamma_{Y_2}$. To prove (a) it suffices to verify that $\Gamma_{Y_1}\Gamma_{Y_2}t\in\operatorname{Loc}\langle\mathcal{K}_{Y_1\cap Y_2}\otimes\mathcal{T}\rangle$ so that $\Gamma_{Y_1\cap Y_2}\Gamma_{Y_1}\Gamma_{Y_2}(t)\simeq\Gamma_{Y_1}\Gamma_{Y_2}(t)$. Note that, by the classification of thick ideals of the rigid tt-category $\mathcal{K}$, $\mathcal{K}_{Y_1\cap Y_2}=\mathcal{K}_{Y_1}\cap\mathcal{K}_{Y_2}$. Thus, we want to show that

$$\Gamma_{Y_1}\Gamma_{Y_2}t\in\operatorname{Loc}\langle\mathcal{K}_{Y_1}\otimes\mathcal{K}_{Y_2}\otimes\mathcal{T}\rangle)$$

for any $t\in\mathcal{T}$. Since $\Gamma_{Y_2}t\in\operatorname{Loc}\langle\mathcal{K}_{Y_2}\otimes\mathcal{T}\rangle$, we have

$$\Gamma_{Y_1}\Gamma_{Y_2}t\in\Gamma_{Y_1}(\operatorname{Loc}\langle\mathcal{K}_{Y_2}\otimes\mathcal{T}\rangle)\subseteq\operatorname{Loc}\langle\Gamma_{Y_1}(\mathcal{K}_{Y_2}\otimes\mathcal{T})\rangle,$$

where the last inclusion uses that Γ_Y preserves coproducts (Remark 10.16). Now using the same Remark 10.16, we see as in (10.21) above that

$$\operatorname{Loc}\langle\Gamma_{Y_1}(\mathcal{K}_{Y_2}\otimes\mathcal{T})\rangle=\operatorname{Loc}\langle\mathcal{K}_{Y_2}\otimes\mathcal{K}_{Y_1}\otimes\mathcal{T}\rangle$$

which is what we wanted to show. This establishes (a).

For part (b), first observe that if $Y\subseteq Y'$ then $\mathrm{L}_{Y'}\simeq\mathrm{L}_{Y'}\mathrm{L}_Y$. Hence $\mathrm{L}_{Y_1\cup Y_2}\simeq\mathrm{L}_{Y_1\cup Y_2}\mathrm{L}_{Y_2}\simeq\mathrm{L}_{Y_1\cup Y_2}\mathrm{L}_{Y_1}\mathrm{L}_{Y_2}$. The proof will be finished if we can show that

$$\mathrm{L}_{Y_1}\mathrm{L}_{Y_2}t\in\operatorname{Loc}\langle\mathcal{K}_{Y_1\cup Y_2}\otimes\mathcal{T}\rangle^{\perp}.$$

For this, it is enough to show that

$$\mathcal{K}_{Y_1\cup Y_2}\otimes\mathrm{L}_{Y_1}\mathrm{L}_{Y_2}t=0\tag{10.23}$$

which, since $\mathcal{K}_{Y_1\cup K_Y}=\operatorname{thickid}\langle\mathcal{K}_{Y_1}\cup\mathcal{K}_{Y_2}\rangle$, is equivalent to

$$\mathcal{K}_{Y_1}\otimes\mathrm{L}_{Y_1}\mathrm{L}_{Y_2}t=0\quad\text{and}\quad\mathcal{K}_{Y_2}\otimes\mathrm{L}_{Y_1}\mathrm{L}_{Y_2}t=0$$

both of which follow from Remark 10.16.

Finally, we prove part (c). It follows from (a) that $\Gamma_{Y_1}\mathrm{L}_{Y_2}t$ is L_{Y_2}-local. Hence $\Gamma_{Y_1}\mathrm{L}_{Y_2}t\xrightarrow{\sim}\mathrm{L}_{Y_2}\Gamma_{Y_1}\mathrm{L}_{Y_2}t$. On the other hand, it also follows from (a) that $\Gamma_{Y_1}t\to\Gamma_{Y_1}\mathrm{L}_{Y_2}t$ is an L_{Y_2}-equivalence, hence

$$\Gamma_{Y_1}\mathrm{L}_{Y_2}t\xrightarrow{\sim}\mathrm{L}_{Y_2}\Gamma_{Y_1}\mathrm{L}_{Y_2}t\xleftarrow{\sim}\mathrm{L}_{Y_2}\Gamma_{Y_1}t.$$ □

10.24. *Definition* (Support for perfectly generated categories). Let $\mathcal{T}$ be a perfectly generated non-closed tensor-triangulated category whose spectrum $\operatorname{Spc}(\mathcal{T}^d)$ is weakly noetherian. Write $\{\mathcal{P}\}=Y_1\cap Y_2^c$ for Thomason subsets $Y_1,Y_2\subseteq\operatorname{Spc}(\mathcal{T}^d)$ and define $\Gamma_{\mathcal{P}}\coloneqq\Gamma_{Y_1}L_{Y_2}$. Armed with Lemma 10.22, one shows that this definition does not depend on the choice of Y_1 and Y_2 by following the ideas of [BF11, Lemma 7.4]. The *Balmer–Favi support* of an object $t\in\mathcal{T}$ is defined as

$$\operatorname{Supp}_{\mathcal{T}}(t)\coloneqq\left\{\,\mathcal{P}\in\operatorname{Spc}(\mathcal{T}^d)\;\middle|\;\Gamma_{\mathcal{P}}(t)\neq0\,\right\}\subseteq\operatorname{Spc}(\mathcal{T}^d).$$

10.25. *Remark.* We warn the reader that it is not necessarily true that $\mathrm{Supp}_{\mathcal{T}}(x) = \mathrm{supp}(x)$ for $x \in \mathcal{T}^d$. That is, the small support does not necessarily recover the universal support of dualizable objects.

10.26. *Definition.* Let $\mathcal{T}$ be a non-closed tensor-triangulated category which has small coproducts. A function $\sigma\colon \mathcal{T} \to \mathcal{P}(X)$ is a *support theory for localizing $\mathcal{T}^d$-submodules* if it satifies the following conditions:

(a) $\sigma(0) = \varnothing$;
(b) $\sigma(\Sigma t) = \sigma(t)$ for every $t \in \mathcal{T}$;
(c) $\sigma(c) \subseteq \sigma(a) \cup \sigma(b)$ for any exact triangle $a \to b \to c \to \Sigma a$ in $\mathcal{T}$.
(d) $\sigma(\coprod_i t_i) = \bigcup_i \sigma(t_i)$ for any set of objects t_i in $\mathcal{T}$.
(e) $\sigma(x \otimes t) \subseteq \sigma(t)$ for any $x \in \mathcal{T}^d$ and $t \in \mathcal{T}$.

These properties are precisely equivalent to the statement that for every subset $Y \subseteq X$, the subcategory $\{\, t \in \mathcal{T} \,|\, \sigma(t) \subseteq Y \,\}$ is a localizing $\mathcal{T}^d$-submodule of $\mathcal{T}$.

10.27. *Definition.* We say that a support theory (X, σ) *stratifies* $\mathcal{T}$ if the map $\mathcal{L} \mapsto \bigcup_{t \in \mathcal{L}} \sigma(t)$ is a bijection from the collection of localizing $\mathcal{T}^d$-submodules of $\mathcal{T}$ to the set of all subsets of X.

10.28. *Remark.* If (X, σ) stratifies $\mathcal{T}$ then the inverse must necessarily be given by $Y \mapsto \{\, t \in \mathcal{T} \,|\, \sigma(t) \subseteq Y \,\}$. It follows that σ provides a lattice isomorphism between the lattice of localizing $\mathcal{T}^d$-submodules and the lattice of all subsets of X. In particular, it preserves joins and meets.

10.29. *Example.* If $\mathrm{Spc}(\mathcal{T}^d)$ is weakly noetherian, then $\mathrm{Supp}_{\mathcal{T}}$ defined in Definition 10.24 is a support function in the sense of Definition 10.26. Axioms (d) and (e) follow from Remark 10.16.

10.30. *Definition.* We say that a perfectly generated non-closed tt-category $\mathcal{T}$ with $\mathrm{Spc}(\mathcal{T}^d)$ weakly noetherian is *stratified* if it is stratified by $\mathrm{Supp}_{\mathcal{T}}$.

10.31. *Example.* Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category in the usual sense and let $\mathcal{K} := \mathcal{T}^d$. The localizing $\mathcal{K}$-submodules of $\mathcal{T}$ are precisely the localizing ideals of $\mathcal{T}$ (Example 10.9). If we apply the construction of Definition 10.24 to $\mathcal{T}$, we are just considering for each Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{K})$, the Bousfield localization of $\mathcal{T}$ whose acyclics are

$$\mathrm{Loc}\langle \mathcal{K}_Y \otimes \mathcal{T} \rangle = \mathrm{Locid}\langle \mathcal{K}_Y \rangle = \mathrm{Locid}\langle e_Y \rangle = \mathrm{Loc}\langle \mathcal{K}_Y \rangle =: \mathcal{T}_Y.$$

This is the usual idempotent triangle

$$e_Y \otimes t \to t \to f_Y \otimes t \to \Sigma e_Y \otimes t$$

on $\mathcal{T}$, and the support is

$$\mathrm{Supp}_{\mathcal{T}}(t) = \{\, \mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,|\, g_{\mathcal{P}} \otimes t \neq 0 \,\}.$$

That is, we obtain the usual Balmer–Favi support of Definition 4.23. Stratification for $\mathcal{T}$ in the sense of Definition 10.30 recovers the notion of stratification studied in [BHS23]. In this example, $\mathrm{Supp}_{\mathcal{T}}(x) = \mathrm{supp}(x)$ for $x \in \mathcal{T}^d$.

10.32. *Example.* Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category in the usual sense. Then $\mathcal{T}^{\mathrm{op}}$ is a perfectly generated non-closed tensor-triangulated category with subcategory of dualizable objects $\mathcal{K} := (\mathcal{T}^{\mathrm{op}})^d \cong \mathcal{T}^d$. The localizing $\mathcal{K}$-submodules of $\mathcal{T}^{\mathrm{op}}$ are precisely the colocalizing coideals of $\mathcal{T}$ (Example 10.10).

If we apply the construction of Definition 10.24 to $\mathcal{T}^{\mathrm{op}}$, we are then considering a Bousfield localization in $\mathcal{T}^{\mathrm{op}}$ for each Thomason subset $Y \subset \mathrm{Spc}(\mathcal{K})$,

$$\Gamma_Y^{\mathrm{op}}(t) \to t \to \mathrm{L}_Y^{\mathrm{op}}(t) \to \Sigma\Gamma_Y^{\mathrm{op}}(t), \tag{10.33}$$

whose acyclics are $\mathrm{Loc}_{\mathcal{T}^{\mathrm{op}}}\langle \mathcal{K}_Y \otimes \mathcal{T}^{\mathrm{op}}\rangle$. This corresponds to a Bousfield localization in $\mathcal{T}$,

$$\mathrm{L}_Y^{\mathrm{op}}(t) \to t \to \Gamma_Y^{\mathrm{op}}(t) \to \Sigma\mathrm{L}_Y^{\mathrm{op}}(t), \tag{10.34}$$

whose *local* objects are

$$\mathrm{Loc}_{\mathcal{T}^{\mathrm{op}}}\langle \mathcal{K}_Y \otimes \mathcal{T}^{\mathrm{op}}\rangle = \mathrm{Coloc}_{\mathcal{T}}\langle \mathcal{K}_Y \otimes \mathcal{T}\rangle.$$

That is, expressed in $\mathcal{T}$, the subcategory of local objects is $\mathrm{Coloc}_{\mathcal{T}}\langle \mathcal{K}_Y \otimes \mathcal{T}\rangle = \mathsf{hom}(e_Y, \mathcal{T}) = (\mathcal{T}_Y)^{\perp\perp}$ using Lemma 4.5. Thus the Bousfield localization (10.34) in $\mathcal{T}$ is the Bousfield localization

$$\mathsf{hom}(f_Y, t) \to t \to \mathsf{hom}(e_Y, t) \to \Sigma\mathsf{hom}(f_Y, t)$$

whose acyclics are $(\mathcal{T}_Y)^{\perp}$. Thus, the original Bousfield localization (10.33) in $\mathcal{T}^{\mathrm{op}}$ is given by

$$\Gamma_Y^{\mathrm{op}}(t) = \mathsf{hom}(e_Y, t) \qquad \text{and} \qquad \mathrm{L}_Y^{\mathrm{op}}(t) = \mathsf{hom}(f_Y, t)$$

and the support function on $\mathcal{T}^{\mathrm{op}}$ is given by

$$\mathrm{Supp}_{\mathcal{T}^{\mathrm{op}}}(t) := \{\, \mathcal{P} \in \mathcal{K} \mid \mathsf{hom}(g_{\mathcal{P}}, t) \neq 0 \,\}.$$

In conclusion, we have established that the support (Definition 10.24) of the opposite category $\mathcal{T}^{\mathrm{op}}$ is precisely Balmer–Favi cosupport (Definition 4.23):

10.35. **Theorem.** *Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category with $\mathrm{Spc}(\mathcal{T}^d)$ weakly noetherian. Then*

$$\mathrm{Cosupp}_{\mathcal{T}}(t) = \mathrm{Supp}_{\mathcal{T}^{\mathrm{op}}}(t)$$

for all $t \in \mathcal{T}$, the colocalizing coideals of $\mathcal{T}$ are precisely the localizing $\mathcal{T}^d$-submodules of $\mathcal{T}^{\mathrm{op}}$, and $\mathcal{T}$ is costratified (Definition 7.1) precisely when $\mathcal{T}^{\mathrm{op}}$ is stratified (Definition 10.30).

10.36. *Remark.* Our perspective has been to regard a non-closed tt-category $\mathcal{T}$ as a module over its rigid tt-subcategory of dualizable objects $\mathcal{T}^d$. There are similarities but also differences between the support theory we have constructed compared to what is achieved in [Ste13]. Our setting is both more general and more specialized. When $\mathcal{T}$ is a rigidly-compactly generated tt-category acting on a compactly generated triangulated category $\mathcal{S}$, Stevenson constructs a notion of support for localizing $\mathcal{T}$-submodules of $\mathcal{S}$ taking values in $\mathrm{Spc}(\mathcal{T}^c)$. In particular, one can let $\mathcal{T}$ act on itself, in which case one recovers the Balmer–Favi support theory. On the other hand, Stevenson's theory applies to non-tensor examples like singularity categories (see, e.g., [Ste14b]). Our setting is more specialized because we require a monoidal structure, but within the monoidal setting our construction is more general in that it only requires a non-closed perfectly generated tt-category; moreover, it zeros in on the significance of the $\mathcal{T}^d$ part of the action. These ideas also provide new examples beyond cosupport, as we now explain:

10.37. *Example.* If $\mathcal{L}$ is any strictly localizing ideal of a rigidly-compactly generated tt-category $\mathcal{T}$ then the corresponding Bousfield localization $\mathcal{T}/\mathcal{L}$ inherits the structure of a tensor-triangulated category and the generators of $\mathcal{T}$ localize to a set of dualizable generators of $\mathcal{T}/\mathcal{L}$. In general, these generators are not compact when $n > 0$. For example, the unit of the $K(n)$-local category is not compact. Nevertheless, the local category $\mathcal{T}/\mathcal{L}$ is well generated hence perfectly generated (see [Kra10, Theorem 7.2.1]) and $(\mathcal{T}/\mathcal{L})^d$ is essentially small (see [Mat16, p. 413]). Thus, Definition 10.24 provides a support theory for any Bousfield localization of a rigidly-compactly generated tt-category which lies in the spectrum of dualizable objects $\mathrm{Spc}((\mathcal{T}/\mathcal{L})^d)$ provided this space is weakly noetherian. We will not pursue this class of examples in this work, but it would be an interesting support theory to study, e.g., for the $K(n)$-local category. Some work in this direction has been carried out in [BHN22].

11. Universality

Our next goal is to establish that our approach to costratification via the cosupport theory defined in Section 4 is in a certain sense the unique or universal such choice. We obtained analogous results for stratification in [BHS23, Section 7]. In fact, given the connection between stratification and costratification established in Section 10, we will proceed by proving a generalized version of that uniqueness result which holds for support theories for perfectly generated categeories.

11.1. *Hypothesis.* Unless otherwise specified, $\mathcal{T}$ will denote a perfectly generated non-closed tensor-triangulated category (Terminology 10.1) and support theory will mean the notion of Definition 10.26.

11.2. **Lemma.** *Let $\sigma\colon \mathcal{T} \to \mathcal{P}(\mathrm{Spc}(\mathcal{T}^d))$ be a support theory (Definition 10.26) which satisfies $\sigma(\Gamma_Y\mathcal{T}) \subseteq Y$ and $\sigma(\mathrm{L}_Y\mathcal{T}) \subseteq Y^c$ for a Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^d)$. Then*

$$\sigma(\Gamma_Y t) = Y \cap \sigma(t) \quad \text{and} \quad \sigma(\mathrm{L}_Y t) = Y^c \cap \sigma(t)$$

for any $t \in \mathcal{T}$.

Proof. This follows in a routine manner from the exact triangle (10.14). $\square$

11.3. **Lemma.** *Assume $\mathrm{Spc}(\mathcal{T}^d)$ is weakly noetherian. For any Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^d)$ and $t \in \mathcal{T}$, we have*

(a) $\mathrm{Supp}_{\mathcal{T}}(\Gamma_Y t) = Y \cap \mathrm{Supp}_{\mathcal{T}}(t)$, and
(b) $\mathrm{Supp}_{\mathcal{T}}(\mathrm{L}_Y t) = Y^c \cap \mathrm{Supp}_{\mathcal{T}}(t)$.

Proof. This is a routine exercise using Lemma 10.22 and the exact triangle (10.14). Note that in light of Lemma 11.2 and Example 10.29, it suffices to establish the $\subseteq$ inclusions in (a) and (b). $\square$

11.4. **Proposition.** *Let $\mathcal{T}$ be a perfectly generated non-closed tensor-triangulated category with $\mathrm{Spc}(\mathcal{T}^d)$ weakly noetherian. The notion of support defined in Definition 10.24 is the only assignment of a subset $\sigma(t) \subseteq \mathrm{Spc}(\mathcal{T}^d)$ to each object $t \in \mathcal{T}$ which can satisfy the following two properties:*

(a) For every $t \in \mathcal{T}$, $\sigma(t) = \varnothing$ implies $t = 0$.
(b) For every Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^d)$, $\sigma(\Gamma_Y t) = \sigma(t) \cap Y$ and $\sigma(\mathrm{L}_Y t) = \sigma(t) \cap Y^c$.

Proof. If $\{\mathcal{P}\} = Y_1 \cap Y_2^c$ then $\Gamma_{Y_1}\mathrm{L}_{Y_2} t = \{\mathcal{P}\} \cap \sigma(t)$. Thus $\mathcal{P} \in \sigma(t)$ if and only if $\sigma(\Gamma_{Y_1}\mathrm{L}_{Y_2} t) \neq \varnothing$ if and only if $\Gamma_{Y_1}\mathrm{L}_{Y_2} t \neq 0$ if and only if $\mathcal{P} \in \mathrm{Supp}_{\mathcal{T}}(t)$. $\square$

11.5. **Theorem.** *Let $\mathcal{T}$ be a perfectly generated non-closed tensor-triangulated category. Suppose $\mathcal{T}$ is stratified (Definition 10.27) by a support function σ (Definition 10.26) taking values in a spectral space X. Suppose σ induces a bijection*

$$Y \mapsto \sigma(\Gamma_Y \mathcal{T})$$

between the set of Thomason subsets of $\mathrm{Spc}(\mathcal{T}^d)$ and the set of Thomason subsets of X. Then there is a unique homeomorphism $f\colon X \xrightarrow{\sim} \mathrm{Spc}(\mathcal{T}^d)$ such that $\sigma(\Gamma_{\mathrm{supp}(a)}\mathcal{T}) = f^{-1}(\mathrm{supp}(a))$ for all $a \in \mathcal{T}^d$. Moreover, if the space X is weakly noetherian then $\sigma(t) = f^{-1}(\mathrm{Supp}_{\mathcal{T}}(t))$ for all $t \in \mathcal{T}$.

Proof. Recall from Remark 10.28 that σ provides a lattice isomorphism. Together with Lemma 10.19 this ensures that the hypothesized bijection $Y \mapsto \sigma(\Gamma_Y\mathcal{T})$ is a lattice isomorphism. Both lattices are coherent frames (see, e.g., [KP17]) hence by Stone duality [Joh82] it corresponds to a homeomorphism $f\colon X \xrightarrow{\sim} \mathrm{Spc}(\mathcal{T}^d)$. By construction this homeomorphism satisfies $f^{-1}(Y) = \sigma(\Gamma_Y\mathcal{T})$ for every Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^d)$. Since σ is a lattice isomorphism, $\Gamma_Y\mathcal{T} \cap \mathrm{L}_Y\mathcal{T} = 0$ and $\Gamma_Y\mathcal{T} \vee \mathrm{L}_Y\mathcal{T} = \mathcal{T}$ which together imply that $\sigma(\mathrm{L}_Y\mathcal{T}) = f^{-1}(Y)^c$. The fact that $f^{-1}(\mathrm{Supp}_{\mathcal{T}}(t)) = \sigma(t)$ for each $t \in \mathcal{T}$ then follows from Lemma 11.2 and Proposition 11.4. The uniqueness statement is standard; see [Bal05, Lemma 3.3] for example. □

11.6. *Remark.* Recall from Remark 10.18 that $\Gamma_Y\mathcal{T}$ is not necessarily the localizing $\mathcal{T}^d$-submodule generated by $\mathcal{T}^d_Y$. For example, the localizing $\mathcal{T}^d$-submodule generated by a dualizable object $a \in \mathcal{T}^d$ is $\mathrm{Loc}\langle a \otimes \mathcal{T}^d\rangle$ while $\Gamma_{\mathrm{supp}(a)}\mathcal{T} = \mathrm{Loc}\langle a \otimes \mathcal{T}\rangle$. This may be helpful in understanding Theorem 11.5: $\sigma(a)$ is the support of the localizing $\mathcal{T}^d$-submodule generated by a, while $\sigma(\Gamma_{\mathrm{supp}(a)}\mathcal{T})$ is the support of a potentially larger localizing $\mathcal{T}^d$-submodule; cf. Remark 10.25.

11.7. *Remark.* The lattice homomorphism $\mathcal{C} \mapsto \mathrm{Loc}\langle \mathcal{C} \otimes \mathcal{T}\rangle$ from thick ideals of $\mathcal{T}^d$ to localizing $\mathcal{T}^d$-submodules of $\mathcal{T}$ (cf. Lemma 10.19) is not obviously injective. In situations where it is known to be injective, the hypothesis of Theorem 11.5 simplifies slightly. This will be the case for our two main examples of interest below.

11.8. **Corollary.** *Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category. Suppose $\mathcal{T}$ is stratified by a support theory σ in a spectral space X. Suppose that $\sigma(a)$ is Thomason closed for every $a \in \mathcal{T}^d$ and that every Thomason closed subset of X arises in this way. If X is weakly noetherian then there is a homeomorphism $X \cong \mathrm{Spc}(\mathcal{T}^d)$ under which $\sigma(t) \cong \mathrm{Supp}_{\mathcal{T}}(t)$.*

Proof. Note that $\Gamma_Y\mathcal{T} = \mathrm{Loc}\langle \mathcal{K}_Y \otimes \mathcal{T}\rangle = \mathrm{Loc}\langle \mathcal{K}_Y \otimes \mathrm{Loc}\langle \mathcal{T}^d\rangle\rangle = \mathrm{Loc}\langle \mathcal{K}_Y\rangle$. Hence $\sigma(\Gamma_Y\mathcal{T}) = \sigma(\mathcal{K}_Y)$. In particular, $\sigma(\Gamma_{\mathrm{supp}(a)}\mathcal{T}) = \sigma(a)$ for any $a \in \mathcal{T}^d$. Similarly, for a Thomason subset $Y = \bigcup_i \mathrm{supp}(a_i)$ we have (invoking Lemma 10.19)

$$\sigma(\Gamma_Y\mathcal{T}) = \sigma(\bigvee_i \Gamma_{\mathrm{supp}(a_i)}\mathcal{T}) = \bigcup_i \sigma(\Gamma_{\mathrm{supp}(a_i)}\mathcal{T}) = \bigcup_i \sigma(a_i).$$

Thus, the hypothesis that $\sigma(a) \subseteq X$ is Thomason for all $a \in \mathcal{T}^d$ implies that $\sigma(\Gamma_Y\mathcal{T})$ is also Thomason for each Thomason subset $Y \subseteq X$. Moreover, every Thomason closed of X is hypothesized to be of the form $\sigma(a)$ for some $a \in \mathcal{T}^d$. Hence every Thomason subset of X is of the form $\bigcup_i \sigma(a_i) = \bigcup_i \sigma(\Gamma_{\mathrm{supp}(a_i)}\mathcal{T}) = \sigma(\bigvee_i \Gamma_{\mathrm{supp}(a_i)}\mathcal{T}) = \sigma(\Gamma_{\cup_i \mathrm{supp}(a_i)}\mathcal{T})$. Moreover, the map $Y \mapsto \Gamma_Y\mathcal{T}$ is injective, because $\Gamma_Y\mathcal{T} \cap \mathcal{T}^d = \mathrm{Loc}\langle \mathcal{K}_Y\rangle \cap \mathcal{T}^d = \mathcal{K}_Y$ by [Nee92b, Lemma 2.2]. Then the hypotheses of Theorem 11.5 hold and we are done. □

11.9. *Remark.* Corollary 11.8 is essentially the uniqueness theorem of [BHS23, Theorem 7.6]. We have included the above arguments to show how it is deduced from the primordial uniqueness Theorem 11.5. The statement in [BHS23, Theorem 7.6] includes some further conditions equivalent to the hypotheses.

