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ON THE ALGEBRAIC K-THEORY OF SIMPLY CONNECTED SPACES

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1. Introduction. This paper combines the cyclotomic trace invariant of [BHM] with the calculus of functors [G1] to evaluate Waldhausen's (reduced) functor $\tilde{A}(X)$ in terms of more familiar objects in algebraic topology, in the case of a simply connected X.

For each prime p, the cyclotomic trace gives a map of spectra

$$\mathrm{Trc}: A(X) \to \mathrm{TC}(X; p).$$

Here $A(X)$ denotes the version of Waldhausen's functor with $\pi_0 A(X) = Z$ rather than $\pi_0 A(X) = K_0(Z\pi_1 X)$.

There is a stable map from $\mathrm{TC}(X; p)$ to the suspension spectrum of the free loop space $\mathscr{L}X = \mathrm{Map}(S^1, X)$, with a disjoint base point added, and the composition with Trc is the topological Dennis trace of [B]. More generally, after p-adic completion the spectrum $\mathrm{TC}(X; p)$ was described completely in [BHM, Sect. 5]. The argument there is only correct in the case when $\pi_1 X$ is finite. However, the result is true in the general case by an argument due to Goodwillie. For a discussion of this, see [M]. We recall the result.

The self-maps of the circle, in particular, the rotation group S^1 and the degree p-map $\Delta_p(z) = z^p$, act on $\mathscr{L}X$ and therefore on the spectrum $\Sigma_+^\infty(\mathscr{L}X)$. After p-completion there is a fiber square (= homotopy cartesian diagram):

$$\begin{array}{ccc} \mathrm{TC}(X;p)_p^\wedge & \xrightarrow{\alpha} & \Sigma_+^\infty(S^1 \wedge (ES^1 \times_{S^1} \mathscr{L}X_+))_p^\wedge \\ \downarrow \beta & & \downarrow \mathrm{Trf} \\ \Sigma_+^\infty(\mathscr{L}X)_p^\wedge & \xrightarrow{1-\Delta_p} & \Sigma_+^\infty(\mathscr{L}X)_p^\wedge . \end{array} \tag{1.1}$$

The right-hand vertical map is the S^1-transfer and $\Sigma_+^\infty(Y)$ denotes the spectrum whose ith space is equal to

$$\lim_{\to} \Omega^n(S^{n+i} \wedge Y_+), \qquad Y_+ = Y \coprod \{+\}.$$

The composition of β and Trc is the p-completion of the topological Dennis trace.

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THEOREM 1.2. *Suppose X is simply connected, of finite type, and p is a prime. Then there is a fiber square*

$$\begin{array}{ccc} A(X)^\wedge_p & \xrightarrow{\text{Trc}} & \text{TC}(X;p)^\wedge_p \\ \downarrow & & \downarrow \\ A(*)^\wedge_p & \xrightarrow{\text{Trc}} & \text{TC}(*;p)^\wedge_p . \end{array}$$

We see that the reduced functors $\tilde{A}(X)$ and $\widetilde{\text{TC}}(X;p)$ agree after p-completion. Moreover, we shall argue that

$$\text{TC}(X;p)^\wedge_p \simeq \Sigma^\infty_+(X)^\wedge_p \times V(X)^\wedge_p$$

with

$$V(X) = \text{fiber}(\Sigma^\infty(S^1 \wedge (ES^1 \times_{S^1} \mathscr{L}X_+)) \to \Sigma^\infty_+(X))$$

the homotopy fiber of the S^1-transfer map composed with the map $e_1 \colon \mathscr{L}X \to X$, which evaluates a free loop at $1 \in S^1$. A theorem of Waldhausen gives a decomposition

$$A(X) \simeq \Sigma^\infty_+(X) \times \text{Wh}(X),$$

and we therefore have the following result.

COROLLARY 1.3. *For a simply connected space of finite type X, and every prime p, $\widetilde{\text{Wh}}(X)^\wedge_p \simeq \tilde{V}(X)^\wedge_p$.*

Remark 1.4. An integral functor $T\mathscr{C}(X)$ was introduced in [G3], and the cyclotomic trace map extended to a map from $A(X)$ to $T\mathscr{C}(X)$. The p-adic completions of $T\mathscr{C}(X)$ and $\text{TC}(X)$ are equivalent, but $T\mathscr{C}(X)$ has the added advantage that its rational type agrees with that of $A(X)$, cf. [Bu]. Using this integral version one gets integral versions of the above.

The results above have consequences for diffeomorphisms of manifolds. We briefly recall the connection. Let $P(M)$ be the space of smooth pseudoisotopies of the manifold M:

$$P(M) = \text{Diff}(M \times I, M \times 0 \cup \partial M \times I).$$

It is the space of diffeomorphisms of $M \times I$, which fix (some tubular neighbourhood of) $M \times 0 \cup \partial M \times I$. There is a suspension map from $P(M)$ to $P(M \times I)$, which essentially consists of crossing a pseudoisotopy with the unit interval (and smoothing corners). The limit

$$\mathscr{P}(M) = \lim_{\to} P(M \times I^k)$$

is the stable pseudoisotopy space of M. It is an infinite loop space, and it depends only on the underlying homotopy type of M. Two main results connect $P(M)$ to $A(M)$. There is Waldhausen's theorem

$$\mathscr{P}(M) \simeq \Omega^2 \mathrm{Wh}(M) \tag{1.5}$$

(cf. [W1], [W2]), and there is the stability result of Igusa [I] that the map

$$P(M) \to \mathscr{P}(M) \tag{1.6}$$

is k-connected whenever $\dim M \geqslant \max(2k + 7, 3k + 4)$.

Let $\tilde{P}(M)$ be the pseudoisotopies of M, which are the identity on a small codim-zero disk in M. It is a consequence of the above for a simply connected manifold M that $\tilde{P}(M)$ and $\Omega^2\tilde{V}(M)$ have the same p-completed homotopy type roughly through dimensions 1/3 dim M. Since $\tilde{P}(M)$ has finite type, this implies that

$$\pi_i \tilde{P}(M) \simeq \pi_{i+2} \tilde{V}(M)$$

throughout the stability range.

One knows from [BHM] that the cyclotomic trace from $A(*)$ to $\mathrm{TC}(*; p)$ defines a split surjection after completion at p, onto a summand of the form $\Sigma^\infty_+(S^1 \wedge BO(2))$, provided p is a regular odd prime in the sense of number theory (i.e., if p does not divide the numerators of the Bernoulli numbers $B_1, \dots, B_{p-3}$). In fact the map might well be a homotopy equivalence at regular primes. In any case, the cyclotomic trace does contain information about the unreduced $P(M)$.

The rational homotopy type of $A(X)$ was calculated in [Bu]. The evaluation of the integral homotopy type in terms of more accessible functors began in [CCGH].

It should be pointed out, however, that there is a serious flaw in [CCGH] in that the trace-type invariant used there is not well defined: the error occurs in the definition of the map τ_y on page 71, where the