11.10. **Corollary.** *Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category. Suppose $\mathcal{T}$ is costratified by a cosupport theory $\mathfrak{C}$ taking values in a spectral space X. The following are equivalent:*

- *(a) For every Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^d)$, $\mathfrak{C}(\mathsf{hom}(e_Y, \mathcal{T}))$ is a Thomason subset of X, and every Thomason subset of X arises in this way.*
- *(b) The map $Y \mapsto \mathfrak{C}(\mathsf{hom}(e_Y, \mathcal{T}))$ is a bijection from the lattice of Thomason subsets of $\mathrm{Spc}(\mathcal{T}^d)$ onto the lattice of Thomason subsets of X.*
- *(c) For every Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^d)$, $\mathfrak{C}(\mathrm{Loc}\langle \mathcal{K}_Y\rangle^{\perp})^c$ is a Thomason subset of X, and every Thomason subset of X arises in this way.*
- *(d) The map $\mathcal{C} \mapsto \mathfrak{C}(\mathrm{Loc}\langle \mathcal{C}\rangle^{\perp})^c$ is a bijection between the thick ideals of $\mathcal{T}^d$ and the Thomason subsets of X.*
- *(e) There is a unique homeomorphism $f\colon X \xrightarrow{\sim} \mathrm{Spc}(\mathcal{T}^d)$ such that*
$$f^{-1}(\mathrm{supp}(a)) = \mathfrak{C}(\mathsf{hom}(a, \mathcal{T}))$$
for each $a \in \mathcal{T}^d$.

If these conditions hold and the spectral space X is weakly noetherian then under the homeomorphism $X \xrightarrow{\sim} \mathrm{Spc}(\mathcal{T}^d)$ the cosupport theory $\mathfrak{C}$ coincides with the Balmer–Favi cosupport Cosupp.

Proof. We wish to apply Theorem 11.5 to $\mathcal{T}^{\mathrm{op}}$. Recall from Example 10.32 that $\Gamma_Y \mathcal{T}^{\mathrm{op}} = \mathsf{hom}(e_Y, \mathcal{T})$. Thus, the hypothesis of Theorem 11.5 (applied to $\mathcal{T}^{\mathrm{op}}$) is that $Y \mapsto \mathfrak{C}(\mathsf{hom}(e_Y, \mathcal{T}))$ is a bijection between the set of Thomason subsets of $\mathrm{Spc}(\mathcal{T}^d)$ and the set of Thomason subsets of X; that is, statement (b). Note that this map is necessarily injective: If $\mathfrak{C}(\mathsf{hom}(e_{Y_1}, \mathcal{T})) = \mathfrak{C}(\mathsf{hom}(e_{Y_2}, \mathcal{T}))$ then $\mathrm{hom}(e_{Y_1}, \mathcal{T}) = \mathrm{hom}(e_{Y_2}, \mathcal{T})$ since $\mathfrak{C}$ costratifies $\mathcal{T}$. Hence
$$\mathrm{Locid}(e_{Y_1}) = {}^{\perp\perp}\mathsf{hom}(e_{Y_1}, \mathcal{T}) = {}^{\perp\perp}\mathsf{hom}(e_{Y_2}, \mathcal{T}) = \mathrm{Locid}(e_{Y_2})$$
so that
$$\mathcal{K}_{Y_1} = \mathcal{T}^d \cap \mathrm{Locid}(e_{Y_1}) = \mathcal{T}^d \cap \mathrm{Locid}(e_{Y_2}) = \mathcal{K}_{Y_2}$$
and hence $Y_1 = Y_2$. Therefore, statement (b) is equivalent to statement (a).

Now recall from Remark 10.28 that the costratifying cosupport $\mathfrak{C}$ is a lattice isomorphism; in particular it preserves joins and meets. If $\mathcal{C}_1 \coloneqq \mathrm{Loc}\langle \mathcal{K}_Y\rangle^{\perp} = f_Y \otimes \mathcal{T} = \mathsf{hom}(f_Y, \mathcal{T})$ and $\mathcal{C}_2 \coloneqq \mathsf{hom}(e_Y, \mathcal{T}) = \mathrm{Loc}\langle \mathcal{K}_Y\rangle^{\perp\perp}$, we have $\mathcal{C}_1 \cap \mathcal{C}_2 = 0$. On the other hand, for any $t \in \mathcal{T}$, there is an exact triangle $\mathsf{hom}(f_Y, t) \to t \to \mathsf{hom}(e_Y, t) \to \Sigma\mathsf{hom}(f_Y, t)$ which shows that $\mathcal{C}_1 \vee \mathcal{C}_2 = \mathcal{T}$. Since $\mathfrak{C}$ preserves joins and meets, we conclude that $\mathfrak{C}(\mathcal{C}_1)^c = \mathfrak{C}(\mathcal{C}_2)$. Thus statement (c) is equivalent to statement (a), and statement (d) is equivalent to statement (b).

Bearing in mind Lemma 4.34 which gives $\mathsf{hom}(a, \mathcal{T}) = \mathsf{hom}(e_{\mathrm{supp}(a)}, \mathcal{T})$, Theorem 11.5 establishes that (b) implies (e). Conversely, suppose (e) holds. For an arbitrary Thomason $Y = \bigcup_{i\in I} \mathrm{supp}(a_i)$ we have $\mathsf{hom}(e_Y, \mathsf{hom}(a_i, t)) = \mathsf{hom}(a_i, t)$ which implies that $f^{-1}(Y) \subseteq \mathfrak{C}(\mathsf{hom}(e_Y, \mathcal{T}))$. Moreover, since $e_Y \in \mathrm{Locid}\langle a_i \mid i \in I\rangle$, we have
$$\mathsf{hom}(e_Y, t) \in \mathsf{hom}(\mathrm{Locid}\langle a_i \mid i \in I\rangle, t) \subseteq \mathrm{Colocid}\langle \mathsf{hom}(a_i, t) \mid i \in I\rangle$$

by (2.8) so that $\mathfrak{C}(\mathsf{hom}(e_Y,\mathcal{T}))\subseteq f^{-1}(Y)$. Thus, $\mathfrak{C}(\mathsf{hom}(e_Y,\mathcal{T}))=f^{-1}(Y)$ which establishes that (a) holds.

Finally note that the last statement (when X is weakly noetherian) is also provided by Theorem 11.5. ☐

11.11. *Remark.* As with Corollary 11.8 (and the original [BHS23, Theorem 7.6]), the point of Corollary 11.10 is that if $\mathcal{T}$ is costratified by a notion of cosupport in a weakly noetherian spectral space such that the costratification is "compatible" with the usual classification of thick ideals of dualizable objects, then it must be achieved by the tensor triangular cosupport theory which lies in the Balmer spectrum.

11.12. **Corollary** (BIK costratification)**.** *Let $\mathcal{T}$ be a rigidly-compactly generated tt-category which, in the terminology of* [BIK12] *is noetherian and costratified by the action of a graded-noetherian ring R. Then the BIK space of cosupports* $\operatorname{cosupp}_R(\mathcal{T})$ *is homeomorphic to* $\operatorname{Spc}(\mathcal{T}^c)$ *and the BIK notion of cosupport in* $\operatorname{cosupp}_R(\mathcal{T})$ *coincides with the notion of cosupport in* $\operatorname{Spc}(\mathcal{T}^c)$ *defined in Definition 4.23.*

Proof. We first note that because $\mathcal{T}$ is costratified by R, it is also stratified by R, see [BIK12, Theorem 9.7]. Moreover, $\operatorname{cosupp}_R(\mathcal{T})=\operatorname{supp}_R(\mathcal{T})$ and for any localizing ideal $\mathcal{L}$, $\operatorname{cosupp}_R(\mathcal{L}^{\perp})=\operatorname{supp}_R(\mathcal{L})^c$ (see the proof of [BIK12, Corollary 9.9]). Invoking [BIK11b, Theorem 6.1] we see that $\mathcal{C}\mapsto\operatorname{cosupp}_R(\operatorname{Loc}\langle\mathcal{C}\rangle^{\perp})^c$ is a bijection between the thick ideals and the Thomason subsets of $\operatorname{supp}_R\mathcal{T}$. Moreover, by [BIK08, Theorem 5.5], $\operatorname{supp}_R(\mathcal{T})=\operatorname{Supp}_R(\mathbb{1})$ is a closed subspace of $\operatorname{Spec}(R)$, hence is itself a noetherian spectral space. Hence, we can invoke Corollary 11.10 with hypothesis (d). Alternatively, we could invoke [BHS23, Corollary 7.11] to conclude that $\operatorname{Spc}(\mathcal{T}^c)\cong\operatorname{supp}_R(\mathcal{T})\cong\operatorname{cosupp}_R(\mathcal{T})$ and invoke Proposition 11.4 noting that the BIK theory of cosupport satisfies conditions (a) and (b) (see in particular [BIK12, Theorem 4.5 and Proposition 4.7]). ☐

11.13. *Remark.* The axioms of a support function for localizing submodules given in Definition 10.26 look slightly weaker than the axioms for a support function for localizing ideals given in [BHS23, Definition 7.1]. The former's axiom (e) states that $\sigma(x\otimes t)\subseteq\sigma(t)$ for any $x\in\mathcal{T}^d$ and $t\in\mathcal{T}$ while the latter's axiom (e) states that $\sigma(t_1\otimes t_2)\subseteq\sigma(t_1)\cap\sigma(t_2)$ for all $t_1,t_2\in\mathcal{T}$. For $\mathcal{T}$ rigidly-compactly generated, this stronger axiom is forced by the axioms of Definition 10.26. Indeed, for any $t\in\mathcal{T}$, $\{\,s\in\mathcal{T}\,|\,\sigma(s\otimes t)\subseteq\sigma(t)\,\}$ is a localizing subcategory which contains the dualizable objects. Similarly, axiom (e) of Definition 3.1 looks slightly stronger than what is provided by Definition 10.26 when applied to $\mathcal{T}^{\mathrm{op}}$, namely it requires $\mathfrak{C}(\hom(s,t))\subseteq\mathfrak{C}(t)$ for all $s,t\in\mathcal{T}$. Again it is implied by the *a priori* weaker axioms by considering $\{\,s\in\mathcal{T}\,|\,\mathfrak{C}(\mathsf{hom}(s,t))\subseteq\mathfrak{C}(t)\,\}$. On the other hand, Definition 10.26 does not include the second part of axiom (a) of Definition 3.1.

11.14. *Remark.* The reader may find it interesting to compare our axioms for cosupport theory with the axioms of [Ver22, Definition 3.2] which explicitly involve Brown–Comenetz duals (which will make an appearance in our work in the next section). Our axioms are *a priori* weaker. In particular, a cosupport datum in the sense of [Ver22, Definition 3.2] is a cosupport function in the sense of Definition 3.1. We claim that our formulation of axiom (e) is morally "correct" due to Remark 3.2; also cf. Definition 10.26(e). In any case, it follows from Proposition 12.9 below that the Balmer–Favi cosupport function satisfies the stronger axioms stated in [Ver22, Definition 3.2]. Hence, by Corollary 11.10 any cosupport theory which costratifies the colocalizing coideals in a weakly noetherian space (in the sense of the corollary) satisfies the stronger axioms of [Ver22, Definition 3.2].

12. Duality

12.1. *Definition.* Let κ be an object in a closed symmetric monoidal category $\mathcal{T}$. We say that κ *dualizes* a full subcategory $\mathcal{T}_0 \subseteq \mathcal{T}$ if $\mathsf{hom}(-,\kappa)$ restricts to an equivalence $\mathcal{T}_0 \xrightarrow{\simeq} \mathcal{T}_0^{\mathrm{op}}$.

12.2. *Remark.* Writing $\Delta_\kappa(x) \coloneqq \mathsf{hom}(x,\kappa)$, observe that the evaluation morphism $\Delta_\kappa(x) \otimes x \to \kappa$ is adjoint to a morphism $\phi_x \colon x \to \Delta_\kappa\Delta_\kappa(x)$. Dinaturality of evaluation implies that ϕ_x is natural in x. Unwinding the definition, we see that κ dualizes $\mathcal{T}_0$ if and only if Δ_κ preserves $\mathcal{T}_0$ and $x \to \Delta_\kappa\Delta_\kappa x$ is an isomorphism for all $x \in \mathcal{T}_0$.

12.3. *Example.* The monoidal unit $\mathbb{1}$ dualizes the full subcategory $\mathcal{T}^d \subseteq \mathcal{T}$ of dualizable objects and also the (potentially larger) full subcategory of reflexive objects of $\mathcal{T}$, i.e., those $x \in \mathcal{T}$ for which the canonical map $x \to (x^\vee)^\vee = \Delta_{\mathbb{1}}\Delta_{\mathbb{1}}(x)$ is an isomorphism. More generally, any $\otimes$-invertible object κ dualizes precisely the same subcategories as the monoidal unit $\mathbb{1}$.

12.4. **Proposition.** *Let $\mathcal{T}$ be a triangulated category with a compatible closed symmetric monoidal structure (as in* [HPS97, Appendix A]*). Let $\kappa \in \mathcal{T}$ be an arbitrary object. It dualizes the full subcategory*

$$\mathcal{T}_\kappa \coloneqq \big\{\, x \in \mathcal{T} \,\big|\, \phi_x \colon x \to \Delta_\kappa\Delta_\kappa(x) \ \textit{is an equivalence} \,\big\}.$$

The full subcategory $\mathcal{T}_\kappa$ is a thick $\mathcal{T}^d$-submodule of $\mathcal{T}$ (that is, a thick subcategory satisfying $\mathcal{T}^d \otimes \mathcal{T}_\kappa \subseteq \mathcal{T}_\kappa$). It contains any full subcategory of $\mathcal{T}$ dualized by κ.

Proof. The functor $\Delta_\kappa \coloneqq \mathsf{hom}(-,\kappa)$ is adjoint to itself

$$\Delta_\kappa : \mathcal{T} \rightleftarrows \mathcal{T}^{\mathrm{op}} : \Delta_\kappa \tag{12.5}$$

with the map ϕ_x of Remark 12.2 serving as both the unit and the counit of the adjunction. The fact that κ dualizes the subcategory $\mathcal{T}_\kappa$ is then just a manifestation of the fact that any adjunction induces an equivalence between the full subcategory of objects on which the unit is an isomorphism and the full subcategory of objects on which the counit is an isomorphism. Since (12.5) is a triangulated adjunction, $\mathcal{T}_\kappa$ is a thick subcategory of $\mathcal{T}$. To see that it is a $\mathcal{T}^d$-submodule just note that for any $x \in \mathcal{T}$ and $y \in \mathcal{T}^d$, we have

$$\Delta_\kappa\Delta_\kappa(x) \otimes y \simeq \Delta_\kappa\Delta_\kappa(x \otimes y)$$

and $\phi_x \otimes \mathrm{id}_y \simeq \phi_{x\otimes y}$. The final statement is clear from Remark 12.2. $\square$

12.6. *Remark.* The fact that *every* object $\kappa \in \mathcal{T}$ dualizes a certain canonical thick $\mathcal{T}^d$-submodule of $\mathcal{T}$ seems not to have been previously noticed (or appreciated) and leads to new approaches to organizing the objects of a tt-category. For example, the Picard group of a rigid tt-category $\mathcal{K}$ describes precisely those objects which dualize $\mathcal{K}$ itself. A more in-depth study of this notion will be the subject of future work. Our present focus is on how this connects with cosupport.

12.7. *Example* (Brown–Comenetz duality). Recall that in Definition 9.4 we constructed an object I_c for each compact object c in a compactly generated triangulated category $\mathcal{T}$. Now suppose $\mathcal{T}$ is a rigidly-compactly generated tt-category. It follows from (9.6) that $I_c = c \otimes I_{\mathbb{1}}$ for any compact object $c \in \mathcal{T}^c$. We define the *Brown–Comenetz dual* of an arbitrary object $t \in \mathcal{T}$, as

$$t^* := \mathsf{hom}(t, I_{\mathbb{1}}).$$

This object represents the functor

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{T}(\mathbb{1}, t \otimes -), \mathbb{Q}/\mathbb{Z}) \colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}\,.$$

Note that for a compact object $c \in \mathcal{T}^c$, we have $c^* = I_{c^\vee}$. In particular, $\mathbb{1}^* = I_{\mathbb{1}}$. This observation, together with (9.6), implies that an object $t = 0$ if and only if $t^* = 0$.

12.8. *Example.* When $\mathcal{T} = \mathrm{SH}$ is the category of spectra, the Brown–Comenetz dual of the sphere $I_{\mathbb{1}}$ dualizes the full subcategory of spectra whose homotopy groups are finite; see [BC76].

12.9. **Proposition.** *For any compact object $c \in \mathcal{T}^c$, we have*

$$\mathrm{Cosupp}(I_c) = \mathrm{supp}(c).$$

For an arbitrary object $t \in \mathcal{T}$, we have

$$\mathrm{Cosupp}(t^*) = \mathrm{Supp}(t).$$

If $t \in \mathcal{T}_{I_{\mathbb{1}}}$, that is, $t \xrightarrow{\sim} t^{**}$ is an isomorphism, then

$$\mathrm{Cosupp}(t) = \mathrm{Supp}(t^*).$$

Proof. Let $c \in \mathcal{T}^c$. Then observe that

$$\begin{aligned} \mathsf{hom}(g_{\mathcal{P}}, I_c) = 0 &\iff \mathcal{T}(d \otimes g_{\mathcal{P}}, I_c) = 0 \text{ for all } d \in \mathcal{T}^c \\ &\iff \mathcal{T}(c, d \otimes g_{\mathcal{P}}) = 0 \text{ for all } d \in \mathcal{T}^c \\ &\iff \mathcal{T}(d^\vee, c^\vee \otimes g_{\mathcal{P}}) = 0 \text{ for all } d \in \mathcal{T}^c \\ &\iff c^\vee \otimes g_{\mathcal{P}} = 0. \end{aligned}$$

Here, the second equivalence relies on the discussion in Remark 9.7. Thus,

$$\mathrm{Cosupp}(I_c) = \mathrm{Supp}(c^\vee) = \mathrm{supp}(c^\vee) = \mathrm{supp}(c).$$

In particular, $\mathrm{Cosupp}(I_{\mathbb{1}}) = \mathrm{Cosupp}(\mathbb{1}^*) = \mathrm{Spc}(\mathcal{T}^c)$. For the second statement, Lemma 4.29 provides the inclusion

$$\mathrm{Cosupp}(t^*) = \mathrm{Cosupp}(\mathsf{hom}(t, \mathbb{1}^*)) \subseteq \mathrm{Supp}(t) \cap \mathrm{Cosupp}(\mathbb{1}^*) = \mathrm{Supp}(t).$$

On the other hand, suppose $\mathcal{P} \in \mathrm{Supp}(t)$, so that $g_{\mathcal{P}} \otimes t \neq 0$. Hence since the I_c cogenerate we have $\mathcal{T}(g_{\mathcal{P}} \otimes t, I_c) \neq 0$ for some compact c. Using $I_c \simeq c \otimes I_{\mathbb{1}}$, we see that $\mathcal{T}(c^\vee, \mathsf{hom}(g_{\mathcal{P}}, \mathsf{hom}(t, I_{\mathbb{1}}))) \neq 0$ so that $\mathcal{P} \in \mathrm{Cosupp}(t^*)$. ∎

12.10. *Example* (Dualizing complexes). A *dualizing complex*[5] for a separated noetherian scheme X is an object $t \in \mathrm{D}^b(\mathrm{coh}\, X)$ which dualizes $\mathrm{D}^b(\mathrm{coh}\, X)$.

12.11. *Definition.* We say that an object $t \in \mathcal{T}$ has *small cosupport* if $\mathrm{Cosupp}(t) \subseteq \mathrm{Supp}(t)$.

12.12. *Example.* Compact objects have small cosupport by Example 4.36.

[5] The classical literature sometimes also requires a "dualizing complex" to have finite injective dimension. Neeman [Nee10] emphasized that this should not be included as part of the definition.

12.13. *Remark.* If $\mathrm{Supp}(t)$ is Thomason and the codetection property holds then t has small cosupport if and only if the canonical map $t \to \mathsf{hom}(e_{\mathrm{Supp}(t)}, t)$ is an isomorphism. In this way, having small cosupport is related to derived notions of being adically complete.

12.14. *Example.* If $X = \mathrm{Spec}(R)$ is a noetherian affine scheme then any $t \in \mathrm{D}^b(\mathrm{coh}\, X)$ has small cosupport. Indeed, the Balmer–Favi support $\mathrm{Supp}(t) \subseteq \mathrm{Spec}(R)$ coincides with the ordinary cohomological support $\bigcup_{i\in\mathbb{Z}} \mathrm{Supp}(H^i(t))$ which is just a finite union of closed sets, hence is itself closed. Denoting this closed set by $V(I)$, the complex t is cohomologically I-adically complete by [PSY15, Theorem 1.21] (see also [SWW17, Theorem 6.7] and [BIK12, Proposition 4.19]) meaning that the canonical map $t \to \mathsf{hom}(e_{V(I)}, t)$ is an isomorphism. Hence $\mathrm{Cosupp}(t) \subseteq V(I) = \mathrm{Supp}(t)$.

12.15. **Proposition.** *Let $\mathcal{T}$ be stratified and suppose $\kappa \in \mathcal{T}$ dualizes the subcategory $\mathcal{T}_0 \subseteq \mathcal{T}$. For any $t \in \mathcal{T}_0$, we have*

$$\mathrm{Cosupp}(t) = \mathrm{Supp}(\Delta_\kappa(t)) \cap \mathrm{Cosupp}(\kappa).$$

In particular, if $\mathbb{1} \in \mathcal{T}_0$ (that is, if $\mathcal{T}^d \subseteq \mathcal{T}_0$) then

$$\mathrm{Cosupp}(\mathbb{1}) = \mathrm{Supp}(\kappa) \cap \mathrm{Cosupp}(\kappa).$$

If the objects of $\mathcal{T}_0$ have small cosupport, then

$$\mathrm{Cosupp}(t) = \mathrm{Supp}(t) \cap \mathrm{Cosupp}(\kappa)$$

for each $t \in \mathcal{T}_0$.

Proof. The first statement follows directly from the isomorphism $t \simeq \Delta_\kappa\Delta_\kappa(t)$ and Theorem 7.15. The second statement is an immediate consequence of the first statement. It follows that if $t \in \mathcal{T}_0$ has small cosupport then $\mathrm{Cosupp}(t) \subseteq \mathrm{Supp}(t) \cap \mathrm{Cosupp}(\kappa)$. On the other hand, if $\Delta_\kappa(t) \in \mathcal{T}_0$ also has small cosupport, then $\mathrm{Supp}(t) \cap \mathrm{Cosupp}(\kappa) = \mathrm{Cosupp}(\Delta_\kappa(t)) \subseteq \mathrm{Supp}(\Delta_\kappa(t))$ and it follows that $\mathrm{Supp}(t) \cap \mathrm{Cosupp}(\kappa) \subseteq \mathrm{Supp}(\Delta_\kappa(t)) \cap \mathrm{Cosupp}(\kappa) = \mathrm{Cosupp}(t)$. $\square$

12.16. *Remark.* In general, we know that $\mathrm{Cosupp}(x) = \mathrm{Supp}(x) \cap \mathrm{Cosupp}(\mathbb{1})$ for any compact object $x \in \mathcal{T}^c$. The point of Proposition 12.15 is that it gives us relations between the support and cosupport for bigger (not necessarily compact) objects. For example:

12.17. **Corollary.** *Let X be a noetherian affine scheme which admits a dualizing complex. Then*

$$\mathrm{Cosupp}(t) = \mathrm{Supp}(t) \cap \mathrm{Cosupp}(\mathbb{1}).$$

for any $t \in \mathrm{D}^b(\mathrm{coh}\, X)$.

Proof. Let $\kappa \in \mathrm{D}^b(\mathrm{coh}\, X)$ be a dualizing complex for X (Example 12.10). We apply Proposition 12.15. Since bounded complexes of coherent sheaves have small cosupport (Example 12.14), we obtain $\mathrm{Cosupp}(t) = \mathrm{Supp}(t) \cap \mathrm{Cosupp}(\kappa)$ for any $t \in \mathrm{D}^b(\mathrm{coh}\, X)$. Note that $\mathrm{D}^b(\mathrm{coh}\, X) \supseteq \mathrm{D}_{\mathrm{qc}}(X)^c$ contains $\mathbb{1}$. Moreover, a dualizing complex is (by definition) itself contained in $\mathrm{D}^b(\mathrm{coh}\, X)$, hence itself has small cosupport. It follows that $\mathrm{Cosupp}(\mathbb{1}) = \mathrm{Cosupp}(\kappa)$ and we are done. $\square$

12.18. *Remark.* Many schemes admit dualizing complexes; for example, complete noetherian local rings, Dedekind domains, and any scheme of finite type over a field (see, e.g., [Har66, Section 10]). Proposition 12.15 and Corollary 12.17

provide a general perspective on (and generalization of) results such as [BIK12, Proposition 4.18]; see also [SWW17, Theorem 1.2]. On the other hand, not every scheme admits a dualizing complex (see, e.g., [Sha79, Kaw02]).

12.19. *Example* (Matlis duality). Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Matlis duality implies that the injective hull $E(k)$ dualizes the full subcategory $\mathrm{thick}\langle k\rangle \subset \mathrm{D}(R) =: \mathcal{T}$. Indeed, this duality can be "lifted" from the ordinary duality on $\mathrm{D}(k)$ along the map $R \to k$; see [BDS16, Example 7.2]. Recall that $\mathrm{thick}\langle k\rangle$ consists of those complexes of R-modules which have only finitely many nonzero homology modules, each of which is a module of finite length (see, e.g., [DGI06]). The objects of $\mathrm{thick}\langle k\rangle$ have small cosupport since a module of finite length is $\mathfrak{m}$-adically complete. Consider $\mathrm{thick}\langle k\rangle \subseteq \mathcal{T}_{E(k)}$. Since $\mathsf{hom}(E(k), E(k))$ is isomorphic to the $\mathfrak{m}$-adic completion of R, we see that $\mathbb{1} \in \mathcal{T}_{E(k)}$ if and only if R is complete, while $\mathbb{1} \in \mathrm{thick}\langle k\rangle$ if and only if R is artinian (and hence complete). Note that the dualizing object $E(k)$ is typically not contained in the subcategory $\mathrm{thick}\langle k\rangle$ which it dualizes. For example, if $R = \mathbb{Z}_{(p)}$ then $E(k) = \mathbb{Z}(p^\infty)$ which is artinian but not noetherian. Note that if R is regular (or, more generally, Gorenstein) of dimension d, then $E(k) = \Sigma^d e_{\{\mathfrak{m}\}}$, so in light of Corollary 8.12, $E(k)$ cannot have small cosupport unless $\dim(R) = 0$, that is, unless R is artinian (in which case $E(k) \in \mathrm{thick}\langle k\rangle$).

Part III. Morphisms and descent

We now return to the world of rigidly-compactly generated tt-categories and study base change results for support and cosupport, as well as descent techniques for establishing stratification and costratification.

13. The image of a geometric functor

We begin with a general observation about the image of the map on Balmer spectra induced by a geometric functor. This seems to have been overlooked in the literature; it gives further evidence for the distinguished role played by the Balmer–Favi notion of support.

13.1. *Terminology.* A coproduct-preserving tt-functor $f^*\colon \mathcal{T} \to \mathcal{S}$ between rigidly-compactly generated tt-categories is called a *geometric functor*. As explained in [BDS16], such a functor admits a right adjoint f_* which itself admits a right adjoint $f^!$. The object $\omega_f \coloneqq f^!(\mathbb{1}_{\mathcal{T}}) \in \mathcal{S}$ is called the relative dualizing object for f^*.

13.2. *Hypothesis.* Throughout this section $f^*\colon \mathcal{T} \to \mathcal{S}$ will denote a geometric functor between rigidly-compactly generated tt-categories and

$$\varphi\colon \operatorname{Spc}(\mathcal{S}^c) \to \operatorname{Spc}(\mathcal{T}^c)$$

will denote the induced map on Balmer spectra. We will further assume throughout that $\operatorname{Spc}(\mathcal{S}^c)$ and $\operatorname{Spc}(\mathcal{T}^c)$ are weakly noetherian.

13.3. *Remark.* The adjoints $f^* \dashv f_* \dashv f^!$ of a geometric functor f^* are related by a number of useful formulas, spelled out in [BDS16, Proposition 2.15]. In particular, the projection formula

$$f_*(f^*(t) \otimes s) \cong t \otimes f_*(s)$$

holds for any $s \in \mathcal{S}$ and $t \in \mathcal{T}$. It follows, using the fact that f_* preserves coproducts, that

$$f_* \operatorname{Locid}\langle f^*(\mathcal{E})\rangle \subseteq \operatorname{Locid}\langle \mathcal{E}\rangle \tag{13.4}$$

for any collection of objects $\mathcal{E} \subseteq \mathcal{T}$.

13.5. *Remark.* While the kernel of f^* is a localizing ideal, the kernel of $f^!$ is a colocalizing coideal. This follows from another useful formula:

$$f^!\mathsf{hom}(t_1, t_2) \cong \mathsf{hom}(f^*t_1, f^!t_2) \tag{13.6}$$

for any $t_1, t_2 \in \mathcal{T}$; see [BDS16, (2.19)].

13.7. *Remark.* For any Thomason subset $Y \subseteq \operatorname{Spc}(\mathcal{T}^c)$, it is established in [BS17, Proposition 5.11] that $f^*(e_Y) = e_{\varphi^{-1}(Y)}$ and $f^*(f_Y) = f_{\varphi^{-1}(Y)}$. In particular, if $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$, then writing $\{\mathcal{P}\} = Y_1 \cap Y_2^c$ for Thomason subsets Y_1, Y_2, we see that the preimage $\varphi^{-1}(\{\mathcal{P}\}) = \varphi^{-1}(Y_1) \cap \varphi^{-1}(Y_2)^c$ is a weakly visible subset, and $f^*(g_{\mathcal{P}}) = e_{\varphi^{-1}(Y_1)} \otimes f_{\varphi^{-1}(Y_2)} = g_{\varphi^{-1}(\{\mathcal{P}\})}$. Hence

$$\operatorname{Supp}_{\mathcal{S}}(f^*(g_{\mathcal{P}})) = \varphi^{-1}(\{\mathcal{P}\}). \tag{13.8}$$

Combined with [BHS23, Lemma 2.13], this generalizes to

$$\operatorname{Supp}_{\mathcal{S}}(f^*(g_{\mathcal{P}}) \otimes s) = \varphi^{-1}(\{\mathcal{P}\}) \cap \operatorname{Supp}_{\mathcal{S}}(s)$$

for any $s \in \mathcal{S}$. These observations will be used repeatedly in the proofs.

13.9. *Remark.* Note that the unit-counit equation

$$\begin{tikzcd} f^*(t) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & f^*f_*f^*(t) \arrow[r] & f^*(t) \end{tikzcd}$$

implies that a right adjoint f_* is conservative on the essential image of its left adjoint f^*. The other unit-counit equation shows that, similarly, a left adjoint is conservative on the essential image of its right adjoint.

13.10. *Definition.* A *weak ring object* in a symmetric monoidal category is an object w which admits a map $\eta\colon \mathbb{1} \to w$ such that $w \otimes \eta\colon w \to w \otimes w$ is split monic. Similarly, a *weak coring object* is an object c which admits a map $\epsilon\colon c \to \mathbb{1}$ such that $c \otimes \epsilon\colon c \otimes c \to c$ is split epi. Note that these notions are left-right agnostic.

13.11. *Example.* The left idempotents e_Y are weak corings and the right idempotents f_Y are weak rings. Indeed, these examples are idempotent (co)rings. More generally, the left and right idempotents of a (not-necessarily-finite) smashing localization are idempotent (co)rings.

13.12. *Remark.* Let $\mathcal{A}$ be a symmetric monoidal additive category in which the tensor product functors $a \otimes -\colon \mathcal{A} \to \mathcal{A}$ are additive functors. Let $F\colon \mathcal{A} \to \mathcal{B}$ be any additive functor. If F admits a left adjoint $L\colon \mathcal{B} \to \mathcal{A}$ then F is conservative on objects of the form $L(b) \otimes w$ where $b \in \mathcal{B}$ is an arbitrary object and $w \in \mathcal{A}$ is a weak ring object. Indeed, if $F(L(b) \otimes w) = 0$ then

$$\mathcal{A}(L(b), L(b) \otimes w) = \mathcal{B}(b, F(L(b) \otimes w)) = 0.$$

Thus, if $\eta\colon \mathbb{1} \to w$ is a unit for the weak ring w then the map $L(b) \xrightarrow{1\otimes\eta} L(b) \otimes w$ vanishes. This implies $L(b) \otimes w = 0$ since the identity of $L(b) \otimes w$ factors as

$$L(b) \otimes w \xrightarrow{1\otimes\eta\otimes 1} L(b) \otimes w \otimes w \to L(b) \otimes w$$

and so must itself vanish. Similarly, if $F\colon \mathcal{A} \to \mathcal{B}$ admits a right adjoint $R\colon \mathcal{B} \to \mathcal{A}$ then F is conservative on objects of the form $R(b) \otimes c$, where b is arbitrary and c is a weak coring.

13.13. **Theorem.** *Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a geometric functor as in Hypothesis 13.2. Then for any weak ring object $w \in \mathcal{S}$,*

$$\varphi(\mathrm{Supp}_{\mathcal{S}}(w)) = \mathrm{Supp}_{\mathcal{T}}(f_*(w))$$

provided either that

(a) the detection property holds for $\mathcal{S}$; or
(b) the weak ring object w is compact.

Proof. Let $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$ and recall from Remark 13.7 that $\operatorname{Supp}(f^*(g_{\mathcal{P}})) = \varphi^{-1}(\{\mathcal{P}\})$. For any object $w \in \mathcal{S}$ consider the following implications:

$$\begin{aligned}
\mathcal{P} \in \operatorname{Supp}_{\mathcal{T}}(f_*(w)) &\iff g_{\mathcal{P}} \otimes f_*(w) \neq 0 \\
&\iff f_*(f^*(g_{\mathcal{P}}) \otimes w) \neq 0 \\
&\underbrace{\implies}_{(\dagger)} f^*(g_{\mathcal{P}}) \otimes w \neq 0 \\
&\underbrace{\impliedby}_{(\ddagger)} \operatorname{Supp}_{\mathcal{S}}(f^*(g_{\mathcal{P}}) \otimes w) \neq \varnothing \\
&\iff \varphi^{-1}(\{\mathcal{P}\}) \cap \operatorname{Supp}_{\mathcal{S}}(w) \neq \varnothing \\
&\iff \mathcal{P} \in \varphi(\operatorname{Supp}_{\mathcal{S}}(w)).
\end{aligned} \tag{13.14}$$

Note that the converse of the ($\ddagger$) implication holds if $\mathcal{S}$ has the detection property. It also holds if w is compact. Indeed, writing $\{\mathcal{P}\} = Y_1 \cap Y_2^c$ and setting $Y_1' \coloneqq \varphi^{-1}(Y_1)$ and $Y_2' \coloneqq \varphi^{-1}(Y_2)$ observe that for any compact $x \in \mathcal{S}^c$, we have

$$\begin{aligned}
\operatorname{Supp}(f^*(g_{\mathcal{P}}) \otimes x) = \varnothing &\iff \operatorname{supp}(x) \cap Y_1' \subseteq Y_2' \\
&\iff e_{\operatorname{supp}(x) \cap Y_1'} \otimes f_{Y_2'} = 0 \\
&\iff e_{\operatorname{supp}(x)} \otimes e_{Y_1'} \otimes f_{Y_2'} = 0 \\
&\iff \operatorname{Locid}\langle e_{\operatorname{supp}(x)} \rangle \otimes e_{Y_1'} \otimes f_{Y_2'} = 0 \\
&\iff \operatorname{Locid}\langle x \rangle \otimes e_{Y_1'} \otimes f_{Y_2'} = 0 \\
&\iff x \otimes e_{Y_1'} \otimes f_{Y_2'} = 0.
\end{aligned}$$

Here we have used [BHS23, Lemma 1.27] several times. In summary, we have established that $\operatorname{Supp}_{\mathcal{T}}(f_*(w)) \subseteq \varphi(\operatorname{Supp}_{\mathcal{S}}(w))$ provided the detection property holds or w is compact. It remains to prove that the converse of the ($\dagger$) implication holds under our assumption that w is a weak ring. This was explained in Remark 13.12: the right adjoint f_* is conservative on objects of the form $f^*(t) \otimes w$ with w a weak ring. ☐

13.15. **Corollary.** *We always have the equality*

$$\operatorname{im} \varphi = \operatorname{Supp}_{\mathcal{T}}(f_*(\mathbb{1}_{\mathcal{S}})).$$

Proof. Apply Theorem 13.13 to the ring object $w \coloneqq \mathbb{1}_{\mathcal{S}}$ and note that $\operatorname{Supp}_{\mathcal{S}}(\mathbb{1}_{\mathcal{S}}) = \operatorname{Spc}(\mathcal{S}^c)$ under our weakly noetherian assumption, e.g., by [BHS23, Lemma 2.18] or Remark 4.8. ☐

13.16. *Remark.* This result provides an unconditional formula for the image of the map on spectra induced by a geometric functor. It improves on [Bal18, Theorem 1.7] which requires the right adjoint f_* to preserve compactness. An analogous result for the homological spectrum is [Bal20, Theorem 5.12].

13.17. *Example.* If $\mathcal{T} \to \mathcal{S}$ is a smashing localization with idempotent triangle $e \to \mathbb{1}_{\mathcal{T}} \to f \to \Sigma e$ in $\mathcal{T}$, then the image of $\varphi\colon \operatorname{Spc}(\mathcal{S}^c) \to \operatorname{Spc}(\mathcal{T}^c)$ is $\operatorname{Supp}(f)$.

13.18. *Example.* Let $\mathcal{C}$ be a rigidly-compactly generated symmetric monoidal stable ∞-category with associated homotopy category $\operatorname{Ho}(\mathcal{C})$. If $\operatorname{Ho}(\mathcal{C}) \to \operatorname{Ho}(\operatorname{Mod}_{\mathcal{C}}(A))$ is extension-of-scalars with respect to a highly structured commutative algebra $A \in \operatorname{CAlg}(\mathcal{C})$, then the image of $\operatorname{Spc}(\operatorname{Ho}(\operatorname{Mod}_{\mathcal{C}}(A))^c) \to \operatorname{Spc}(\operatorname{Ho}(\mathcal{C})^c)$ is $\operatorname{Supp}(A)$.

13.19. **Corollary.** *If $f^*\colon \mathcal{T} \to \mathcal{S}$ is conservative then $\varphi\colon \operatorname{Spc}(\mathcal{S}^c) \to \operatorname{Spc}(\mathcal{T}^c)$ is surjective.*

Proof. Indeed, $0 \neq g_{\mathcal{P}}$ implies $0 \neq f^*(g_{\mathcal{P}})$ and thus $0 \neq f_*f^*(g_{\mathcal{P}}) \simeq f_*(\mathbb{1}_{\mathcal{S}}) \otimes g_{\mathcal{P}}$ by Remark 13.9. Therefore $\operatorname{Supp}(f_*(\mathbb{1}_{\mathcal{S}})) = \operatorname{Spc}(\mathcal{T}^c)$ and we invoke Corollary 13.15. □

13.20. *Remark.* In contrast, the conservativity of

$$f^*|_{\mathcal{T}^c}\colon \mathcal{T}^c \to \mathcal{S}^c$$

(that is, conservativity of f^* on compact objects) is equivalent to the image of φ containing all the closed points; see [Bal18, Theorem 1.2]. In fact, Corollary 13.19 and [Bal18, Theorem 1.4] establish that if $f^*\colon \mathcal{T} \to \mathcal{S}$ is conservative then it satisfies a nilpotence theorem for morphisms between compact objects (see *loc. cit.* for more details). In fact, [BCH$^+$23a, Theorem 2.25] deduces a stronger nilpotence theorem from the conservativity of f^* in which only the source is assumed to be compact. Combined with [Bal18, Theorem 1.3], this gives another proof of Corollary 13.19; see [BCH$^+$23a, Corollary 2.26].

13.21. **Proposition.** *If $f^!\colon \mathcal{T} \to \mathcal{S}$ is conservative then $f^*\colon \mathcal{T} \to \mathcal{S}$ is conservative.*

Proof. If $f^*(t) = 0$ then $f^!\mathsf{hom}(t,t) \cong \mathsf{hom}(f^*(t), f^!(t)) = 0$ by (13.6). Hence $\mathsf{hom}(t,t) = 0$ so that $t = 0$. □

13.22. *Remark.* We will later prove that the converses to Corollary 13.19 and Proposition 13.21 hold when $\mathcal{T}$ is stratified; see Corollary 14.24. Another situation in which the converses hold will be given in Proposition 13.33 below. As we shall see, these conservativity properties are closely related to the following notion, whose name is motivated by terminology used in [Mat16]:

13.23. *Definition.* We say that a geometric functor $f^*\colon \mathcal{T} \to \mathcal{S}$ is *weakly descendable* if $\mathbb{1}_{\mathcal{T}} \in \operatorname{Locid}\langle f_*(\mathbb{1}_{\mathcal{S}})\rangle$.

13.24. *Remark.* If $\mathbb{1}_{\mathcal{T}} \in \operatorname{Locid}\langle f_*(\mathbb{1}_{\mathcal{S}})\rangle$ then it follows from (2.6) and (2.8) that

$$t \in \operatorname{Locid}\langle f_*f^*t\rangle \quad \text{and} \quad t \in \operatorname{Colocid}\langle f_*f^!t\rangle$$

for all $t \in \mathcal{T}$. Hence f^* and $f^!$ are both conservative.

13.25. *Remark.* For any compact object $x \in \mathcal{S}^c$, Theorem 13.13 implies that

$$\varphi(\operatorname{supp}(x)) = \operatorname{Supp}_{\mathcal{T}}(f_*(x \otimes x^\vee)).$$

Indeed just recall that $x \otimes x^\vee \cong \mathsf{hom}(x,x)$ is the endomorphism ring object and that $\operatorname{supp}(x) = \operatorname{supp}(x \otimes x^\vee)$. For an arbitrary object $s \in \mathcal{S}$, we can also consider the endomorphism ring object $\mathsf{hom}(s,s)$. The theorem provides

$$\varphi(\operatorname{Supp}_{\mathcal{S}}(\mathsf{hom}(s,s))) = \operatorname{Supp}_{\mathcal{T}}(f_*\mathsf{hom}(s,s))$$

assuming $\mathcal{S}$ satisfies the detection property.

13.26. *Remark.* Note that if $f_*(\mathbb{1}_{\mathcal{S}})$ is compact then Corollary 13.15 implies that the image $\operatorname{im}\varphi = \operatorname{Supp}(f_*(\mathbb{1}_{\mathcal{S}})) = \operatorname{supp}(f_*(\mathbb{1}_{\mathcal{S}}))$ is closed. Moreover, if f_* preserves all compact objects (not just $\mathbb{1}_{\mathcal{S}}$) then Remark 13.25 shows that φ maps every Thomason closed subset to a Thomason closed subset. It follows that φ is a closed map (see, e.g., [DST19, Theorem 5.3.3]). This leads to the following terminology:

13.27. *Definition.* We will say that $f^*\colon \mathcal{T} \to \mathcal{S}$ is *weakly closed* if $f_*(\mathbb{1}_{\mathcal{S}})$ is compact and *strongly closed* if f_* preserves compact objects. Note that if $\mathcal{S}$ is monogenic then a weakly closed morphism is automatically strongly closed.

13.28. *Remark.* The above terminology is given more for grammatical and linguistic convenience rather than deep mathematical significance. We do not claim that it captures the correct geometric notion of "closed morphism" in tensor triangular geometry. For example, a closed immersion $\mathrm{Spec}(R/I) \hookrightarrow \mathrm{Spec}(R)$ of affine schemes induces a geometric functor $\mathrm{D}(R) \to \mathrm{D}(R/I)$ which is often not (weakly or strongly) closed in the sense of Definition 13.27; see [San19, Example 7.8].

13.29. *Remark.* In the terminology of [BDS16], a geometric functor f^* is strongly closed if and only if it satisfies *Grothendieck–Neeman duality* (or GN-duality, for short). As explained in [BDS16], a morphism f^* satisfies GN-duality if and only if it preserves products if and only if it has a left adjoint if and only if the trio of functors $f^* \dashv f_* \dashv f^!$ extends on both sides to a sequence of five adjoints:

$$f_! \dashv f^* \dashv f_* \dashv f^! \dashv f_{(-1)}. \tag{13.30}$$

There is then an analogue of Theorem 13.13 for the left adjoint $f_!$:

13.31. **Theorem.** *Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a geometric functor as in Hypothesis 13.2. If f^* is strongly closed then for any weak coring object $w \in \mathcal{S}$,*

$$\varphi(\mathrm{Supp}_{\mathcal{S}}(w)) = \mathrm{Supp}_{\mathcal{T}}(f_!(w))$$

provided either that

(a) the detection property holds for $\mathcal{S}$; or
(b) the weak coring object is compact.

Proof. One proceeds through the argument as in the proof of Theorem 13.13 replacing f_* with $f_!$. Note that the left projection formula holds by [BDS16, (3.11)]. To obtain the right-to-left implication of the analogue of (†), namely that $f^*(g_{\mathcal{P}}) \otimes w \neq 0$ implies $f_!(f^*(g_{\mathcal{P}}) \otimes w) \neq 0$ one again uses the fact explained in Remark 13.12 that the left adjoint $f_!$ is conservative on objects of the form $f^*(t) \otimes w$ for w a weak coring. ☐

13.32. *Remark.* According to the ur-Wirthmüller isomorphism of [BDS16, (3.10)], $f_!(s) \simeq f_*(s \otimes \omega_f)$ for any $s \in \mathcal{S}$. In particular, applied to the weak coring $s := \mathbb{1}_{\mathcal{S}}$, Theorem 13.31 implies that $\mathrm{im}\,\varphi = \mathrm{Supp}(f_*(\omega_f))$. Thus, $f_*(\omega_f)$ and $f_*(\mathbb{1}_{\mathcal{S}})$ have the same support if f^* is strongly closed. These objects can be very different for a general geometric functor f^*. For example, if f^* is a smashing localization as in Example 13.17 then $f_*(\mathbb{1}_{\mathcal{S}}) = f$ while $f_*(\omega_f) = \mathsf{hom}(f, \mathbb{1}_{\mathcal{T}})$. Of course, it is rare for a smashing localization to be strongly closed. On the other hand, according to [San22] a necessary condition for f^* to be finite étale is that $f_*(\mathbb{1}_{\mathcal{S}}) \simeq f_*(\omega_f)$. We have shown that these two objects have the same support for any strongly closed functor.

13.33. **Proposition.** *Suppose $f^*\colon \mathcal{T} \to \mathcal{S}$ is weakly closed (Definition 13.27). The following are equivalent:*

(a) f^* is weakly descendable (Definition 13.23);
(b) $f^!$ is conservative;
(c) f^* is conservative;
(d) $\varphi\colon \mathrm{Spc}(\mathcal{S}^c) \to \mathrm{Spc}(\mathcal{T}^c)$ is surjective.

Proof. The implication $(a) \Rightarrow (b)$ follows from Remark 13.24, $(b) \Rightarrow (c)$ is Proposition 13.21 and $(c) \Rightarrow (d)$ is Corollary 13.19. For $(d) \Rightarrow (a)$ note that the surjectivity of φ implies that $\mathrm{Supp}_{\mathcal{T}}(f_*(\mathbb{1}_{\mathcal{S}})) = \mathrm{Spc}(\mathcal{T}^c)$ by Corollary 13.15 and $f_*(\mathbb{1}_{\mathcal{S}})$ is compact by assumption. Thus, by the classification of thick ideals of $\mathcal{T}^c$, $\mathbb{1}_{\mathcal{T}} \in \mathrm{thickid}\langle f_*(\mathbb{1}_{\mathcal{S}})\rangle \subseteq \mathrm{Locid}\langle f_*(\mathbb{1}_{\mathcal{S}})\rangle$. ☐

14. Base change for support and cosupport

We now turn to base change formulas that are valid for arbitrary objects. The cost will be some additional hypotheses on our functors or on our categories.

14.1. *Hypothesis.* We continue to let $f^* \colon \mathcal{T} \to \mathcal{S}$ denote an arbitrary geometric functor and assume both $\mathrm{Spc}(\mathcal{T}^c)$ and $\mathrm{Spc}(\mathcal{S}^c)$ are weakly noetherian. We write

$$\varphi\colon\ \mathrm{Spc}(\mathcal{S}^c) \to \mathrm{Spc}(\mathcal{T}^c)$$

for the induced map on spectra.

14.2. **Proposition.** *Suppose $f^* \colon \mathcal{T} \to \mathcal{S}$ is as above (Hypothesis 14.1) and let $s \in \mathcal{S}$.*

(a) If $\mathcal{S}$ has the detection property, then

$$\mathrm{Supp}_{\mathcal{T}}(f_*(s)) \subseteq \varphi(\mathrm{Supp}_{\mathcal{S}}(s))$$

with equality when the functor f_* is conservative.

(b) If $\mathcal{S}$ has the codetection property, then

$$\mathrm{Cosupp}_{\mathcal{T}}(f_*(s)) \subseteq \varphi(\mathrm{Cosupp}_{\mathcal{S}}(s))$$

with equality when the functor f_* is conservative.

Proof. The set-up is the same as the proof of Theorem 13.13. Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$. For part (a) consider the implications (13.14) displayed in the proof of Theorem 13.13. Note that the left-to-right direction of (‡) holds if $\mathcal{S}$ has the detection property. Hence, going from left-to-right we obtain the inclusion in part (a). On the other hand, the right-to-left direction of (†) holds if f_* is conservative.

For part (b), consider the analogous series of implications:

$$\begin{aligned} \mathcal{P} \in \mathrm{Cosupp}_{\mathcal{T}}(f_*(s)) &\iff \mathsf{hom}(g_{\mathcal{P}}, f_*(s)) \neq 0 \\ &\iff f_*\mathsf{hom}(f^*(g_{\mathcal{P}}), s) \neq 0 \\ &\underbrace{\Longrightarrow}_{(\dagger)} \mathsf{hom}(f^*(g_{\mathcal{P}}), s) \neq 0 \\ &\underbrace{\Longleftarrow}_{(\ddagger)} \mathrm{Cosupp}_{\mathcal{S}}(\mathsf{hom}(f^*(g_{\mathcal{P}}), s)) \neq \varnothing \\ &\iff \varphi^{-1}(\{\mathcal{P}\}) \cap \mathrm{Cosupp}_{\mathcal{S}}(s) \neq \varnothing \\ &\iff \mathcal{P} \in \varphi(\mathrm{Cosupp}_{\mathcal{S}}(s)). \end{aligned} \tag{14.3}$$

Again, the converse of (‡) holds when $\mathcal{S}$ has the codetection property, and the converse of (†) holds if f_* is conservative. ☐

14.4. *Remark.* If the right adjoint f_* is not conservative then there exist objects for which the inequalities of Proposition 14.2 are strict.

14.5. *Remark.* It is a well-known exercise (recall Remark 9.12) that the right adjoint f_* is conservative if and only if f^* sends a set of compact generators of $\mathcal{T}$ to a set of (compact) generators of $\mathcal{S}$, in other words, if and only if $\mathcal{S}^c = \mathrm{thick}\langle f^*(\mathcal{T}^c)\rangle$.

For example, if $\mathcal{S}$ is monogenic then every geometric functor $f^*\colon \mathcal{T} \to \mathcal{S}$ has a conservative right adjoint.

14.6. *Remark.* The right adjoint f_* of a geometric functor is always "weakly conservative" in the following sense: For any nonzero $s \in \mathcal{S}$ there always exists a compact object $c \in \mathcal{S}^c$ such that $f_*(c \otimes s) \neq 0$. Indeed, since $\mathcal{S}$ is compactly generated, if $s \neq 0$ then $\mathcal{S}(c, s) \neq 0$ for some compact $c \in \mathcal{S}^c$. Replacing c by its dual, we can assert that there exists a compact $c \in \mathcal{S}^c$ such that $\mathcal{S}(\mathbb{1}_\mathcal{S}, c \otimes s) \neq 0$. Since $\mathbb{1}_\mathcal{S} = f^*(\mathbb{1}_\mathcal{T})$, adjunction implies $\mathcal{T}(\mathbb{1}_\mathcal{T}, f_*(c \otimes s)) \neq 0$ hence $f_*(c \otimes s) \neq 0$. One way of appreciating this phenomenon is to recognize that while the kernel of f_* is a localizing subcategory, it need not be an ideal. Indeed, the above argument shows that the largest localizing ideal contained in it, namely

$$\left\{ s \in \mathcal{S} \,\middle|\, f_*(c \otimes s) = 0 \ \forall c \in \mathcal{S}^c \right\}$$

is always the zero ideal. In other words, the right adjoint f_* of a geometric functor is conservative precisely when the kernel of f_* is an ideal.

14.7. **Proposition.** *Suppose $f^*\colon \mathcal{T} \to \mathcal{S}$ is as above (Hypothesis 14.1) and let $t \in \mathcal{T}$.*

(a) *We always have inclusions*

$$\varphi(\mathrm{Supp}_\mathcal{S}(f^*(t))) \subseteq \mathrm{Supp}_\mathcal{T}(t \otimes f_*(\mathbb{1}_\mathcal{S})) \subseteq \mathrm{Supp}_\mathcal{T}(t) \cap \mathrm{im}\,\varphi.$$

(b) *If $\mathcal{S}$ has the detection property, then*

$$\varphi(\mathrm{Supp}_\mathcal{S}(f^*(t))) = \mathrm{Supp}_\mathcal{T}(t \otimes f_*(\mathbb{1}_\mathcal{S})).$$

(c) *If $\mathcal{S}$ has the detection property and $\mathcal{T}$ is stratified, then*

$$\varphi(\mathrm{Supp}_\mathcal{S}(f^*(t))) = \mathrm{Supp}_\mathcal{T}(t) \cap \mathrm{im}\,\varphi.$$

(d) *If f^* is conservative, then*

$$\mathrm{Supp}_\mathcal{T}(t \otimes f_*(\mathbb{1}_\mathcal{S})) = \mathrm{Supp}_\mathcal{T}(t),$$

hence

$$\varphi(\mathrm{Supp}_\mathcal{S}(f^*(t)) = \mathrm{Supp}_\mathcal{T}(t)$$

if $\mathcal{S}$ also has the detection property.

Proof. Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$ and write $\{\mathcal{P}\} = Y_1 \cap Y_2^c$ with Y_1 and Y_2 Thomason. Recall Remark 13.7. For any object $t \in \mathcal{T}$ consider the following implications:

$$\begin{aligned}
\mathcal{P} \in \varphi(\mathrm{Supp}_\mathcal{S}(f^*(t))) &\iff \varphi^{-1}(\{\mathcal{P}\}) \cap \mathrm{Supp}_\mathcal{S}(f^*(t)) \neq \varnothing \\
&\iff \mathrm{Supp}_\mathcal{S}(f^*(g_\mathcal{P}) \otimes f^*(t)) \neq \varnothing \\
&\underbrace{\implies}_{(\dagger)} f^*(g_\mathcal{P}) \otimes f^*(t) \neq 0 \\
&\iff f^*(g_\mathcal{P} \otimes t) \neq 0 \\
&\underbrace{\impliedby}_{(\ddagger)} f_*(f^*(g_\mathcal{P} \otimes t)) \neq 0 \\
&\iff g_\mathcal{P} \otimes t \otimes f_*(\mathbb{1}_\mathcal{S}) \neq 0 \\
&\iff \mathcal{P} \in \mathrm{Supp}_\mathcal{T}(t \otimes f_*(\mathbb{1}_\mathcal{S})).
\end{aligned} \tag{14.8}$$

Note that the converse of (‡) holds by Remark 13.9. Hence, going from left to right establishes the first inclusion in part (a), while the second inclusion in part (a) is due to Corollary 13.15 and [BHS23, Remark 2.12(e)]. This establishes (a). Now the converse of (†) holds when we have the detection property. Hence going from right to left gives (b). Part (c) follows from part (b) by invoking the tensor product formula ([BHS23, Theorem 8.2]), which holds because $\mathcal{T}$ is stratified, and Corollary 13.15: $\operatorname{Supp}_{\mathcal{T}}(t \otimes f_*(\mathbb{1}_{\mathcal{S}})) = \operatorname{Supp}_{\mathcal{T}}(t) \cap \operatorname{Supp}_{\mathcal{T}}(f_*(\mathbb{1}_{\mathcal{S}})) = \operatorname{Supp}_{\mathcal{T}}(t) \cap \operatorname{im}\varphi$. For part ($d$) it suffices to show that $\operatorname{Supp}_{\mathcal{T}}(t) \subseteq \operatorname{Supp}_{\mathcal{T}}(t \otimes f_*(\mathbb{1}_{\mathcal{S}}))$ when f^* is conservative. Indeed if f^* is conservative then so is the composite f_*f^* (Remark 13.9), hence $0 \neq g_{\mathcal{P}} \otimes t$ implies $0 \neq f_*f^*(g_{\mathcal{P}} \otimes t) \simeq g_{\mathcal{P}} \otimes t \otimes f_*(\mathbb{1}_{\mathcal{S}})$. □

14.9. *Remark.* If $\mathcal{S}$ has the detection property then the equality

$$\varphi(\operatorname{Supp}_{\mathcal{S}}(f^*(t))) = \operatorname{Supp}_{\mathcal{T}}(t) \cap \operatorname{im}\varphi$$

also holds (without further hypotheses on f^* or $\mathcal{S}$) if the object t is compact, or a left idempotent e_Y, or a right idempotent f_Y, or an object $g_{\mathcal{P}}$. Indeed, in these cases the inclusion $\operatorname{Supp}_{\mathcal{T}}(t \otimes f_*(\mathbb{1}_{\mathcal{S}})) \subseteq \operatorname{Supp}_{\mathcal{T}}(t) \cap \operatorname{Supp}_{\mathcal{T}}(f_*(\mathbb{1}_{\mathcal{S}}))$ is an equality by [BHS23, Lemma 2.13 and Lemma 2.18].

14.10. **Proposition.** *Suppose $f^*\colon \mathcal{T} \to \mathcal{S}$ is as above (Hypothesis 14.1) and let $t \in \mathcal{T}$.*

(a) We always have inclusions

$$\varphi(\operatorname{Cosupp}_{\mathcal{S}}(f^!(t))) \subseteq \operatorname{Cosupp}_{\mathcal{T}}(\mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t)) \subseteq \operatorname{Cosupp}_{\mathcal{T}}(t) \cap \operatorname{im}\varphi.$$

(b) If $\mathcal{S}$ has the codetection property, then

$$\varphi(\operatorname{Cosupp}_{\mathcal{S}}(f^!(t))) = \operatorname{Cosupp}_{\mathcal{T}}(\mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t)).$$

(c) If $\mathcal{S}$ has the codetection property and $\mathcal{T}$ is stratified, then

$$\varphi(\operatorname{Cosupp}_{\mathcal{S}}(f^!(t))) = \operatorname{Cosupp}_{\mathcal{T}}(t) \cap \operatorname{im}\varphi.$$

(d) If $f^!$ is conservative, then

$$\operatorname{Cosupp}_{\mathcal{T}}(\mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t)) = \operatorname{Cosupp}_{\mathcal{T}}(t),$$

hence

$$\varphi(\operatorname{Cosupp}_{\mathcal{S}}(f^!(t)) = \operatorname{Cosupp}_{\mathcal{T}}(t)$$

if $\mathcal{S}$ also has the codetection property.

Proof. The proof is similar to the proof of Proposition 14.7. Let $\mathcal{P} \in \operatorname{Spc}(\mathcal{T}^c)$ and write $\{\mathcal{P}\} = Y_1 \cap Y_2^c$ with Y_1 and Y_2 Thomason. For any object $t \in \mathcal{T}$ consider the following implications analogous to those of (13.14):

$$\begin{aligned}
\mathcal{P} \in \varphi(\operatorname{Cosupp}_{\mathcal{S}}(f^!(t))) &\iff \varphi^{-1}(\{\mathcal{P}\}) \cap \operatorname{Cosupp}_{\mathcal{S}}(f^!(t)) \neq \varnothing \\
&\iff \operatorname{Cosupp}_{\mathcal{S}}(\mathsf{hom}(f^*(g_{\mathcal{P}}), f^!(t))) \neq \varnothing \\
&\underbrace{\Longrightarrow}_{(\dagger)} \mathsf{hom}(f^*(g_{\mathcal{P}}), f^!(t)) \neq 0 \\
&\underbrace{\Longleftarrow}_{(\ddagger)} f_*\mathsf{hom}(f^*(g_{\mathcal{P}}), f^!(t)) \neq 0 \\
&\iff \mathsf{hom}(g_{\mathcal{P}}, f_*f^!(t))) \neq 0 \\
&\iff \mathsf{hom}(g_{\mathcal{P}}, \mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t)) \neq 0 \\
&\iff \mathcal{P} \in \operatorname{Cosupp}_{\mathcal{T}}(\mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t)).
\end{aligned} \tag{14.11}$$

Here we have invoked the adjunction isomorphisms $f_*\mathsf{hom}(f^*(a), b) \cong \mathsf{hom}(a, f_*(b))$ and $f_*f^!(t) \cong f_*\mathsf{hom}(\mathbb{1}_{\mathcal{S}}, f^!(t)) \cong \mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t)$ established in [BDS16]. Moreover, the isomorphism $f^!\mathsf{hom}(a, b) \cong \mathsf{hom}(f^*(a), f^!(b))$ from (13.6) shows that $\mathsf{hom}(f^*(g_{\mathcal{P}}), f^!(t))$ is in the essential image of $f^!$ and hence the converse of ($\ddagger$)

holds by Remark 13.9. Hence, going from left to right we obtain the first inclusion in part (a), while the second inclusion is a standard property of cosupport (4.30) together with Corollary 13.15. This establishes (a). Now the converse of (†) holds when we have codetection property. Hence going from right to left gives (b). Part (c) follows from part (b) by invoking the hom formula (Theorem 7.15), which uses that $\mathcal{T}$ is stratified, and Corollary 13.15. For part (d) it suffices to show that $\mathrm{Cosupp}_{\mathcal{T}}(t) \subseteq \mathrm{Cosupp}_{\mathcal{T}}(\mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t))$ when $f^!$ is conservative. Indeed, if $\mathsf{hom}(g_{\mathcal{P}}, t) \neq 0$ then $f_* f^! \mathsf{hom}(g_{\mathcal{P}}, t) \neq 0$ by Remark 13.9. That is, $0 \neq f_* f^! \mathsf{hom}(g_{\mathcal{P}}, t) \cong f_* \mathsf{hom}(f^*(g_{\mathcal{P}}), f^!(t)) \cong \mathsf{hom}(g_{\mathcal{P}}, f_* f^!(t))$ and recall $f_* f^!(t) \cong \mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t)$. ☐

14.12. **Corollary.** *Suppose f^* is weakly closed (Definition 13.27). Then:*

(a) If $\mathcal{S}$ has the detection property then

$$\varphi(\mathrm{Supp}_{\mathcal{S}}(f^*(t)) = \mathrm{Supp}_{\mathcal{T}}(t) \cap \operatorname{im} \varphi$$

for every $t \in \mathcal{T}$.

(b) If $\mathcal{S}$ has the codetection property then

$$\varphi(\mathrm{Cosupp}_{\mathcal{S}}(f^!(t)) = \mathrm{Cosupp}_{\mathcal{T}}(t) \cap \operatorname{im} \varphi$$

for every $t \in \mathcal{T}$.

Proof. This follows from parts (a) and (b) of Proposition 14.7 and Proposition 14.10 together with the half-tensor and half-hom theorems ([BHS23, Lemma 2.18] and Proposition 4.35). ☐

14.13. *Remark.* It follows from Proposition 14.7(a) and Proposition 14.10(a) that the following inclusions hold for any $t \in \mathcal{T}$:

$$\mathrm{Supp}_{\mathcal{S}}(f^*(t)) \subseteq \varphi^{-1}(\mathrm{Supp}_{\mathcal{T}}(t)), \text{ and}$$
$$\mathrm{Cosupp}_{\mathcal{S}}(f^!(t)) \subseteq \varphi^{-1}(\mathrm{Cosupp}_{\mathcal{T}}(t)).$$

Our next goal is to establish that these are equalities when $\mathcal{T}$ is stratified; see Corollary 14.19 below.

14.14. **Lemma.** *Let $\mathcal{Q} \in \mathrm{Spc}(\mathcal{S}^c)$. For any $s \in \mathcal{S}$ we have*

$$\mathrm{Supp}_{\mathcal{T}}(f_*(s \otimes g_{\mathcal{Q}})) \subseteq \{\varphi(\mathcal{Q})\}.$$

Proof. Suppose $\mathcal{P} \in \mathrm{Supp}_{\mathcal{T}}(f_*(s \otimes g_{\mathcal{Q}}))$. Then

$$f_*(f^*(g_{\mathcal{P}}) \otimes s \otimes g_{\mathcal{Q}}) \simeq g_{\mathcal{P}} \otimes f_*(s \otimes g_{\mathcal{Q}}) \neq 0$$

so that $f^*(g_{\mathcal{P}}) \otimes g_{\mathcal{Q}} \neq 0$. That is, $\mathcal{Q} \in \mathrm{Supp}_{\mathcal{S}}(f^*(g_{\mathcal{P}})) = \varphi^{-1}(\{\mathcal{P}\})$ by Remark 13.7. Thus $\mathcal{P} = \varphi(\mathcal{Q})$. ☐

14.15. **Theorem** (Local Avrunin–Scott identities)**.** *Let $f^* \colon \mathcal{T} \to \mathcal{S}$ be a geometric functor as in Hypothesis 14.1 and let $t \in \mathcal{T}$. Suppose the localizing ideals $\Gamma_{\mathcal{P}}\mathcal{T}$ are minimal for all $\mathcal{P} \in \mathrm{Supp}_{\mathcal{T}}(t)$. Then we have:*

$$\mathrm{Supp}_{\mathcal{S}}(f^*(t)) = \varphi^{-1}(\mathrm{Supp}_{\mathcal{T}}(t)); \text{ and} \tag{14.16}$$

$$\mathrm{Cosupp}_{\mathcal{S}}(f^!(t)) = \varphi^{-1}(\mathrm{Cosupp}_{\mathcal{T}}(t)). \tag{14.17}$$

Proof. The $\subseteq$ inclusions always hold (Remark 14.13). To establish the $\supseteq$ inclusion in (14.16), suppose $\mathcal{Q} \in \varphi^{-1}(\mathrm{Supp}_{\mathcal{T}}(t))$ so that $\varphi(\mathcal{Q}) \in \mathrm{Supp}_{\mathcal{T}}(t)$. Since $\Gamma_{\varphi(\mathcal{Q})}\mathcal{T}$ is minimal, it follows that $g_{\varphi(\mathcal{Q})} \in \mathrm{Locid}\langle t\rangle$. Hence $f^*(g_{\varphi(\mathcal{Q})}) \in \mathrm{Locid}\langle f^*(t)\rangle$. Therefore

$$\mathcal{Q} \in \varphi^{-1}(\{\varphi(\mathcal{Q})\}) = \mathrm{Supp}_{\mathcal{S}}(f^*(g_{\varphi(\mathcal{Q})})) \subseteq \mathrm{Supp}_{\mathcal{S}}(f^*(t)),$$

where the first equality is because of Remark 13.7. It remains to establish the $\supseteq$ inclusion in (14.17). Suppose $\mathcal{Q} \notin \mathrm{Cosupp}_{\mathcal{S}}(f^!(t))$. We will show that $\mathcal{Q} \notin \varphi^{-1}(\mathrm{Cosupp}_{\mathcal{T}}(t))$. By Remark 14.6 and Lemma 14.14, there exists a compact $c \in \mathcal{S}^c$ such that $\mathrm{Supp}_{\mathcal{T}}(f_*(c \otimes g_{\mathcal{Q}})) = \{\varphi(\mathcal{Q})\}$. Since $\Gamma_{\varphi(\mathcal{Q})}\mathcal{T}$ is minimal, we conclude that

$$g_{\varphi(\mathcal{Q})} \in \mathrm{Locid}\langle f_*(c \otimes g_{\mathcal{Q}})\rangle. \tag{14.18}$$

Now, since $\mathcal{Q} \notin \mathrm{Cosupp}_{\mathcal{S}}(f^!(t))$ by hypothesis, we have $\mathsf{hom}(g_{\mathcal{Q}}, f^!(t)) = 0$. Hence $\mathsf{hom}(c \otimes g_{\mathcal{Q}}, f^!(t)) = 0$ and so

$$\mathsf{hom}(f_*(c \otimes g_{\mathcal{Q}}), t) \simeq f_*\mathsf{hom}(c \otimes g_{\mathcal{Q}}, f^!(t)) = 0.$$

Therefore, $\mathsf{hom}(g_{\varphi(\mathcal{Q})}, t) = 0$ by (14.18). That is, $\varphi(\mathcal{Q}) \notin \mathrm{Cosupp}_{\mathcal{T}}(t)$. $\square$

14.19. **Corollary** (Global Avrunin–Scott identities). *If $\mathcal{T}$ is stratified, then for any $t \in \mathcal{T}$ we have:*

$$\mathrm{Supp}_{\mathcal{S}}(f^*(t)) = \varphi^{-1}(\mathrm{Supp}_{\mathcal{T}}(t)); \text{ and} \tag{14.20}$$

$$\mathrm{Cosupp}_{\mathcal{S}}(f^!(t)) = \varphi^{-1}(\mathrm{Cosupp}_{\mathcal{T}}(t)). \tag{14.21}$$

Proof. Indeed, [BHS23, Theorem 4.1] shows that stratification of $\mathcal{T}$ gives minimality at every point of its spectrum, so the result follows from Theorem 14.15. $\square$

14.22. *Remark.* In a tensor triangular setting, the Avrunin–Scott identities for support and cosupport were studied (under more restrictive hypotheses) in [BCHV19, Proposition 3.14]. They originate in the work of Avrunin–Scott [AS82] on support varieties for representations of finite groups.

14.23. *Remark.* If we assume that the right adjoint f_* is conservative, then we can prove the Avrunin–Scott identities of Corollary 14.19 under slightly weaker hypotheses on the category $\mathcal{T}$ by modifying the proof of [BCHV19, Proposition 3.14]; namely, the identity for f^* holds if $\mathcal{T}$ has the tensor product formula, while the identity for $f^!$ holds if $\mathcal{T}$ has the Hom formula. Recall that the former is implied by stratification ([BHS23, Theorem 8.2]) and the latter is equivalent to stratification (Theorem 7.15).

14.24. **Corollary.** *Let $\mathcal{T}$ be stratified. For $f^*\colon \mathcal{T} \to \mathcal{S}$ as in Hypothesis 14.1, the following conditions are equivalent:*

(a) f^* is weakly descendable (Definition 13.23);
(b) $f^!$ is conservative;
(c) f^* is conservative;
(d) $\varphi\colon \mathrm{Spc}(\mathcal{S}^c) \to \mathrm{Spc}(\mathcal{T}^c)$ is surjective.

Proof. The implication $(a) \Rightarrow (b)$ follows from Remark 13.24, $(b) \Rightarrow (c)$ is Proposition 13.21 and $(c) \Rightarrow (d)$ is Corollary 13.19. For $(d) \Rightarrow (a)$ note that the surjectivity of φ implies that $\mathrm{Supp}_{\mathcal{T}}(f_*(\mathbb{1}_{\mathcal{S}})) = \mathrm{Spc}(\mathcal{T}^c)$ by Corollary 13.15. If $\mathcal{T}$ is stratified, this implies $\mathbb{1}_{\mathcal{T}} \in \mathrm{Locid}\langle f_*(\mathbb{1}_{\mathcal{S}})\rangle$. $\square$

14.25. *Example.* Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring whose maximal ideal $\mathfrak{m}$ is nilpotent (that is, R is a commutative artinian local ring). Corollary 14.24 implies that the functor $\mathrm{D}(R) \to \mathrm{D}(k)$ is conservative. Hence, if M is an arbitrary flat R-module, we conclude that $M/\mathfrak{m}M = 0$ implies $M = 0$. This is a well-known variant of Nakayama's Lemma which replaces finite generation of the module M with flatness together with nilpotence of the ideal $\mathfrak{m}$; cf. [Mat89, Theorem 7.10].

14.26. *Example.* Let A be the ring denoted A in [Kel94]. It is a non-discrete valuation domain of rank 1 whose value group is $\mathbb{Z}[1/\ell] \subset \mathbb{Q}$; see [FS01, Theorem II.3.8]. The quotient $R := A/xA$ by any nontrivial principal ideal is then a non-noetherian local ring whose spectrum is a single point. The map to the residue field $f : R \to k$ induces a functor $f^* : \mathrm{D}(R) \to \mathrm{D}(k)$ which is surjective on spectra. However, the maximal ideal of R satisfies $\mathfrak{m}^2 = \mathfrak{m}$ and is flat as an R-module. Hence $M := \mathfrak{m}$ is a nonzero R-module which is annihilated by f^*. We conclude that $\mathrm{D}(R)$ cannot be stratified, since this would contradict Corollary 14.24. Alternatively, it follows from the work of Bazzoni–Šťovíček (cf. [BŠ17, Example 5.24]) that $\mathrm{D}(R) \to \mathrm{D}(k)$ is a smashing localization which has no finite acyclics. Hence $\mathrm{D}(R)$ does not satisfy the telescope conjecture and we can alternatively conclude that it is not stratified by invoking [BHS23, Theorem 9.11].

14.27. *Example.* Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring and let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. The induced functor $\mathrm{D}(R) \to \mathrm{D}(\widehat{R})$ is conservative. Indeed, $R \to \widehat{R}$ is faithfully flat, so $\mathrm{Spec}(\widehat{R}) \to \mathrm{Spec}(R)$ is surjective.

15. Descending the local-to-global principle

We now provide a version of descent for the local-to-global principle.

15.1. **Proposition.** *Let $f^* : \mathcal{T} \to \mathcal{S}$ be as above (Hypothesis 14.1). Assume that $\mathcal{S}$ satisfies the local-to-global principle. Then*

$$t \in \mathrm{Locid}\langle t \otimes g_{\mathcal{P}} \mid \mathcal{P} \in \mathrm{Supp}_{\mathcal{T}}(t)\rangle$$

for any object $t \in \mathcal{T}$ such that $t \in \mathrm{Locid}\langle f_* f^* t\rangle$.

Proof. By the local-to-global principle in $\mathcal{S}$, we have

$$\begin{aligned}\mathrm{Locid}\langle f^* g_{\mathcal{P}}\rangle &= \mathrm{Locid}\langle f^* g_{\mathcal{P}} \otimes g_{\mathcal{Q}} \mid \mathcal{Q} \in \varphi^{-1}(\{\mathcal{P}\})\rangle \\ &= \mathrm{Locid}\langle g_{\varphi^{-1}(\{\mathcal{P}\})} \otimes g_{\mathcal{Q}} \mid \mathcal{Q} \in \varphi^{-1}(\{\mathcal{P}\})\rangle \\ &= \mathrm{Locid}\langle g_{\mathcal{Q}} \mid \mathcal{Q} \in \varphi^{-1}(\{\mathcal{P}\})\rangle\end{aligned}$$

for any $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$, where the last equality uses Remark 4.8. Hence

$$\mathrm{Locid}\langle f^* t \otimes f^* g_{\mathcal{P}}\rangle = \mathrm{Locid}\langle f^* t \otimes g_{\mathcal{Q}} \mid \mathcal{Q} \in \varphi^{-1}(\{\mathcal{P}\})\rangle \tag{15.2}$$

by [BHS23, Lemma 3.6], for example. Then, by the local-to-global principle for $f^* t$, we have

$$\begin{aligned}f^* t &\in \mathrm{Locid}\langle f^* t \otimes g_{\mathcal{Q}} \mid \mathcal{Q} \in \mathrm{Supp}_{\mathcal{S}}(f^* t)\rangle \\ &\subseteq \mathrm{Locid}\langle f^* t \otimes g_{\mathcal{Q}} \mid \mathcal{Q} \in \varphi^{-1}(\mathrm{Supp}_{\mathcal{T}}(t))\rangle && \text{(Remark 14.13)} \\ &= \mathrm{Locid}\langle f^* t \otimes f^* g_{\mathcal{P}} \mid \mathcal{P} \in \mathrm{Supp}_{\mathcal{T}}(t)\rangle && \text{(by (15.2))}.\end{aligned}$$

Using (13.4), we obtain

$$f_* f^* t \in \mathrm{Locid}\langle t \otimes g_{\mathcal{P}} \mid \mathcal{P} \in \mathrm{Supp}_{\mathcal{T}}(t)\rangle$$

which establishes the claim. □

15.3. *Remark.* It follows from Corollary 13.15 that objects satisfying $t \in \operatorname{Locid}\langle f_* f^* t\rangle$ have their support contained in the image of φ. For some functors, the converse holds: If $\operatorname{Supp}_{\mathcal{T}}(t) \subseteq \operatorname{im}\varphi$ then $t \in \operatorname{Locid}\langle f_* f^* t\rangle$. This is the case for the following two examples:

(a) A finite localization $f^*\colon \mathcal{T} \to \mathcal{T}(U)$ under the assumption that $\mathcal{T}$ has the detection property. In this case, $\operatorname{im}\varphi = U$ and $f_* f^* t \cong t$ for all t which are supported in U.
(b) A weakly closed functor $f^*\colon \mathcal{T} \to \mathcal{S}$ under the assumption that $\mathcal{T}$ has the detection property. This follows from [BHS23, Lemma 3.7].

Intuitively we can think of Proposition 15.1 as saying that the local-to-global principle partially descends along f^* to objects supported in the image of φ. However, note that if $f^*\colon \mathcal{T} \to \mathcal{S}$ is a functor which is not conservative and yet for which φ is surjective (such as the functor described in Example 14.26) then there exists an object $t \in \mathcal{T}$ with $\operatorname{Supp}_{\mathcal{T}}(t) \subseteq \operatorname{im}\varphi$ and yet with $t \notin \operatorname{Locid}\langle f_* f^* t\rangle$, so the above intuition is not completely accurate.

15.4. *Remark.* It follows from Remark 14.13 that the following hold:

(a) If $f^*\colon \mathcal{T} \to \mathcal{S}$ is conservative then the detection property descends from $\mathcal{S}$ to $\mathcal{T}$.
(b) If $f^!\colon \mathcal{T} \to \mathcal{S}$ is conservative then the codetection property descends from $\mathcal{S}$ to $\mathcal{T}$.

Since the codetection property is equivalent to the (co)local-to-global principle (by Theorem 6.4) we see that the local-to-global principle descends from $\mathcal{S}$ to $\mathcal{T}$ whenever $f^!$ is conservative.

15.5. *Example.* If $f^*\colon \mathcal{T} \to \mathcal{S}$ is fully faithful then the local-to-global principle descends from $\mathcal{S}$ to $\mathcal{T}$. This follows from Proposition 15.1 since $t \cong f_* f^* t$ for all $t \in \mathcal{T}$. It also follows from Remark 15.4 since if f^* is fully faithful then $f^!$ is conservative. Indeed, in general $f_* f^!(t) \cong \mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t)$, which becomes $f_* f^!(t) \cong t$ when f^* is fully faithful since $f_*(\mathbb{1}_{\mathcal{S}}) \cong \mathbb{1}_{\mathcal{T}}$ (see, e.g., [San22, Remark 4.13]).

15.6. *Example.* If $f^*\colon \mathcal{T} \to \mathcal{S}$ is conservative and weakly closed then the local-to-global principle descends from $\mathcal{S}$ to $\mathcal{T}$. This follows from Proposition 13.33 and Remark 15.4. It can also be obtained as an application of Proposition 15.1.

16. Local cogeneration

We have seen in Section 9 that a compactly generated category is perfectly cogenerated by the Brown–Comenetz duals of its compact objects. We now explain how these can be used to construct suitable *local* cogenerators in tt-geometry. This will be an important ingredient in our bootstrap result for descending costratification in Section 17.

16.1. **Proposition.** *Let $\mathcal{T}$ be a rigidly-compactly generated tt-category.*

(a) Let $Y \subseteq \operatorname{Spc}(\mathcal{T}^c)$ be a Thomason subset. The subcategory $\mathsf{hom}(e_Y, \mathcal{T})$ is perfectly cogenerated by $\{\, I_c \mid \operatorname{supp}(c) \subseteq Y \,\}$. In particular,

$$\mathsf{hom}(e_Y, \mathcal{T}) = \operatorname{Coloc}\langle I_c \mid \operatorname{supp}(c) \subseteq Y\rangle.$$

(b) Let $Y_1, Y_2 \subseteq \mathrm{Spc}(\mathcal{T}^c)$ be Thomason subsets and consider the weakly visible subset $W := Y_1 \cap Y_2^c$. The subcategory $\Lambda^W \mathcal{T}$ is perfectly cogenerated by $\{\, \Lambda^W I_c \,|\, \mathrm{supp}(c) \subseteq Y_1 \,\}$. In particular,

$$\Lambda^W \mathcal{T} = \mathrm{Coloc}\langle \Lambda^W I_c \mid \mathrm{supp}(c) \subseteq Y_1 \rangle.$$

Proof. Let $\mathcal{K} := \mathcal{T}^c$ and let $\mathcal{K}_Y := \{\, x \in \mathcal{K} \,|\, \mathrm{supp}(x) \subseteq Y \,\}$. By Example 12.7 and Proposition 9.9, $\mathcal{T}$ is perfectly cogenerated by $\{\, I_c \,|\, c \in \mathcal{K} \,\} = \mathcal{K} \otimes I_{\mathbb{1}}$. Recall from Lemma 4.5 that $\mathsf{hom}(e_Y, \mathcal{T}) = \mathrm{Coloc}\langle \mathcal{K}_Y \otimes \mathcal{T} \rangle$. Hence Lemma 10.12 (applied to $\mathcal{T}^{\mathrm{op}}$) implies that $\mathsf{hom}(e_Y, \mathcal{T})$ is perfectly cogenerated by $\mathcal{K}_Y \otimes \mathcal{K} \otimes I_{\mathbb{1}} = \mathcal{K}_Y \otimes I_{\mathbb{1}} = \{\, I_c \,|\, c \in \mathcal{K}_Y \,\}$. This establishes (a).

We obtain part (b) by applying Proposition 9.11 to part (a). The functor

$$\mathsf{hom}(f_{Y_2}, -) \colon \mathsf{hom}(e_{Y_1}, \mathcal{T}) \to \mathsf{hom}(g_W, \mathcal{T}) \tag{16.2}$$

preserves products. Hence, since the domain category is perfectly cogenerated (by part (a)), we know it must have a left adjoint. Indeed, as observed in Remark 5.11, it has the fully faithful left adjoint

$$- \otimes f_{Y_2} \colon \mathsf{hom}(g_W, \mathcal{T}) \to \mathsf{hom}(e_{Y_1}, \mathcal{T}).$$

But this functor is naturally isomorphic to

$$\mathsf{hom}(e_{Y_1}, -) \colon \mathsf{hom}(g_W, \mathcal{T}) \to \mathsf{hom}(e_{Y_1}, \mathcal{T})$$

which evidently preserves products. Thus, the functor (16.2) has a conservative left adjoint, which itself preserves products. Hence we can invoke Proposition 9.11 to conclude that $\Lambda^W := \mathsf{hom}(g_W, \mathcal{T})$ is perfectly cogenerated by

$$\{\, \mathsf{hom}(f_{Y_2}, I_c) \,|\, \mathrm{supp}(c) \subseteq Y_1 \,\}.$$

Note that $\mathsf{hom}(f_{Y_2}, I_c) = \mathsf{hom}(g_W, I_c) = \Lambda^W I_c$ since $I_c \in \mathsf{hom}(e_{Y_1}, \mathcal{T})$ already. ☐

16.3. *Remark.* The statement in part (a) of Proposition 16.1 is the special case of part (b) when $Y_2 := \varnothing$. Indeed, as explained in the proof, if $\mathrm{supp}(c) \subseteq Y$ then $I_c \cong \mathsf{hom}(e_Y, I_c) = \Lambda^Y I_c$. We have formulated the proposition as we have, since the special case is used to prove the more general statement.

16.4. *Remark.* The reader may find it interesting to compare with [BIK12, Prop. 5.4].

16.5. *Remark.* The compact generators of $e_Y \otimes \mathcal{T} = \mathrm{Loc}\langle c \mid \mathrm{supp}(c) \subseteq Y \rangle$ similarly pushdown to a set of perfect generators of $\Gamma_W \mathcal{T} = \mathrm{Loc}\langle \Gamma_W(c) \mid \mathrm{supp}(c) \subseteq Y_1 \rangle$. Indeed, one uses the second adjunction displayed in Remark 5.11, observing that the fully faithful right adjoint $\mathsf{hom}(f_{Y_2}, -)$ is naturally isomorphic to $e_{Y_1} \otimes -$, which preserves coproducts, whence one can invoke Proposition 9.11.

16.6. **Proposition.** *Let $f^* \colon \mathcal{T} \to \mathcal{S}$ be as above (Hypothesis 14.1) and let $W \subseteq \mathrm{Spc}(\mathcal{T}^c)$ be a weakly visible subset. The trio of adjoints $f^* \dashv f_* \dashv f^!$ induces a trio of adjoints*

$$\begin{tikzcd}
\Gamma_W \mathcal{T} \arrow[r, phantom, "\cong"] \arrow[d, shift right=1.5ex, "f^*"'] \arrow[d, phantom, "\dashv"] & \Lambda^W \mathcal{T} \arrow[d, shift left=1.5ex, "f^!"] \arrow[d, phantom, "\dashv"] \\
\Gamma_{\varphi^{-1}(W)} \mathcal{S} \arrow[r, phantom, "\cong"] \arrow[u, shift right=1.5ex, "f_*"'] & \Lambda^{\varphi^{-1}(W)} \mathcal{S} \arrow[u, shift left=1.5ex, "f_*"]
\end{tikzcd}$$

where the middle square commutes and uses the equivalences of Remark 5.11.

Proof. It follows from the definitions and the standard isomorphisms from [BDS16] together with Remark 13.7 that the functors f^*, f_* and $f^!$ restrict to the four functors in the statement, and one verifies directly that we have the two displayed adjunctions $f^* \dashv f_*$ and $f_* \dashv f^!$. One then checks that the middle square commutes from the definitions of the "stalk-costalk equivalences" in Remark 5.11. □

17. Bootstrap for costratification

We now provide descent techniques for establishing costratification.

17.1. **Lemma.** *Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a geometric functor. For any $d \in \mathcal{T}^c$, there is an isomorphism $I_{f^*d} \cong f^! I_d$.*

Proof. For any $s \in \mathcal{S}$, the defining property of the Brown–Comenetz dual (Definition 9.4) together with adjunction provides natural isomorphisms

$$\begin{aligned}\mathcal{S}(s, I_{f^*d}) &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(f^*d, s), \mathbb{Q}/\mathbb{Z})\\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{T}(d, f_*s), \mathbb{Q}/\mathbb{Z})\\ &\cong \mathcal{T}(f_*s, I_d)\\ &\cong \mathcal{S}(s, f^! I_d)\end{aligned}$$

and we summon Yoneda. □

17.2. **Theorem.** *Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a geometric functor as in Hypothesis 14.1 and consider a prime $\mathcal{P} \in \mathrm{im}\,\varphi$. Assume that*

(a) $\mathcal{T}$ is stratified; and
(b) $\Lambda^{\mathcal{Q}}\mathcal{S}$ is a minimal colocalizing coideal of $\mathcal{S}$ for all $\mathcal{Q} \in \varphi^{-1}(\{\mathcal{P}\})$.

Then $\Lambda^{\mathcal{P}}\mathcal{T}$ is a minimal colocalizing coideal of $\mathcal{T}$.

Proof. Let $t \in \Lambda^{\mathcal{P}}\mathcal{T}$ be a nonzero object. Hence $\mathrm{Cosupp}_{\mathcal{T}}(t) = \{\mathcal{P}\}$ by the codetection property (which holds by Theorem 6.4). Since $\mathcal{T}$ is stratified, we have

$$\mathrm{Cosupp}_{\mathcal{S}}(f^!(t)) = \varphi^{-1}\,\mathrm{Cosupp}_{\mathcal{T}}(t) = \varphi^{-1}(\{\mathcal{P}\})$$

by Corollary 14.19. Then, since $\mathcal{S}$ has cominimality at all primes $\mathcal{Q} \in \varphi^{-1}(\{\mathcal{P}\})$, we have

$$\mathrm{Colocid}\langle \Lambda^{\mathcal{Q}}\mathcal{S} \mid \mathcal{Q} \in \varphi^{-1}(\{\mathcal{P}\})\rangle \subseteq \mathrm{Colocid}\langle f^!(t)\rangle.$$

Now, $\mathrm{Colocid}\langle f^!(t)\rangle = \mathrm{Coloc}\langle \mathsf{hom}(\mathcal{S}, f^!(t))\rangle$ and so, since f_* preserves products, we obtain an inclusion

$$f_*\Lambda^{\mathcal{Q}}\mathcal{S} \subseteq \mathrm{Coloc}\langle f_*\mathsf{hom}(\mathcal{S}, f^!(t))\rangle.$$

Using the adjunction isomorphism [BDS16, (2.18)], we conclude that

$$f_*\Lambda^{\mathcal{Q}}\mathcal{S} \subseteq \mathrm{Coloc}\langle \mathsf{hom}(f_*\mathcal{S}, t)\rangle \subseteq \mathrm{Colocid}\langle t\rangle \tag{17.3}$$

for each $\mathcal{Q} \in \varphi^{-1}(\{\mathcal{P}\})$.

Combining Lemma 14.14 and Remark 14.6, there exists $c \in \mathcal{S}^c$ (possibly depending on $\mathcal{Q} \in \varphi^{-1}(\{\mathcal{P}\})$) with $\mathrm{supp}(f_*(\Gamma_{\mathcal{Q}}c)) = \{\mathcal{P}\}$. On the one hand, stratification of $\mathcal{T}$ shows that $\Gamma_{\mathcal{P}}\mathbb{1} \in \mathrm{Locid}\langle f_*\Gamma_{\mathcal{Q}}c\rangle$. Hence by (2.8) we have for every $d \in \mathcal{T}^c$:

$$\Lambda^{\mathcal{P}} I_d \cong \mathsf{hom}(\Gamma_{\mathcal{P}}\mathbb{1}, I_d) \in \mathrm{Colocid}\langle \mathsf{hom}(f_*\Gamma_{\mathcal{Q}}c, I_d)\rangle.$$

On the other hand, there are isomorphisms

$$\begin{aligned} f_*(\Lambda^{\mathcal{Q}} I_{c^\vee \otimes f^*(d)}) &\cong f_* \mathsf{hom}(\Gamma_{\mathcal{Q}} \mathbb{1}, c^\vee \otimes I_{f^* d}) \\ &\cong f_* \mathsf{hom}(\Gamma_{\mathcal{Q}} c, I_{f^* d}) \\ &\cong \mathsf{hom}(f_* \Gamma_{\mathcal{Q}} c, I_d), \end{aligned}$$

where the fourth isomorphism uses Lemma 17.1. Combining these two observations with (17.3), we see that

$$\Lambda^{\mathcal{P}} I_d \in \mathrm{Colocid}\langle f_*(\Lambda^{\mathcal{Q}} I_{c^\vee \otimes f^*(d)}) \rangle \subseteq \mathrm{Colocid}\langle t \rangle.$$

Proposition 16.1 then implies that $\Lambda^{\mathcal{P}} \mathcal{T} \subseteq \mathrm{Colocid}\langle t \rangle$. This establishes cominimality of $\mathcal{T}$ at $\mathcal{P}$. ☐

17.4. **Corollary** (Bootstrap)**.** *Suppose $f^* \colon \mathcal{T} \to \mathcal{S}$ is a geometric functor as above (Hypothesis 14.1) such that*

(a) $\mathcal{T}$ is stratified;
(b) $\mathcal{S}$ is costratified; and
(c) f^* is conservative.

Then $\mathcal{T}$ is costratified.

Proof. By Theorem 7.7 and Theorem 6.4, we need to establish the minimality of $\Lambda^{\mathcal{P}} \mathcal{T}$ for each $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$. By Corollary 13.19, conservativity of f^* implies that φ is surjective. Therefore, Theorem 17.2 applies at all points of $\mathrm{Spc}(\mathcal{T}^c)$. ☐

17.5. *Remark.* In the situation of Corollary 17.4, assuming (a), condition (c) is equivalent to the statement that the map φ is surjective, and also equivalent to $f^!$ being conservative, see Corollary 14.24.

17.6. *Remark.* Theorem 17.2 becomes especially useful in combination with descent results for stratification. If $\mathcal{S}$ is costratified then it is also stratified (Theorem 7.19) and one can then try to apply one of the stratification descent techniques to show that $\mathcal{T}$ is stratified. Corollary 17.4 implies that, under the mild hypothesis that the induced map on spectra is surjective, costratification then descends as well. Thus we view it as a bootstrap technique. We give three instances of this idea below, suggestively called

- Zariski descent (Corollary 17.14);
- Quasi-finite descent (Corollary 17.17); and
- Nil-descent (Corollary 17.22).

The following variant of the bootstrap theorem will also be useful:

17.7. **Theorem.** *Suppose that there exist geometric functors $f_i^* \colon \mathcal{T} \to \mathcal{S}_i$ (Hypothesis 14.1) to costratified categories $\mathcal{S}_i$ such that the induced maps φ_i on spectra are jointly surjective:*

$$\bigcup_i \mathrm{im}\, \varphi_i = \mathrm{Spc}(\mathcal{T}^c).$$

Then $\mathcal{T}$ is stratified if and only if $\mathcal{T}$ is costratified.

Proof. That costratification for $\mathcal{T}$ implies stratification for $\mathcal{T}$ is Theorem 7.19. For the converse, we apply Theorem 17.2 to the functors $f_i^* \colon \mathcal{T} \to \mathcal{S}_i$. Because $\mathcal{S}_i$ is costratified, the theorem implies that $\mathcal{T}$ satisfies cominimality at $\mathcal{P}$ for every $\mathcal{P} \in \mathrm{im}\, \varphi_i$. Varying over i, we see that cominimality holds at all primes of $\mathrm{Spc}(\mathcal{T}^c)$. Since $\mathcal{T}$ is stratified by assumption the local-to-global principle holds, and hence so does the colocal-to-global principle (Theorem 6.4). Therefore $\mathcal{T}$ is costratified by Theorem 7.7. $\square$

17.8. *Remark.* In [Ste13, Theorem 8.11] and [BHS23, Corollary 5.5], it was shown that stratification is—in an appropriate sense—a Zariski-local property of a rigidly-compactly generated tt-category. We complement this result by establishing the analogous statement (Corollary 17.14) for costratification.

17.9. **Proposition.** *Let $\mathcal{T}$ be a rigidly-compactly generated tt-category with* $\mathrm{Spc}(\mathcal{T}^c)$ *weakly noetherian.*

(a) If $\mathcal{T}$ satisfies the (co)local-to-global principle, then so does each finite localization $\mathcal{T}(V)$.

(b) If $\mathrm{Spc}(\mathcal{T}^c) = V_1 \cup \ldots \cup V_r$ is a finite cover by complements of Thomason sets such that $\mathcal{T}(V_i)$ satisfies the (co)local-to-global principle, then so does $\mathcal{T}$.

Proof. The statements for the local-to-global principle were proven in [BHS23, Corollary 3.13] and [BHS23, Proposition 3.17], respectively. We established that the local-to-global principle and the colocal-to-global principle are equivalent in Theorem 6.4. $\square$

17.10. *Remark.* Stevenson proves in [Ste13, Proposition 8.4] that the finite localization functor $\mathcal{T} \to \mathcal{T}(Y^c)$ associated with a Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ induces an equivalence of stalks

$$\Gamma_{\mathcal{P}}\mathcal{T} \xrightarrow{\sim} \Gamma_{\mathcal{P}}(\mathcal{T}(Y^c))$$

for any $\mathcal{P} \in Y^c$. His result was stated under the assumption that $\mathrm{Spc}(\mathcal{T}^c)$ is noetherian, but the argument works when the space is weakly noetherian. The corresponding statement for costalks holds as well:

17.11. **Proposition.** *Suppose $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ is a Thomason subset and let $\mathcal{P} \in Y^c$. We have an equivalence of costalks*

$$\Lambda^{\mathcal{P}}\mathcal{T} \xrightarrow{\sim} \Lambda^{\mathcal{P}}(\mathcal{T}(Y^c)).$$

Proof. Let $f^*\colon \mathcal{T} \to \mathcal{T}(Y^c)$ denote the finite localization functor. As in Proposition 16.6, the adjunction $f_* \dashv f^!$ restricts to an adjunction

$$f_*\colon \Lambda^{\mathcal{P}}(\mathcal{T}(Y^c)) \rightleftarrows \Lambda^{\mathcal{P}}\mathcal{T} : f^!.$$

The left adjoint is fully faithful (being the restriction of a fully faithful functor). Hence it remains to prove that the counit $f_*f^!(x) \to x$ is an isomorphism for each $x \in \Lambda^{\mathcal{P}}\mathcal{T}$. Indeed, $f_*f^!(\Lambda^{\mathcal{P}}t) \simeq \mathsf{hom}(f_Y, \Lambda^{\mathcal{P}}t) \simeq \Lambda^{\mathcal{P}}t$ for all $t \in \mathcal{T}$, which gives the desired claim. $\square$

17.12. **Proposition.** *Let $U \subseteq \mathrm{Spc}(\mathcal{T}^c)$ be the complement of a Thomason subset. If $\mathcal{P} \in U$ is a weakly visible point, then $\mathcal{T}$ satisfies cominimality at $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$ if and only if the localized category $\mathcal{T}(U)$ satisfies cominimality at $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}(U)^c)$.*

Proof. Let $f^*\colon \mathcal{T} \to \mathcal{T}(U)$ denote the localization functor. Proposition 17.11 establishes that the induced adjunction $f_* : \Lambda^{\mathcal{P}}\mathcal{T}(U) \rightleftarrows \Lambda^{\mathcal{P}}\mathcal{T} : f^!$ of Proposition 16.6 is an adjoint equivalence. We thus have an inclusion-preserving bijection between the colocalizing subcategories of $\Lambda^{\mathcal{P}}\mathcal{T}(U)$ and the colocalizing subcategories of $\Lambda^{\mathcal{P}}\mathcal{T}$ given by $\mathcal{C} \mapsto \mathcal{C}' := f_*(\mathcal{C})$ with inverse $\mathcal{C}' \mapsto \mathcal{C} := f^!(\mathcal{C}')$. Moreover, since the inclusion of the costalk $\Lambda^{\mathcal{P}}\mathcal{T} \hookrightarrow \mathcal{T}$ preserves products, the colocalizing subcategories of $\Lambda^{\mathcal{P}}\mathcal{T}$ are precisely the colocalizing subcategories of $\mathcal{T}$ which are contained in $\Lambda^{\mathcal{P}}\mathcal{T}$ — and similarly for the costalk $\Lambda^{\mathcal{P}}\mathcal{T}(U) \hookrightarrow \mathcal{T}(U)$. We claim that under this correspondence $\mathcal{C} \mapsto \mathcal{C}'$ of colocalizing subcategories, $\mathcal{C}$ is a coideal of $\mathcal{T}(U)$ if and only if $\mathcal{C}'$ is a coideal of $\mathcal{T}$. This follows from the adjunction isomorphisms $\mathsf{hom}_{\mathcal{T}}(t, f_*(s)) \simeq f_*\mathsf{hom}_{\mathcal{T}(U)}(f^*(t), s)$ and $f_*\mathsf{hom}_{\mathcal{T}(U)}(s, f^!(t)) \simeq \mathsf{hom}_{\mathcal{T}}(f_*(s), t)$. This establishes an inclusion-preserving correspondence between the colocalizing coideals of $\mathcal{T}(U)$ contained in $\Lambda^{\mathcal{P}}\mathcal{T}(U)$ and the colocalizing coideals of $\mathcal{T}$ contained in $\Lambda^{\mathcal{P}}\mathcal{T}$. Hence $\Lambda^{\mathcal{P}}\mathcal{T}(U)$ is minimal among colocalizing coideal of $\mathcal{T}(U)$ if and only if $\Lambda^{\mathcal{P}}\mathcal{T}$ is minimal among colocalizing coideals of $\mathcal{T}$. □

17.13. **Corollary.** *The category $\mathcal{T}$ satisfies cominimality at a point $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$ if and only if the local category $\mathcal{T}/\langle\mathcal{P}\rangle$ satisfies cominimality at its unique closed point.*

Proof. Apply Proposition 17.12 with $U = \mathrm{gen}(\mathcal{P})$. □

17.14. **Corollary** (Zariski descent)**.** *Let $\mathcal{T}$ be a rigidly-compactly generated tt-category with $\mathrm{Spc}(\mathcal{T}^c)$ weakly noetherian and which satisfies the local-to-global principle. Suppose $\mathrm{Spc}(\mathcal{T}^c) = \bigcup_{i\in I} V_i$ is a cover by complements of Thomason subsets. Then $\mathcal{T}$ is costratified if and only if each finite localization $\mathcal{T}(V_i)$ is costratified.*

Proof. Each $\mathcal{T}(V_i)$ has the colocal-to-global principle by Proposition 17.9 and Theorem 6.4. Hence it suffices to consider cominimality (Theorem 7.7) and we can invoke Proposition 17.12. □

17.15. *Remark.* We end this section with two descent techniques for stratification which empower our bootstrap theorem (cf. Remark 17.6). The first is a modification of the argument in [Bar21, Section 2.2.2] establishing finite étale descent, which in turn is a generalization of [BHS23, Theorem 6.4].

17.16. **Theorem** (Quasi-finite descent)**.** *Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a geometric functor as in Hypothesis 14.1. Suppose f^* is strongly closed and φ is surjective with discrete fibers. If $\mathcal{S}$ is stratified then so is $\mathcal{T}$.*

Proof. Since f^* is weakly closed, the surjectivity assumption on φ implies that both $f^!$ and f^* are conservative (recall Proposition 13.33). Hence the local-to-global principle descends from $\mathcal{S}$ to $\mathcal{T}$ by Example 15.6 and it suffices to check minimality at each prime in $\mathcal{T}$. The hypotheses on the functor f^* are local in the target (cf. [BHS23, Proposition 1.30]) and hence it suffices to assume that $\mathcal{T}$ is local and check minimality at the unique closed point $\mathfrak{m} \in \mathrm{Spc}(\mathcal{T}^c)$.

Since the Thomason closed subset $\varphi^{-1}(\{\mathfrak{m}\})$ is discrete (by hypothesis), an elementary topological argument verifies that the fiber $\varphi^{-1}(\{\mathfrak{m}\})$ consists of finitely many visible closed points. Now consider an object $t \in \mathcal{T}$. Recall that $\mathfrak{m} \in \mathrm{Supp}_{\mathcal{T}}(t)$ if and only if $\mathfrak{m} \in \mathrm{Cosupp}_{\mathcal{T}}(t)$ (Theorem 8.3). We claim that if $\mathfrak{m} \in \mathrm{Supp}_{\mathcal{T}}(t)$ then $\varphi^{-1}(\{\mathfrak{m}\}) \subseteq \mathrm{Supp}_{\mathcal{S}}(f^*(t)) \cap \mathrm{Cosupp}_{\mathcal{S}}(f^!(t))$. To this end, let $\mathcal{Q} \in \varphi^{-1}(\{\mathfrak{m}\})$. Since $\mathcal{Q}$ is a visible closed point, there is some $z \in \mathcal{S}^c$ such that $\mathrm{supp}(z) = \{\mathcal{Q}\}$. Moreover, since the right adjoint f_* is weakly conservative (Remark 14.6) there exists some $x \in \mathcal{S}^c$ such that $f_*(z \otimes x) \neq 0$; but $\mathrm{supp}(z \otimes x) = \{\mathcal{Q}\}$, so replacing z by $z \otimes x$ if necessary we may assume without loss of generality that $f_*(z) \neq 0$. Note that $\varnothing \neq \mathrm{Supp}(f_*(z)) \subseteq \varphi(\mathrm{Supp}(z)) = \{\mathfrak{m}\}$ by Proposition 14.2 so that $\mathrm{Supp}(f_*(z)) = \{\mathfrak{m}\}$. Since $f_*(z)$ is compact, $\mathrm{Supp}(t \otimes f_*(z)) = \mathrm{Supp}(t) \cap \mathrm{Supp}(f_*(z)) = \{\mathfrak{m}\}$ by [BHS23, Lemma 2.18]. In particular, $t \otimes f_*(z) \neq 0$ so that $f^*(t) \otimes z \neq 0$ and thus $\mathcal{Q} \in \mathrm{Supp}_{\mathcal{S}}(f^*(t))$. This establishes that $\varphi^{-1}(\{\mathfrak{m}\}) \subseteq \mathrm{Supp}_{\mathcal{T}}(t)$. Now consider $\mathrm{Cosupp}_{\mathcal{S}}(f^!(t))$. Again, since $f_*(z)$ is compact we have $\mathrm{Cosupp}_{\mathcal{T}}(\mathsf{hom}(f_*(z), t)) =$

$\mathrm{Supp}_{\mathcal{T}}(f_*(z)) \cap \mathrm{Cosupp}_{\mathcal{T}}(t) = \{\mathfrak{m}\}$ by Proposition 4.35. Hence $f_*\mathsf{hom}(z, f^!(t)) = \mathsf{hom}(f_*(z), t) \neq 0$ so that $\mathsf{hom}(z, f^!(t)) \neq 0$ and thus $\mathcal{P} \in \mathrm{Cosupp}_{\mathcal{T}}(f^!(t))$.

In summary, if $t_1, t_2 \in \mathcal{T}$ are objects with $\mathfrak{m}$ contained in their (co)support then

$$\begin{aligned} \varnothing \neq \varphi^{-1}(\{\mathfrak{m}\}) &\subseteq \mathrm{Supp}_{\mathcal{S}}(f^*(t_1)) \cap \mathrm{Cosupp}_{\mathcal{S}}(f^!(t_2)) \\ &= \mathrm{Cosupp}_{\mathcal{S}}(\mathsf{hom}(f^*(t_1), f^!(t_2))), \end{aligned}$$

where the equality uses the assumption that $\mathcal{S}$ is stratified and applies Theorem 7.15. Therefore, $f^!\mathsf{hom}(t_1, t_2) = \mathsf{hom}(f^*(t_1), f^!(t_2)) \neq 0$ and hence $\mathsf{hom}(t_1, t_2) \neq 0$. This establishes minimality at $\mathfrak{m}$ by Lemma 7.12. $\square$

17.17. **Corollary.** *If f^* is strongly closed, φ is surjective with discrete fibers, and $\mathcal{S}$ is costratified, then $\mathcal{T}$ is costratified as well.*

Proof. Since $\mathcal{S}$ is costratified, it is also stratified by Theorem 7.19. It then follows from Theorem 17.16 that $\mathcal{T}$ is also stratified. We can now invoke Corollary 17.4 since f^* is conservative due to Proposition 13.33. $\square$

17.18. *Remark.* The name of Theorem 17.16 is motivated by algebraic geometry. A finite type morphism $f\colon X \to Y$ of schemes is quasi-finite in the sense of [Gro61, Définition 6.2.3] if and only if it has discrete fibers.

17.19. *Remark.* The next result is inspired by work of Shaul and Williamson [SW20] in the context of BIK-stratification and BIK-costratification. The following tt-geometric version strengthens [Bar21, Theorem 2.24].

17.20. **Theorem** (Nil-descent)**.** *Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a geometric functor as in Hypothesis 14.1. Suppose $f^!$ is conservative and φ is injective. If $\mathcal{S}$ is stratified then so is $\mathcal{T}$.*

Proof. Since $f^!$ is conservative, $\mathcal{T}$ inherits the local-to-global principle from $\mathcal{S}$ (Remark 15.4). To establish that $\mathcal{T}$ is stratified we will use Theorem 7.15. To this end, let $t_1, t_2 \in \mathcal{T}$ be objects with $\mathsf{hom}(t_1, t_2) = 0$. Our goal is to prove that $\mathrm{Supp}(t_1) \cap \mathrm{Cosupp}(t_2) = \varnothing$. We have $0 = f^!\mathsf{hom}(t_1, t_2) = \mathsf{hom}(f^*t_1, f^!t_2)$ by (13.6), hence

$$\mathrm{Supp}(f^*t_1) \cap \mathrm{Cosupp}(f^!t_2) = \varnothing$$

since $\mathcal{S}$ is assumed stratified. The assumption that φ is injective then implies

$$\varphi(\mathrm{Supp}(f^*t_1)) \cap \varphi(\mathrm{Cosupp}(f^!t_2)) = \varnothing. \tag{17.21}$$

Since $\mathcal{S}$ has detection and codetection, Proposition 14.7 and Proposition 14.10 imply that (17.21) is the same as

$$\mathrm{Supp}(t_1 \otimes f_*(\mathbb{1}_{\mathcal{S}})) \cap \mathrm{Cosupp}(\mathsf{hom}(f_*\mathbb{1}, t_2)) = \varnothing.$$

Since $f^!$ is conservative, f^* is also conservative by Proposition 13.21. Proposition 14.7 and Proposition 14.10 then imply that $\mathrm{Supp}(t_1 \otimes f_*(\mathbb{1}_{\mathcal{S}})) = \mathrm{Supp}(t_1)$ and $\mathrm{Cosupp}(\mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), t_2)) = \mathrm{Cosupp}(t_2)$, which completes the proof. $\square$

17.22. **Corollary.** *If $f^!$ is conservative and φ is injective (hence bijective) and $\mathcal{S}$ is costratified, then $\mathcal{T}$ is costratified as well.*

Proof. Note that f^* is also conservative by Proposition 13.21. The claim then follows from Theorem 7.19, Theorem 17.20, and Corollary 17.4. $\square$

17.23. *Remark.* If f^* is weakly closed and φ is a bijection then we can invoke Theorem 17.20 and Corollary 17.22 because of Proposition 13.33.

Part IV. Applications and examples

We now turn to applications and examples. We show that a pure-semisimple tt-category (in particular, any tt-field) is costratified (Theorem 18.4) and that any affine weakly regular tt-category is costratified (Theorem 18.15). We also show that the derived category of a (topologically weakly noetherian) quasi-compact and quasi-separated scheme is costratified if and only if it is stratified (Theorem 19.5). In representation theory, we show that the category of k-linear representations $\mathrm{Rep}(G,k)$ and the derived category of k-linear permutation modules $\mathrm{DPerm}(G,k)$ is costratified for any finite group G and field k (Theorem 19.10, Theorem 19.13 and Theorem 20.14). In homotopy theory, we give a new proof that the category of $E(n)$-local spectra is costratified (Theorem 20.19) and prove that certain cochain algebras have costratified derived categories (Theorem 20.27). We establish that the category of rational G-spectra is costratified for any compact Lie group G (Theorem 20.48) and show that, for any finite group G, the category of spectral G-Mackey functors valued in a commutative algebra $\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp})$ is costratified whenever $\mathrm{D}(\mathbb{E})$ itself is costratified and has a noetherian spectrum (Theorem 20.31). As a special case, this shows that Kaledin's category of derived Mackey functors is costratified (Corollary 20.36).

18. Tensor triangular examples

Pure-semisimple categories and tt-fields.

18.1. **Lemma.** *Let $\mathcal{T}$ be a rigidly-compactly generated tt-category with weakly noetherian spectrum. Suppose that every colocalizing coideal of $\mathcal{T}$ is also a localizing ideal. Then $\mathcal{T}$ is stratified if and only if it is costratified.*

Proof. In light of Theorem 6.4 and Theorem 7.19, it suffices to show that if minimality holds at a point $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$ then cominimality also holds at $\mathcal{P}$. Under the equivalence $\Lambda^{\mathcal{P}}\mathcal{T} \cong \Gamma_{\mathcal{P}}\mathcal{T}$ of Remark 5.11, the thick subcategories $\mathcal{C}$ of $\Lambda^{\mathcal{P}}\mathcal{T}$ correspond bijectively with the thick subcategories $\mathcal{C}'$ of $\Gamma_{\mathcal{P}}\mathcal{T}$. Moreover, one readily checks that $\mathcal{T}^d \otimes \mathcal{C} \subseteq \mathcal{C}$ if and only if $\mathcal{T}^d \otimes \mathcal{C}' \subseteq \mathcal{C}'$. By our hypothesis, the costalk $\Lambda^{\mathcal{P}}\mathcal{T}$ is a localizing ideal of $\mathcal{T}$. In particular, the inclusion $\Lambda^{\mathcal{P}}\mathcal{T} \hookrightarrow \mathcal{T}$ preserves coproducts. It follows that if $\mathcal{C} \subseteq \Lambda^{\mathcal{P}}\mathcal{T}$ is a localizing ideal of $\mathcal{T}$ then its corresponding $\mathcal{C}' \subseteq \Gamma_{\mathcal{P}}\mathcal{T}$ is also a localizing ideal of $\mathcal{T}$. Armed with these observations, suppose $\mathcal{C} \subsetneq \Lambda^{\mathcal{P}}\mathcal{T}$ is a proper colocalizing coideal. By hypothesis, it is a localizing ideal of $\mathcal{T}$, hence corresponds to a proper localizing ideal $\mathcal{C}' \subsetneq \Gamma_{\mathcal{P}}\mathcal{T}$. Minimality at $\mathcal{P}$ implies that $\mathcal{C}' = 0$ and hence $\mathcal{C} = 0$, so that we have cominimality at $\mathcal{P}$. □

18.2. *Definition* (Beligiannis [Bel00], Krause [Kra00]). A compactly generated triangulated category $\mathcal{T}$ is said to be *pure-semisimple* if every pure monomorphism splits, where $f\colon x \to y$ is a pure monomorphism if the induced map $\mathcal{T}(c,x) \to \mathcal{T}(c,y)$ is a monomorphism for all compact objects $c \in \mathcal{T}^c$. This is equivalent to a host of other conditions, for example that any object of $\mathcal{T}$ is a coproduct of compact objects, that $\mathcal{T}$ has all filtered colimits, or that $\mathcal{T}$ is phantomless; see [Bel00, Theorem 9.3] and [Kra00, Theorem 2.10].

18.3. **Proposition.** *If $\mathcal{T}$ is a rigidly-compactly generated pure-semisimple tt-category then* $\mathrm{Spc}(\mathcal{T}^c)$ *is a finite discrete space.*

Proof. We first prove that every Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ is both open and closed. Since $\mathcal{T}$ is pure-semisimple, we can write $f_Y = \coprod_{i\in I} x_i$ as a coproduct of compact objects. Note that each x_i is contained in $\ker(-\otimes e_Y) = \mathrm{Locid}\langle f_Y\rangle$; see Remark 4.1. Since this is a localizing ideal, it also contains the duals $x_i^\vee$, and hence

$$\mathcal{T}(f_Y, \Sigma e_Y) = \prod_{i\in I} \mathcal{T}(x_i, \Sigma e_Y) = \prod_{i \in I} \mathcal{T}(\mathbb{1}, \Sigma e_Y \otimes x_i^\vee) = 0.$$

Thus Proposition 8.10 establishes that Y is both open and closed.

Next we establish that the specialization order is trivial (cf. Remark 8.5). To this end, suppose there exists an inclusion of primes $\mathcal{P} \subseteq \mathcal{Q}$ with $\mathcal{P} \neq \mathcal{Q}$. Then $\mathcal{Q} \not\subseteq \mathcal{P}$, i.e., $\mathcal{Q} \notin \overline{\{\mathcal{P}\}}$. Hence there exists a Thomason closed subset Z which contains $\mathcal{P}$ but does not contain $\mathcal{Q}$. This is a contradiction, since the Thomason subset Z is open (by the first part of the proof) and hence closed under generalization.

It follows that $\{\mathcal{P}\} = \mathrm{gen}(\mathcal{P})$ is open since it is the complement of a Thomason subset. Since every point is open, the topology is discrete. Moreover, a discrete quasi-compact space is necessarily finite. ☐

18.4. **Theorem.** *If $\mathcal{T}$ is a rigidly-compactly generated pure-semisimple tt-category, then $\mathrm{Spc}(\mathcal{T}^c)$ is a finite discrete space and $\mathcal{T}$ is costratified and stratified.*

Proof. Proposition 18.3 establishes that the Balmer spectrum is finite and discrete. We first show that $\mathcal{T}$ is stratified. By [BHS23, Theorem 3.21] the local-to-global principle holds for $\mathcal{T}$. We establish minimality using Lemma 7.12. To that end, fix $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$ and consider nonzero objects $t_1, t_2 \in \Gamma_{\mathcal{P}}\mathcal{T}$. Since $\mathcal{T}$ is pure-semisimple, we have $t_1 = \coprod_{i\in I} x_i$ and $t_2 = \coprod_{j \in J} y_j$ for nonzero compact objects x_i, y_j. Since x_i (respectively, y_j) is a retract of t_1 (respectively, t_2), we see that each x_i and y_j is in $\Gamma_{\mathcal{P}}\mathcal{T}$ as well. Thus $\mathrm{supp}(x_i) = \mathrm{supp}(y_j) = \{\mathcal{P}\}$ and hence $\mathsf{hom}(x_i, y_j) \neq 0$. It follows that $\mathsf{hom}(t_1, t_2) \neq 0$, as required.

For any collection of objects $\{t_i\}_{i\in I}$ in a pure-semisimple tt-category $\mathcal{T}$, the canonical map $\coprod_{i\in I} t_i \to \prod_{i \in I} t_i$ from the coproduct to the product splits; see [Bel00, Theorem 9.3]. In particular, any colocalizing subcategory of $\mathcal{T}$ is localizing. Moreover, since a colocalizing subcategory of $\mathcal{T}$ is a coideal if and only if it is closed under tensoring with dualizable objects (Example 10.10), we conclude that any colocalizing coideal is a localizing ideal. Hence we can invoke Lemma 18.1 to conclude that $\mathcal{T}$ is costratified. ☐

18.5. *Example.* Let A be an Artin algebra which is derived equivalent to a hereditary algebra of Dynkin type. Then the homotopy category $\mathrm{K}(\mathrm{Inj}\, A)$ of complexes of injective A-modules is pure-semisimple [Zhe13, Proposition 4.4] and hence stratified and costratified by Theorem 18.4.

18.6. *Remark.* As a consequence of Proposition 18.3, we see that an infinite product of nontrivial pure-semisimple tt-categories cannot be pure-semisimple. This should be compared with the following observation: An infinite product of nontrivial pure-semisimple commutative rings is never pure-semisimple. Indeed, a pure-semisimple ring is always artinian [Cha60, Theorem 4.4] and, in particular, its Zariski spectrum is always finite discrete. We note also that if $\mathrm{D}(R)$ is pure-semisimple in the sense of Definition 18.2 then R is pure-semisimple [GP05, Corollary 7.2]. We thus see that for a field k, neither $\prod_{i=1}^\infty \mathrm{D}(k)$ nor $\mathrm{D}(\prod_{i=1}^\infty k)$ is pure-semisimple, even though $\mathrm{D}(k)$ is pure-semisimple.

18.7. *Example.* In [BKS19, Definition 1.1], Balmer–Krause–Stevenson define a *tt-field* to be a rigidly-compactly generated tt-category $\mathcal{F}$ for which every object $X \in \mathcal{F}$ is a coproduct $X \simeq \coprod_{i\in I} x_i$ of compact-rigid objects $x_i \in \mathcal{F}^c$ and for which each object $X \in \mathcal{F}$ is $\otimes$-faithful ($X \otimes f = 0 \implies f = 0$). A tt-field $\mathcal{F}$ is pure-semisimple and $\mathrm{Spc}(\mathcal{F}^c) = \{*\}$ is a single point; see [BKS19, Proposition 5.1].

18.8. *Example.* The stable module category $\mathcal{T} = \mathrm{StMod}(kC_{p^n})$ is a pure-semisimple triangulated category which, if $n \geq 2$, is not a tt-field; see [BKS19, Example 5.11].

18.9. **Corollary.** *Any tt-field $\mathcal{F}$ is both stratified and costratified.*

Affine weakly regular tt-categories.

18.10. *Definition* (Dell'Ambrogio–Stanley [DS16])*.* A tensor-triangulated category $\mathcal{T}$ is said to be *affine weakly regular* if it satisfies the following two conditions:

(a) (affine) $\mathcal{T}$ is compactly generated by its tensor unit $\mathbb{1}$.
(b) (weakly regular) The graded endomorphism ring $R \coloneqq \mathrm{Hom}^*_{\mathcal{T}}(\mathbb{1}, \mathbb{1})$ is a graded noetherian ring concentrated in even degrees, and for every homogeneous prime ideal $\mathfrak{p}$ of R, the maximal ideal of the local ring $R_\mathfrak{p}$ is generated by a (finite) regular sequence of homogeneous non-zero-divisors.

The first axiom ensures that $\mathcal{T}$ is a rigidly-compactly generated tt-category and that every (co)localizing subcategory is a (co)ideal (cf. Remark 2.4).

18.11. *Remark.* Given an affine weakly regular tt-category we set

$$\pi_*(X) \coloneqq \mathrm{Hom}^*_{\mathcal{T}}(\mathbb{1}, X).$$

For any prime ideal $\mathfrak{p} \in \mathrm{Spec}^h(R)$, there is a residue field object $K(\mathfrak{p}) \in \mathcal{T}$ with the property that

$$\pi_*(K(\mathfrak{p})) \simeq \kappa(\mathfrak{p})$$

where $\kappa(\mathfrak{p}) \coloneqq R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ denotes the algebraic residue field; see [DS16, §3]. This object plays a role analogous to $g_\mathfrak{p}$; in particular, $\mathrm{Loc}\langle g_\mathfrak{p}\rangle = \mathrm{Loc}\langle K(\mathfrak{p})\rangle$. The following theorem states the main results of [DS16]:

18.12. **Theorem** (Dell'Ambrogio–Stanley)**.** *Let $\mathcal{T}$ be an affine weakly regular tt-category. There is a homeomorphism $\mathrm{Spc}(\mathcal{T}^c) \cong \mathrm{Spec}^h(R)$ and $\mathcal{T}$ is stratified.*

18.13. **Lemma.** *For any set I and integers $\{d_i\}_{i\in I}$, the natural map*

$$\coprod_{i\in I} \Sigma^{d_i} K(\mathfrak{p}) \longrightarrow \prod_{i\in I} \Sigma^{d_i} K(\mathfrak{p}) \tag{18.14}$$

is a split monomorphism.

Proof. Applying π_* to the morphism (18.14) gives the monomorphism of graded $\kappa(\mathfrak{p})$-modules $\bigoplus_{i\in I} \Sigma^{d_i}\kappa(\mathfrak{p}) \to \prod_{i\in I} \Sigma^{d_i}\kappa(\mathfrak{p})$. We can extend the standard basis of $\bigoplus_{i\in I} \Sigma^{d_i}\kappa(\mathfrak{p})$ to a basis $\mathcal{B}$ of $\prod_{i\in I} \Sigma^{d_i}\kappa(\mathfrak{p})$. As $\prod_{i\in I} \Sigma^{d_i} K(\mathfrak{p}) \simeq K(\mathfrak{p}) \otimes \prod_{i\in I} \Sigma^{d_i}\mathbb{1}$, Proposition 3.6 of [DS16] provides an isomorphism

$$\coprod_{b\in\mathcal{B}} \Sigma^{|b|} K(\mathfrak{p}) \simeq \prod_{i\in I} \Sigma^{d_i} K(\mathfrak{p})$$

where $|b|$ denotes the homological degree of b. Any set-theoretic splitting of $I \subseteq \mathcal{B}$ then lifts to a left-inverse of (18.14). $\square$

18.15. **Theorem.** *Let $\mathcal{T}$ be an affine weakly regular tt-category. Then $\mathcal{T}$ is costratified.*

Proof. The proof follows [BIK12, Theorem 10.3]. The colocal-to-global principle holds by Corollary 6.5, so we only need to establish cominimality at every $\mathfrak{p} \in \mathrm{Spec}^h(R)$. To that end, let $0 \neq t \in \Lambda^{\mathfrak{p}}\mathcal{T}$. Then, since $\mathcal{T}$ is stratified we have

$$\mathrm{Coloc}\langle t\rangle = \mathrm{Coloc}\langle \mathsf{hom}(g_{\mathfrak{p}}, t)\rangle = \mathrm{Coloc}\langle \mathsf{hom}(K(\mathfrak{p}), t)\rangle$$

where the last step uses that $\mathrm{Loc}\langle g_{\mathfrak{p}}\rangle = \mathrm{Loc}\langle K(\mathfrak{p})\rangle$ and Lemma 4.34. Now a similar argument to [DS16, Proposition 3.6] shows that

$$\mathsf{hom}(K(\mathfrak{p}), t) \simeq \coprod\nolimits_{\alpha} \Sigma^{|\alpha|} K(\mathfrak{p})$$

where α runs through a graded $\kappa(\mathfrak{p})$-vector space basis of $\pi_*\mathsf{hom}(K(\mathfrak{p}), t)$. It follows that $K(\mathfrak{p}) \in \mathrm{Coloc}\langle \mathsf{hom}(K(\mathfrak{p}), t)\rangle$. Moreover, by Lemma 18.13 we see that $\coprod_{\alpha} \Sigma^{|\alpha|} K(\mathfrak{p})$ is a retract of $\prod_{\alpha} \Sigma^{|\alpha|} K(\mathfrak{p})$, hence we have

$$\mathsf{hom}(K(\mathfrak{p}), t) \simeq \coprod\nolimits_{\alpha} \Sigma^{|\alpha|} K(\mathfrak{p}) \in \mathrm{Coloc}\langle K(\mathfrak{p})\rangle.$$

Therefore, we obtain

$$\mathrm{Coloc}\langle t\rangle = \mathrm{Coloc}\langle \mathsf{hom}(K(\mathfrak{p}), t)\rangle = \mathrm{Coloc}\langle K(\mathfrak{p})\rangle.$$

If $\mathrm{Coloc}\langle t\rangle$ were a proper subcategory of $\Lambda^{\mathfrak{p}}\mathcal{T}$, then any object s in the complement would have to satisfy $\mathrm{Coloc}\langle t\rangle = \mathrm{Coloc}\langle s\rangle$, which is absurd. Therefore, $\Lambda^{\mathfrak{p}}\mathcal{T}$ is minimal as a colocalizing subcategory of $\mathcal{T}$, as required. □

Finite products of tt-categories.

18.16. *Example.* Let $\{\mathcal{T}_i\}_{i=1}^n$ be a finite collection of rigidly-compactly generated tensor-triangulated categories. Their product $\prod_{i=1}^n \mathcal{T}_i$ is again rigidly-compactly generated and $(\prod_{i=1}^n \mathcal{T}_i)^c = \prod_{i=1}^n \mathcal{T}_i^c$. Moreover,

$$\mathrm{Spc}(\prod_{i=1}^n \mathcal{T}_i^c) \cong \coprod_{i=1}^n \mathrm{Spc}(\mathcal{T}_i^c)$$

where the right-hand side has the disjoint union topology. Indeed, each prime ideal of $\prod_{i=1}^n \mathcal{T}_i^c$ is of the form $\prod_{i=1}^n \mathcal{P}_i$, where for some k, $\mathcal{P}_k$ is a prime ideal of $\mathcal{T}_k^c$, and for $i \neq k$, $\mathcal{P}_i = \mathcal{T}_i^c$. Thus $\prod_{i=1}^n \mathcal{T}_i$ has (weakly) noetherian spectrum if and only if each $\mathcal{T}_i$ has (weakly) noetherian spectrum. Moreover, by Zariski descent (Proposition 17.9 and Corollary 17.13), $\prod_{i=1}^n \mathcal{T}_i$ is costratified if and only if each $\mathcal{T}_i$ is costratified.

18.17. *Example.* Let $\mathcal{T}$ be any rigidly-compactly generated tt-category. Let $\mathcal{T} \times_{\mathbb{Z}_2} \mathcal{T}$ denote the product category $\mathcal{T} \times \mathcal{T}$ with the $\mathbb{Z}_2$-graded tensor-triangulated structure described in [San22, Example 4.23]. It is again rigidly-compactly generated and the inclusion $a \mapsto (a, 0)$ is a fully faithful geometric functor $f^* \colon \mathcal{T} \hookrightarrow \mathcal{T} \times \mathcal{T}$. The projection onto the first coordinate is both left and right adjoint to f^*. Hence $f^! = f^*$. The prime ideals of $(\mathcal{T} \times \mathcal{T})^c = \mathcal{T}^c \times \mathcal{T}^c$ are $\mathcal{P} \times \mathcal{P}$ for $\mathcal{P}$ a prime ideal of $\mathcal{T}^c$, and the induced map

$$\mathrm{Spc}(f^*) \colon \mathrm{Spc}(\mathcal{T}^c \times \mathcal{T}^c) \to \mathrm{Spc}(\mathcal{T}^c)$$

given by $\mathcal{P} \times \mathcal{P} \mapsto \mathcal{P}$ is a homeomorphism. Under this identification, one readily checks that $\mathrm{Supp}_{\mathcal{T}\times\mathcal{T}}((t_0, t_1)) = \mathrm{Supp}_{\mathcal{T}}(t_0) \cup \mathrm{Supp}_{\mathcal{T}}(t_1)$ and $\mathrm{Cosupp}_{\mathcal{T}\times\mathcal{T}}((t_0, t_1)) = \mathrm{Cosupp}_{\mathcal{T}}(t_0) \cup \mathrm{Cosupp}_{\mathcal{T}}(t_1)$. It is then straightforward to establish that if $\mathcal{T}$ is costratified then $\mathcal{T} \times_{\mathbb{Z}_2} \mathcal{T}$ is costratified. The converse follows from Theorem 17.20 and Corollary 17.4.

19. Algebraic examples

Commutative rings and schemes.

19.1. **Proposition.** *Let R be a commutative ring with* $\mathrm{Spec}(R)$ *weakly noetherian. Then* $\mathrm{D}(R)$ *is stratified if and only if it is costratified.*

Proof. Recall that $\mathrm{Spc}(\mathrm{D}(R)^c) \cong \mathrm{Spec}(R)$ by Thomason's theorem [Tho97]; see [BHS23, Example 1.36]. For each $\mathfrak{p} \in \mathrm{Spec}(R)$, we consider the base-change functor $f_{\mathfrak{p}}^*\colon \mathrm{D}(R) \to \mathrm{D}(\kappa(\mathfrak{p}))$, where $\kappa(\mathfrak{p})$ denotes the residue field at $\mathfrak{p}$. This is a geometric functor whose target is a tt-field and the induced map on spectra sends the unique point $*$ to $\mathfrak{p}$. The result then follows from Theorem 17.7 and Corollary 18.9. □

19.2. *Example.* For any commutative noetherian ring R, we then deduce the main result of [Nee11] from the original [Nee92a]: $\mathrm{D}(R)$ is costratified.

19.3. *Example.* Let R be an absolutely flat ring which is not noetherian. Stevenson [Ste14a, Ste17] proves that R is semi-artinian $\iff$ the local-to-global principle for $\mathrm{D}(R)$ holds $\iff$ $\mathrm{D}(R)$ is stratified. By Proposition 19.1, this is also equivalent to $\mathrm{D}(R)$ being costratified. For example, this applies to the subring of $\prod_{\mathbb{N}} \mathbb{F}_p$ consisting of those sequences which are eventually constant; cf. [BHS23, Example 3.25].

19.4. *Remark.* We can extend Proposition 19.1 to derived categories of schemes. For a quasi-compact and quasi-separated scheme X, let $\mathrm{D}_{\mathrm{qc}}(X)$ denote the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology. It is rigidly-compactly generated and its subcategory of rigid-compact objects $\mathrm{D}_{\mathrm{qc}}(X)^c = \mathrm{D}_{\mathrm{perf}}(X)$ is the derived category of perfect complexes. A fundamental result concerning the Balmer spectrum is that $\mathrm{Spc}(\mathrm{D}_{\mathrm{perf}}(X)) \cong X$. See [Bal05, Theorem 6.3], [BKS07, Theorem 9.5], and [Tho97].

19.5. **Theorem.** *Let X be a quasi-compact and quasi-separated scheme which is topologically weakly noetherian. The derived category* $\mathrm{D}_{\mathrm{qc}}(X)$ *is stratified if and only if it is costratified.*

Proof. One direction is Theorem 7.19. For the converse, suppose that $\mathrm{D}_{\mathrm{qc}}(X)$ is stratified. It satisfies the local-to-global principle by [BHS23, Theorem 4.1] and hence the colocal-to-global principle by Theorem 6.4. Now take an open affine cover of X by subsets $V_i = \mathrm{Spec}(A_i)$. Each of these affine schemes is also topologically weakly noetherian (cf. [BHS23, Remark 2.6]) and we have the implications $\mathrm{D}_{\mathrm{qc}}(X)$ is stratified $\iff$ each $\mathrm{D}(A_i)$ is stratified ([BHS23, Corollary 5.5]) $\iff$ each $\mathrm{D}(A_i)$ is costratified (Proposition 19.1) $\iff$ $\mathrm{D}_{\mathrm{qc}}(X)$ is costratified (Corollary 17.14). □

19.6. *Example.* If X is noetherian, then it follows from [BHS23, Corollary 5.10] that $\mathrm{D}_{\mathrm{qc}}(X)$ is stratified and hence is also costratified. For such X, the costratification of $\mathrm{D}_{\mathrm{qc}}(X)$ has also been obtained in recent work of Verasdanis [Ver22].

Derived categories of representations.

19.7. *Definition.* For a finite group G and commutative ring R, we let

$$\mathrm{Rep}(G, R) \coloneqq \mathrm{Ind}\,\mathrm{Fun}(BG, \mathrm{Perf}(R))$$

denote the derived ∞-category of R-linear G-representations, where Ind denotes ind-completion ([Lur09, Section 5.3.5]), and let

$$\mathrm{StMod}(G, R) \coloneqq \mathrm{Ind}\,\mathrm{Fun}(BG, \mathrm{Perf}(R))/\,\mathrm{Perf}(R[G]).$$

By construction, both of these categories are rigidly-compactly generated symmetric monoidal stable ∞-categories; passage to their homotopy categories yields the corresponding rigidly-compactly generated tt-categories. Moreover, up to idempotent completion, the category of compact objects in $\mathrm{StMod}(G,R)$ is obtained as a finite localization of the compact objects in $\mathrm{Rep}(G,R)$.

19.8. *Remark.* If $R = k$ is a field, then the category $\mathrm{Rep}(G,k)$ is equivalent to $\mathrm{K}(\mathrm{Inj}\, k[G])$, the homotopy category of unbounded complexes of injective $k[G]$-modules, as studied in [BK08], and the category $\mathrm{StMod}(G,k)$ agrees with the usual stable module category.

19.9. *Remark.* The next result is originally due to Benson–Iyengar–Krause [BIK12, Theorem 11.6] with the computation of the spectrum due to Benson–Carlson–Rickard [BCR97]. We will give an alternative proof via our bootstrap theorem, relying on a result we prove at the end of Section 20 using Galois ascent as in [Mat15].

19.10. **Theorem** (Benson–Iyengar–Krause)**.** *For any finite group G and field k, the category $\mathrm{Rep}(G,k)$ is costratified with spectrum $\mathrm{Spc}(\mathrm{Rep}(G,k)^c) \cong \mathrm{Proj}\, H^*(G;k)$.*

Proof. By Chouinard's theorem, as given in [BIK11a, Proposition 9.6], the functor

$$\mathrm{Res}^G_E \colon \mathrm{Rep}(G,k) \to \prod_{E \le G} \mathrm{Rep}(E,k)$$

given by the product of restriction functors is a conservative geometric functor. Moreover, the target category $\prod_{E\le G} \mathrm{Rep}(E,k)$ is costratified by Theorem 20.14 below and Example 18.16, and $\mathrm{Rep}(G,k)$ is stratified by [BIK11a, Theorem 10.1]. It follows from Corollary 17.4 that $\mathrm{Rep}(G,k)$ is costratified as well. □

19.11. *Remark.* The forthcoming [BBI+23] will establish stratification of $\mathrm{Rep}(G,R)$ for all finite groups G and all noetherian commutative rings R, and then deduce costratification via the bootstrap theorem. The computation of $\mathrm{Spc}(\mathrm{Rep}(G,R)^c)$ for any commutative ring R is due to Lau [Lau21].

19.12. *Remark.* There is an enlargement of the category $\mathrm{Rep}(G,R)$ given by the derived category of permutation modules $\mathrm{DPerm}(G,R)$, for any (pro)finite group G and commutative ring R. For the construction of this category as well as its relation to Artin motives and Mackey functors, we refer to [BG21].

19.13. **Theorem.** *The derived category of permutation modules $\mathrm{DPerm}(G,k)$ is costratified for any finite group G and field k of characteristic p dividing the order of G.*

Proof. In [BG22], the authors construct geometric functors

$$\check{\Psi}_H \colon \mathrm{DPerm}(G,k) \to \mathrm{Rep}(W_G(H),k)$$

indexed by all conjugacy classes of p-subgroups H of G, where $W_G(H)$ denotes the Weyl group of H in G. In [BG22, Theorem 5.12], they prove that these functors are jointly conservative, while [BG22, Theorem 8.11] establishes stratification for $\mathrm{DPerm}(G,k)$. It then follows from our bootstrap theorem Corollary 17.4 together with Theorem 19.10 that $\mathrm{DPerm}(G,k)$ is costratified as well. □

20. Homotopical examples

Galois descent and ascent.

20.1. *Definition.* Let G be a compact Lie group and consider a morphism $f\colon A \to B$ of commutative ring spectra, where B is equipped with an A-linear G-action. Following Rognes [Rog08], the map f is a *Galois extension with Galois group* G (or simply a G*-Galois extension*) if it satisfies the following two conditions:

(a) the canonical map $A \to B^{hG}$ is an equivalence;

(b) the canonical map $B \otimes_A B \to \mathsf{hom}(G_+, B)$ is an equivalence.

A Galois extension $f\colon A \to B$ is said to be *faithful* if $f^*\colon \operatorname{Mod}(A) \to \operatorname{Mod}(B)$ is conservative.

20.2. *Remark.* In forthcoming joint work with Naumann and Pol [BCH+23b], we investigate general descent properties for stratification along conservative geometric functors. Combined with our bootstrap theorem, we obtain Galois descent for costratification:

20.3. **Proposition** (Galois descent)**.** *Let $f\colon A \to B$ be a faithful G-Galois extension for G a compact Lie group. If* $\operatorname{Mod}(B)$ *is costratified, then so is* $\operatorname{Mod}(A)$*.*

Proof. Since $\operatorname{Mod}(B)$ is costratified, it is also stratified by Theorem 7.19. It then follows from Galois descent for stratification, proved in [BCH+23b], that $\operatorname{Mod}(A)$ is also stratified. The bootstrap theorem (Corollary 17.4) thus gives the claim. □

20.4. *Example.* There are numerous examples that this result applies to. For example, the complexification map $KO \to KU$ from real K-theory to complex K-theory is a faithful C_2-Galois extension [Rog08, Proposition 5.3.1] and $\operatorname{Mod}(KU)$ is costratified by Theorem 18.15. Hence, costratification descends, and we deduce that $\operatorname{Mod}(KO)$ is costratified. Similarly, if E_n denotes the Lubin–Tate spectrum, then $\operatorname{Mod}(E_n)$ is costratified by Theorem 18.15 (see Remark 20.16 below) and we deduce from [Rog08, Theorem 5.4.4] and [HMS17, Proposition 3.6] that $\operatorname{Mod}(E_n^{hG})$ is costratified for any finite subgroup $G \subseteq \mathbb{G}_n$ of the Morava stabilizer group.

20.5. *Remark.* Under stronger conditions on the Galois group, there is a converse to Proposition 20.3 which we establish as Proposition 20.12 below. The proof follows the strategy used in [Bar22] to establish the analogous result for stratification. This in turn was inspired by ideas developed in [Mat15]. First we need some general lemmas.

20.6. *Notation.* Let $\mathcal{T}$ be a rigidly-compactly generated tt-category and write $\mathcal{C}\mathrm{olocid}(\mathcal{T})$ for the class of colocalizing coideals of $\mathcal{T}$. Note that for any functor $F\colon \mathcal{T} \to \mathcal{S}$, we have a function

$$\mathcal{C}\mathrm{olocid}(\mathcal{T}) \xrightarrow{F} \mathcal{C}\mathrm{olocid}(\mathcal{S})$$

which sends a colocalizing coideal $\mathcal{C}$ to the colocalizing coideal generated by $F(\mathcal{C})$.

20.7. *Remark.* If $G\colon \mathcal{S} \to \mathcal{R}$ is a functor with the property that colocalizing coideals of $\mathcal{R}$ pull back to colocalizing coideals of $\mathcal{S}$, then $G(\operatorname{Colocid}\langle\mathcal{E}\rangle) \subseteq \operatorname{Colocid}\langle G(\mathcal{E})\rangle$ for any collection of objects $\mathcal{E} \subseteq \mathcal{S}$. It follows that for any functor $F\colon \mathcal{T} \to \mathcal{S}$, the diagram

$$\begin{tikzcd} \mathcal{C}\mathrm{olocid}(\mathcal{T}) \arrow[r, "F"'] \arrow[rr, bend left, "G\circ F"] & \mathcal{C}\mathrm{olocid}(\mathcal{S}) \arrow[r, "G"'] & \mathcal{C}\mathrm{olocid}(\mathcal{R}) \end{tikzcd}$$

commutes.

20.8. *Example.* Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a geometric functor. The isomorphism (13.6) implies that colocaling coideals pull back along the product-preserving exact functor $f^!\colon \mathcal{T} \to \mathcal{S}$. If f_* is conservative, then the same is true for the functor $f_*\colon \mathcal{S} \to \mathcal{T}$. Indeed, given a colocalizing coideal $\mathcal{C}$ of $\mathcal{T}$, we have

$$\begin{aligned} \mathsf{hom}(\mathcal{S}, f_*^{-1}(\mathcal{C})) &= \mathsf{hom}(\mathrm{Loc}\langle f^*(\mathcal{T}^c)\rangle, f_*^{-1}(\mathcal{C})) && \text{(Remark 14.5)} \\ &\subseteq \mathrm{Coloc}\langle \mathsf{hom}(f^*(\mathcal{T}^c), f_*^{-1}(\mathcal{C}))\rangle && (2.8) \\ &\subseteq \mathrm{Coloc}\langle f_*^{-1}(\mathcal{C})\rangle = f_*^{-1}(\mathcal{C}) \end{aligned}$$

where the last inclusion uses the isomorphism $f_*\mathsf{hom}(f^*(a), b) \simeq \mathsf{hom}(a, f_*(b))$ from [BDS16, (2.17)].

20.9. **Lemma.** *Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a weakly descendable (Definition 13.23) geometric functor whose right adjoint f_* is conservative. Then the composite*

$$\mathcal{C}\mathrm{olocid}(\mathcal{T}) \xrightarrow{f^!} \mathcal{C}\mathrm{olocid}(\mathcal{S}) \xrightarrow{f_*} \mathcal{C}\mathrm{olocid}(\mathcal{T})$$

is the identity.

Proof. Since f_* is conservative, Remark 20.7 and Example 20.8 imply that the above composite sends a colocalizing coideal $\mathcal{D}$ of $\mathcal{T}$ to

$$\mathrm{Colocid}\langle f_*f^!(\mathcal{D})\rangle = \mathrm{Colocid}\langle \mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), \mathcal{D})\rangle.$$

Certainly $\mathrm{Colocid}\langle \mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), \mathcal{D})\rangle \subseteq \mathcal{D}$. On the other hand, if f^* is weakly descendable, then

$$\mathcal{D} = \mathsf{hom}(\mathbb{1}_{\mathcal{T}}, \mathcal{D}) \subseteq \mathsf{hom}(\mathrm{Locid}\langle f_*(\mathbb{1}_{\mathcal{S}})\rangle, \mathcal{D}) \subseteq \mathrm{Colocid}\langle \mathsf{hom}(f_*(\mathbb{1}_{\mathcal{S}}), \mathcal{D})\rangle$$

by (2.8) and the proof is complete. ∎

20.10. *Example.* A morphism $f\colon A \to B$ of commutative ring spectra is called *descendable* if $\mathrm{thickid}\langle B\rangle = \mathrm{Mod}(A)$ in $\mathrm{Mod}(A)$. Then the base-change functor $f^*\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is weakly descendable in the sense of Definition 13.23 and its right adjoint is conservative. Hence Lemma 20.9 applies. For example, a faithful G-Galois extension $f\colon A \to B$ is descendable. This follows from the fact that B is dualizable as an A-module (by [Rog08, Proposition 6.2.1]). Since $B \otimes -$ is conservative on $\mathrm{Mod}(A)$, $B \otimes f_{\mathrm{supp}(B)} = 0$ implies $f_{\mathrm{supp}(B)} = 0$ so that $\mathrm{supp}(B) = \mathrm{Spc}(\mathrm{Mod}(A)^c)$; hence $A \in \mathrm{thickid}\langle B\rangle$ follows from the classification of thick ideals of compact objects. A more high-powered argument is given in [Mat16, Theorem 3.38].

20.11. **Lemma.** *Suppose $f\colon A \to B$ is a faithful G-Galois extension with G a connected compact Lie group. Then $f^!$ induces a bijection*

$$f^!\colon \mathcal{C}\mathrm{olocid}(\mathrm{Mod}(A)) \xrightarrow{\cong} \mathcal{C}\mathrm{olocid}(\mathrm{Mod}(B)).$$

Proof. The following pushout square of commutative ring spectra induces a commutative diagram of forgetful functors, as displayed on the right:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "g_1"] \\ B \arrow[r, "g_2"'] & B \otimes_A B \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Mod}(B \otimes_A B) \arrow[r, "(g_1)_*"] \arrow[d, "(g_2)_*"'] & \mathrm{Mod}(B) \arrow[d, "f_*"] \\ \mathrm{Mod}(B) \arrow[r, "f_*"'] & \mathrm{Mod}(A). \end{tikzcd}$$

The latter square is horizontally right adjointable, i.e., the corresponding Beck–Chevalley transformation $(g_2)_* \circ g_1^! \to f^! \circ f_*$ is a natural equivalence. Hence the following diagram is commutative

$$\begin{tikzcd} \mathrm{Mod}(B) \arrow[r, "g_1^!"] \arrow[d, "f_*"'] & \mathrm{Mod}(B \otimes_A B) \arrow[d, "(g_2)_*"] \\ \mathrm{Mod}(A) \arrow[r, "f^!"'] & \mathrm{Mod}(B). \end{tikzcd}$$

By Remark 20.7 and Example 20.8, it induces a commutative square

$$\begin{tikzcd} \mathcal{C}\mathrm{olocid}(\mathrm{Mod}(B)) \arrow[r, "g_1^!"] \arrow[d, "f_*"'] & \mathcal{C}\mathrm{olocid}(\mathrm{Mod}(B \otimes_A B)) \arrow[d, "(g_2)_*"] \\ \mathcal{C}\mathrm{olocid}(\mathrm{Mod}(A)) \arrow[r, "f^!"'] & \mathcal{C}\mathrm{olocid}(\mathrm{Mod}(B)). \end{tikzcd}$$

The morphism f is descendable (Example 20.10) and consequently g_1 and g_2 are descendable, too. Hence, by Lemma 20.9, the maps $f^!$ and $g_1^!$ are (split) injective while f_* and $(g_2)_*$ are (split) surjective. The assumption on G guarantees that the canonical morphism $h\colon B \otimes_A B \to B$ is descendable as well (see [Mat16, Proposition 3.36]), so h_* is (split) surjective using Lemma 20.9 once more. Furthermore, $h \circ g_2 \simeq \mathrm{id}$ on B, so $(g_2)_* \circ h_* = \mathrm{id}$ as maps on $\mathcal{C}\mathrm{olocid}(\mathrm{Mod}(B))$. This shows that h_* is also injective, hence both $(g_2)_*$ and h_* are in fact bijections. It follows from the commutative square above that f_* is a bijection as well. □

20.12. **Proposition** (Galois ascent). *Let $f\colon A \to B$ be a faithful G-Galois extension with G a connected compact Lie group. Then f^* induces a homeomorphism*

$$\varphi\colon \mathrm{Spc}(\mathrm{Mod}(B)^c) \xrightarrow{\cong} \mathrm{Spc}(\mathrm{Mod}(A)^c).$$

Moreover, if $\mathrm{Mod}(A)$ is costratified, then so is $\mathrm{Mod}(B)$.

Proof. The statement about the Balmer spectra is part of [BCH+23b], so assume that $\mathrm{Mod}(A)$ is costratified. We have a diagram

$$\begin{tikzcd} \mathcal{C}\mathrm{olocid}(\mathrm{Mod}(A)) \arrow[r, "f^!"] \arrow[d, "\mathrm{Cosupp}"', "\cong"] & \mathcal{C}\mathrm{olocid}(\mathrm{Mod}(B)) \arrow[d, "\mathrm{Cosupp}"] \\ \mathcal{P}(\mathrm{Spc}(\mathrm{Mod}(A)^c)) \arrow[r, "\cong", "\varphi^{-1}"'] & \mathcal{P}(\mathrm{Spc}(\mathrm{Mod}(B)^c)), \end{tikzcd}$$

which commutes by Corollary 14.19 and Theorem 7.19. The top horizontal map is a bijection by Lemma 20.9, hence so is the right vertical map. In other words, $\mathrm{Mod}(B)$ is costratified. □

20.13. *Remark.* As a consequence we can complete our proof of Theorem 19.10 by giving the following alternative proof of [BIK12, Theorem 11.6]:

20.14. **Theorem** (Benson–Iyengar–Krause). *Let E be an elementary abelian p-group and k a field of characteristic p. Then the category $\mathrm{Rep}(E, k)$ is costratified, with spectrum $\mathrm{Spc}(\mathrm{Rep}(E,k)^c) \cong \mathrm{Proj}\, H^*(E;k)$.*

Proof. Let E be an elementary abelian subgroup of rank $r \geq 1$. By [Mat15, Proposition 3.9], the inclusion $E \cong (\mathbb{Z}/p)^{\times} \subseteq (S^1)^{\times r} =: \mathbb{T}$ induces a faithful $\mathbb{T}$-Galois extension $f \colon k^{h\mathbb{T}} \to k^{hE}$. Note that

$$\pi_* k^{h\mathbb{T}} \cong k[x_1, \ldots, x_r]$$

with all generators x_i in degree 2, so $\mathrm{Mod}(k^{h\mathbb{T}})$ is an affine weakly regular tt-category. Therefore, $\mathrm{Mod}(k^{h\mathbb{T}})$ is costratified by Theorem 18.15. Since f satisfies the assumptions of Proposition 20.12, Galois ascent implies that $\mathrm{Mod}(k^{hE})$ is costratified as well. Finally, we observe that there is an equivalence of tt-categories

$$\mathrm{Mod}(k^{hE}) \simeq \mathrm{Rep}(E, k),$$

so $\mathrm{Rep}(E, k)$ is costratified, too. □

Chromatic homotopy theory.

20.15. *Definition.* For a fixed prime number p, let $\mathcal{S}$ denote the p-local stable homotopy category, let E_n denote the nth *Lubin–Tate spectrum*, and let $\mathcal{S}_{E_n}$ denote the category of E_n-local spectra. Recall that E_n is a commutative ring spectrum Bousfield equivalent to the nth Johnson–Wilson spectrum $E(n)$. For further background material on chromatic homotopy theory, we refer the interested reader to [BB19].

20.16. *Remark.* We write $L_n \colon \mathcal{S} \to \mathcal{S}_{E_n}$ for the corresponding Bousfield localization functor. This is a smashing localization, and as such, we have $\mathcal{S}_{E_n} \simeq \mathrm{Mod}(L_n S^0)$ and the localization $L_n \colon \mathcal{S} \to \mathrm{Mod}(L_n S^0)$ is given by base change along $S^0 \to L_n S^0$.

20.17. *Remark.* We have

$$\pi_*(E_n) \cong W(\mathbb{F}_{p^n})[\![u_1, \ldots, u_n]\!][u^{\pm 1}]$$

where $|u_i| = 0$ and $|u| = -2$. In particular, $\mathrm{Mod}(E_n)$ is affine weakly regular (Definition 18.10) and hence $\mathrm{Mod}(E_n)$ is costratified by Theorem 18.15.

20.18. *Notation.* For each $0 \leq h \leq n$, we define

$$\mathcal{P}_h \coloneqq \left\{ x \in \mathcal{S}_{E_n}^c \,\middle|\, K(h)_*(x) = 0 \right\},$$

a prime ideal of $\mathcal{S}_{E_n}^c$. The following is a restatement of [HS99, Theorem 6.9], as given in [BHN22, Proposition 3.5].

20.19. **Theorem** (Hovey–Strickland)**.** *The spectrum*

$$\mathrm{Spc}(\mathcal{S}_{E_n}^c) = \mathcal{P}_n - \cdots - \mathcal{P}_1 - \mathcal{P}_0$$

is a local irreducible space consisting of $n+1$ points, where closure goes to the left: $\overline{\{\mathcal{P}_h\}} = \left\{ \mathcal{P}_k \,\middle|\, h \leq k \leq n \right\}$.

20.20. *Remark.* The following result recovers the classification of colocalizing subcategories of $\mathcal{S}_{E_n}$ given in [HS99, Theorem 6.14].

20.21. **Theorem** (Hovey–Strickland)**.** *The category of E_n-local spectra $\mathcal{S}_{E_n}$ is costratified.*

Proof. Consider the geometric functor $f^* \colon \mathcal{S}_{E_n} \to \mathrm{Mod}(E_n)$ given by base-change. It is a consequence of the Hopkins–Ravenel smash product theorem [Rav92, Chapter 8] that this functor is conservative (see also [Mat16, Prop. 3.18 and Thm. 4.17]). Moreover, $\mathrm{Mod}(E_n)$ is costratified (Remark 20.17) and $\mathcal{S}_{E_n}$ is stratified by [BHS23, Theorem 10.14]. By Corollary 17.4, we conclude that $\mathcal{S}_{E_n}$ is costratified. □

20.22. *Remark* (Chromatic cosupport). In [BHS23, Proposition 10.12], we established an isomorphism $g_{\mathcal{P}_k} \simeq M_k S^0$ between the Balmer–Favi idempotent and the fiber of $L_k S^0 \to L_{k-1} S^0$. It follows that

$$\begin{aligned}\mathrm{Cosupp}(t) &= \{\mathcal{P}_k \in \mathrm{Spc}(\mathcal{S}^c_{E(n)}) \mid \mathsf{hom}(g_{\mathcal{P}_k}, t) \neq 0\}\\ &= \{k \in \{0, \dots, n\} \mid \mathsf{hom}(M_k S^0, t) \neq 0\}.\end{aligned}$$

As recalled in Example 3.6, Hovey–Strickland define the chromatic cosupport by

$$\text{co-supp}(t) = \left\{ k \in \{0, \dots, n\} \,\middle|\, \mathsf{hom}(K(k), t) \neq 0 \right\}.$$

We claim that $\text{co-supp}(t) = \mathrm{Cosupp}(t)$ for all $t \in \mathcal{S}_{E_n}$. Indeed, suppose that $\mathsf{hom}(M_k S^0, t) = 0$. Then

$$\mathsf{hom}(K(k), \mathsf{hom}(M_k S^0, t)) \simeq \mathsf{hom}(K(k) \otimes M_k S^0, t) = 0$$

but $K(k) \otimes M_k S^0 \simeq M_k K(k) \simeq K(k)$, so $\mathsf{hom}(K(k), t) = 0$. Conversely, suppose that $\mathsf{hom}(K(k), t) = 0$. The collection of $Y \in \mathcal{S}_{E_n}$ for which $\mathsf{hom}(Y, t) = 0$ is a localizing subcategory which contains $K(k)$, and hence also contains $M_k S^0$ by [HS99, Proposition 6.17]. Therefore, $\mathsf{hom}(M_k S^0, t) = 0$ as well. Therefore the Balmer–Favi notion of cosupport agrees with the usual version of chromatic cosupport. Alternatively, one can prove this using Corollary 11.10 and [HS99, Theorem 6.14].

Cochain algebras.

20.23. *Definition.* For a space X, we let $C^*(X; \mathbb{F}_p)$ denote the function spectrum $F(\Sigma^\infty_+ X, \mathbb{F}_p)$.

20.24. *Remark.* In [BCHV22] we investigated when $\mathrm{Mod}(C^*(X; \mathbb{F}_p))$ is stratified in the sense of BIK. This used a certain category $\mathcal{E}(X)$ associated to X, whose objects are (isomorphism classes of) pairs (V, ϕ) consisting of an elementary abelian p-group E and a finite morphism $\phi\colon H^*(X; \mathbb{F}_p) \to H^*(BE; \mathbb{F}_p)$ of unstable algebras over the dual Steenrod algebra. We showed that these maps ϕ lift to maps on the level of cochains, and therefore we have a map of commutative ring spectra

$$\rho_X \colon C^*(X; \mathbb{F}_p) \to \prod_{(E,\phi)\in\mathcal{E}(X)} C^*(BE; \mathbb{F}_p). \tag{20.25}$$

20.26. *Definition.* A p-good[6] connected topological space X with noetherian mod p cohomology is said to satisfy *Chouinard's condition* if induction and coinduction along the map (20.25) are conservative.

20.27. **Theorem.** *Let X be a p-good connected space with noetherian mod p cohomology. Then the following are equivalent:*

(a) $\mathrm{Mod}(C^*(X; \mathbb{F}_p))$ *is stratified.*
(b) $\mathrm{Mod}(C^*(X; \mathbb{F}_p))$ *is costratified.*
(c) X *satisfies Chouinard's condition.*

If any of these holds, then $\mathrm{Spc}(\mathrm{Mod}(C^*(X; \mathbb{F}_p))^c) \cong \mathrm{Spec}^h(H^*(X; \mathbb{F}_p))$.

Proof. We have that (a) is equivalent to (c) by [BCHV22, Theorem 5.12] and (b) implies (a) by Theorem 7.19. These also imply claim (b). Assume then that (a), and hence (c), hold and consider the base change functor

$$f^* \colon \mathrm{Mod}(C^*(X; \mathbb{F}_p)) \to \prod_{(E,\phi)\in\mathcal{E}(X)} \mathrm{Mod}(C^*(BE; \mathbb{F}_p)).$$

[6] in the sense of Bousfield–Kan [BK72].

Since the product is finite (which is a consequence of [Rec84, p. 194]), the target category is costratified by [BIK12, Theorem 11.6] and Example 18.16. Moreover, f^* is a conservative geometric functor and $\mathrm{Mod}(C^*(X;\mathbb{F}_p))$ is stratified. Now apply Corollary 17.4. ☐

20.28. *Example.* Let $X = S^3\langle 3\rangle$ denote the 3-connected cover of S^3. In this case, there is a single non-trivial object in the category $\mathcal{E}(S^3\langle 3\rangle)$, namely the pair coming from the composite $B\mathbb{Z}/p \to BS^1 \to S^3\langle 3\rangle$. In [BCHV19, Example 5.16] it is shown that

$$\mathrm{Mod}(C^*(S^3\langle 3\rangle;\mathbb{F}_p)) \to \mathrm{Mod}(C^*(B\mathbb{Z}/p;\mathbb{F}_p))$$

satisfies Chouinard's condition. We deduce that $\mathrm{Mod}(C^*(S^3\langle 3\rangle;\mathbb{F}_p))$ is costratified. Note that in our previous work, we were unable to prove this; see [BCHV19, Section 4.5] and the discussion therein.

20.29. *Example.* If X is a noetherian H-space, then X satisfies Chouniard's condition by [BCHV19, Theorem 5.15]. Therefore, $\mathrm{Mod}(C^*(X;\mathbb{F}_p))$ is costratified.

Equivariant homotopy theory.

20.30. *Notation.* Let Sp_G denote the ∞-category of genuine G-spectra. For a commutative algebra $\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp}_G)$, we write $\mathrm{Mod}_{\mathrm{Sp}_G}(\mathbb{E})$ for the ∞-category of modules and $\mathrm{D}_G(\mathbb{E})$ for the associated homotopy category.

20.31. **Theorem.** *Let G be a finite group and let $\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp}_G)$. Suppose that the following conditions hold for all subgroups $H \le G$:*

(a) $\mathrm{Spc}(\mathrm{D}(\Phi^H\mathbb{E})^c)$ *is noetherian; and*
(b) $\mathrm{D}(\Phi^H\mathbb{E})$ *is costratified.*

Then $\mathrm{D}_G(\mathbb{E})$ is costratified and $\mathrm{Spc}(\mathrm{D}_G(\mathbb{E})^c)$ is noetherian.

Proof. We first observe that by Theorem 7.19, condition (b) implies that each $\mathrm{D}(\Phi^H\mathbb{E})$ is stratified. Then [BCH+23a, Theorem 3.33] shows that $\mathrm{D}_G(\mathbb{E})$ is stratified with noetherian spectrum $\mathrm{Spc}(\mathrm{D}_G(\mathbb{E})^c)$. In order to show that it is costratified, we use the bootstrap theorem (Corollary 17.4). Indeed, let

$$\Phi\colon\ \mathrm{D}_G(\mathbb{E}) \xrightarrow{\{\Phi^H\}_H} \prod_{H\le G} \mathrm{D}(\Phi^H\mathbb{E})$$

denote the product of geometric fixed point functors. By [BCH+23a, Proposition 3.25], Φ is a conservative geometric functor. By assumption, each $\mathrm{D}(\Phi^H\mathbb{E})$ is costratified, and hence so is the product (Example 18.16) and we have already seen that $\mathrm{D}_G(\mathbb{E})$ is stratified. Now apply Corollary 17.4. ☐

20.32. *Remark.* The generalized Quillen stratification theorem proven in [BCH+23a, Theorem 4.3] describes the underlying set of $\mathrm{Spc}(\mathrm{D}_G(\mathbb{E})^c)$ in terms of the spectra of the geometric fixed points $\mathrm{D}(\Phi^H\mathbb{E})$ along with the Weyl group actions.

20.33. *Remark.* Let $\mathcal{F}$ denote a family of subgroups of G. Recall that $\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp}_G)$ is $\mathcal{F}$-nilpotent if $\mathbb{E}$ is in the thick ideal of Sp_G generated by $\{\, G/H_+ \mid H \in \mathcal{F}\,\}$. In this case, we have $\Phi^H\mathbb{E} = 0$ whenever $\mathbb{E} \notin \mathcal{F}$ by [MNN17, Theorem 6.41], and in particular it suffices to check the conditions of Theorem 20.31 for $H \in \mathcal{F}$. There is always a minimal such family $\mathcal{F}$ known as the *derived defect base* of E. See [MNN19] for a computation of the derived defect base of many equivariant spectra.

20.34. *Example.* There is a canonical geometric functor $\mathrm{triv}_G\colon \mathrm{Sp} \to \mathrm{Sp}_G$ that sends a spectrum to the corresponding G-spectrum with trivial G-action. See [PSW22, Section 3] for further details. For $\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp})$, let $\mathbb{E}_G \coloneqq \mathrm{triv}_G\,\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp}_G)$ denote the corresponding commutative algebra in genuine G-spectra. For each $H \le G$, we have that $\Phi^H \mathbb{E}_G \simeq \mathbb{E}$. In particular, Theorem 20.31 implies the following:

20.35. **Corollary.** *Let $\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp})$ be a commutative ring spectrum such that $\mathrm{Spc}(\mathrm{D}(\mathbb{E})^c)$ is noetherian. If $\mathrm{D}(\mathbb{E})$ is costratified, then $\mathrm{D}_G(\mathbb{E}_G)$ is costratified for any finite group G.*

20.36. **Corollary.** *For any finite group G and discrete commutative ring R, the category of derived Mackey functors $\mathrm{D}_G(HR_G)$ is costratified.*

Proof. Using Example 19.2, we can apply Corollary 20.35 to $\mathbb{E} = HR$. □

20.37. *Remark.* By [PSW22, Proposition 4.9] and [BHS23, Proposition 14.3], there is an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Mack}_G(\mathbb{E}) \simeq \mathrm{Mod}_{\mathrm{Sp}_G}(\mathbb{E}_G)$$

where $\mathrm{Mack}_G(\mathbb{E})$ denotes Barwick's category of spectral G-Mackey functors with $\mathbb{E}$-coefficients [Bar17]. In particular, the previous result shows that HR-valued spectral Mackey functors are costratified for any discrete noetherian commutative ring R. For $R = \mathbb{Z}$, the corresponding spectrum was determined completely in [PSW22].

20.38. *Example.* Taking $\mathbb{E} = L_n S^0$ (Remark 20.16) we obtain the category $\mathrm{SH}_{G,E_n} \coloneqq \mathrm{D}_G(\mathbb{E}_G)$ of E_n-local spectral Mackey functors; see [BHS23, Example 13.15]. Equivalently, this is the category of spectral Mackey functors with coefficients in $\mathcal{S}_{E_n}$; see [BHS23, Example 14.4]. As explained in *loc. cit.*, the underlying set of the spectrum of SH_{G,E_n} is known, thanks to Theorem 20.19.

20.39. **Corollary.** *For any finite group G, prime number p, and $0 \le n < \infty$, the category of E_n-local spectral Mackey functors SH_{G,E_n} is costratified.*

Proof. This follows from Corollary 20.35 with $\mathbb{E} = L_n S^0$, using Theorem 20.21. □

20.40. *Remark.* Given a non-equivariant spectrum $\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp})$, there is another way to produce an equivariant spectrum, namely by taking the associated Borel equivariant spectrum $b_G\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp}_G)$; see [MNN17, Section 6.3].

20.41. **Theorem.** *For any finite group G, the category $\mathrm{D}_G(b_G E_n)$ is costratified by $\mathrm{Spc}(\mathrm{D}_G(b_G E_n)^c) \cong \mathrm{Spec}(E_n^0(BG))$.*

Proof. It is shown in [BCH+23a, Theorem 6.14] that $\Phi^H(b_G E_n)$ is nonzero only when H is an abelian p-group of rank at most n, in which case its homotopy groups are a regular noetherian even-periodic ring. In particular, each $\mathrm{D}(\Phi^H(b_G E_n))$ is costratified with noetherian spectrum by Theorem 18.15. By Theorem 20.31, $\mathrm{D}_G(b_G E_n)$ is costratified by $\mathrm{Spc}(\mathrm{D}_G(b_G E_n)^c)$, which is homeomorphic to $\mathrm{Spec}(E^0(BG))$ by [BCH+23a, Lemma 7.3]. □

20.42. *Remark.* Our next example comes from the category of modules associated to equivariant complex K-theory KU_G for a finite group G; see [Seg68]. We let $R(G)$ denote the complex representation ring associated to G.

20.43. **Theorem.** *For any finite group G, the category $\mathrm{D}_G(KU_G)$ is costratified by $\mathrm{Spc}(\mathrm{D}_G(KU_G)^c) \cong \mathrm{Spec}(R(G))$.*

Proof. This is similar to the previous proof. It is shown in [BCH⁺23a, Lemma 8.6] that $\Phi^H(KU_G)$ is nonzero only when H is a cyclic subgroup of G, in which case its homotopy groups are a regular noetherian even-periodic ring. In particular, each $\mathrm{D}(\Phi^H(KU_G))$ is costratified with noetherian spectrum by Theorem 18.15. By Theorem 20.31, $\mathrm{D}_G(KU_G)$ is costratified by $\mathrm{Spc}(\mathrm{D}_G(KU_G)^c)$, which is homeomorphic to $\mathrm{Spec}(R(G))$ by [BCH⁺23a, Lemma 8.11]. □

20.44. *Remark.* For a compact Lie group G, let $\mathrm{SH}_{G,\mathbb{Q}}$ denote the stable homotopy category of rational G-equivariant spectra. This is a rigidly-compactly generated tt-category which is stratified; see [Gre19] or [BHS23, Theorem 12.22]. Our final goal is to prove that it is costratified.

20.45. *Remark.* Recall that, by definition, L is *cotoral* in K if L is a normal subgroup of K and K/L is a torus. Moreover, for each $H \le G$ we have geometric fixed point functors

$$\Phi^H\colon \mathrm{SH}_{G,\mathbb{Q}} \to \mathrm{SH}_{\mathbb{Q}}$$

which are jointly conservative (for example, [Sch18, Proposition 3.3.10]). Since the spectrum of $\mathrm{SH}_{\mathbb{Q}} \simeq \mathrm{D}(\mathbb{Q})$ is a single point, we obtain a prime ideal $\mathfrak{p}_H \in \mathrm{Spc}(\mathrm{SH}_{G,\mathbb{Q}})$ for each $H \le G$. Up to conjugacy, these turn out to be all the prime ideals:

20.46. **Theorem** (Greenlees)**.** *Let G be a compact Lie group. Then as a set*

$$\mathrm{Spc}(\mathrm{SH}^c_{G,\mathbb{Q}}) = \{\, \mathfrak{p}_H \mid (H) \text{ conjugacy class of closed subgroups in } G \,\}.$$

The specialization order is determined by cotoral inclusions:

$$\mathfrak{p}_K \subseteq \mathfrak{p}_H \text{ if and only if } K \text{ is conjugate to a subgroup cotoral in } H.$$

The topology on $\mathrm{Spc}(\mathrm{SH}^c_{G,\mathbb{Q}})$ is the “Zariski topology on the f-topology” of [Gre98].

20.47. *Remark.* The space $\mathrm{Spc}(\mathrm{SH}^c_{G,\mathbb{Q}})$ is weakly noetherian by [BHS23, Lem. 12.12].

20.48. **Theorem.** *For any compact Lie group G, the category of rational G-spectra is costratified.*

Proof. We apply bootstrap to the collection of geometric fixed point functors $\Phi^H\colon \mathrm{SH}_{G,\mathbb{Q}} \to \mathrm{SH}_{\mathbb{Q}} \simeq \mathrm{D}(\mathbb{Q})$. The target category is costratified (Corollary 18.9), while $\mathrm{SH}_{G,\mathbb{Q}}$ is stratified (Remark 20.44). Therefore, by Theorem 17.7 and Theorem 20.46 we deduce that $\mathrm{Spc}(\mathrm{SH}_{G,\mathbb{Q}})$ is costratified. □

21. Open questions

We collect here some questions we do not know the answer to.

21.1. *Question.* Does the detection property always hold? (See Remark 6.6.)

21.2. *Question.* Does stratification imply costratification? (See Remark 7.22.)

21.3. *Question.* What is the cosupport of $g_{\mathcal{P}}$? More precisely, in what generality is it true that $\mathrm{Cosupp}(g_{\mathcal{P}}) = \mathrm{gen}(\mathcal{P})$? (See Lemma 4.26.)

21.4. *Question.* Does the (co)detection property always hold for weak (co)rings?

21.5. *Question.* A rigidly-compactly generated tt-category $\mathcal{T}$ satisfies the telescope conjecture if it is stratified with generically noetherian spectrum; cf. [BHS23, Theorem 9.11]. Can this result be improved if $\mathcal{T}$ is costratified?

21.6. *Question.* Does there exist a geometric functor $f^*\colon \mathcal{T} \to \mathcal{S}$ between rigidly-compactly generated tt-categories such that f^* is conservative but $f^!$ is not? (See Corollary 14.24 and the results cited in its proof.)

21.7. *Question.* Is it true that if $f^*\colon \mathcal{T} \to \mathcal{S}$ is any geometric functor and $\mathcal{S}$ satisfies the local-to-global principle, then $\mathcal{T}$ satisfies the local-to-global principle for objects $t \in \mathcal{T}$ with $\operatorname{Supp}(t) \subseteq \operatorname{im}\varphi$? It suffices to prove it under the additional assumption that f_* is conservative. (See Section 15.)

21.8. *Question.* Does the local-to-global principle hold for an object $t \in \mathcal{T}$ if its support $\operatorname{Supp}(t)$ is noetherian?

21.9. *Question.* Is there a characterization of when the local-to-global principle holds purely in terms of the topology of $\operatorname{Spc}(\mathcal{T}^c)$?

21.10. *Question.* Does stratification *always* descend along a conservative geometric functor $f^*\colon \mathcal{T} \to \mathcal{S}$ between rigidly-compactly generated tt-categories? If so, our bootstrap theorem (Corollary 17.4) implies that costratification always descends as well.

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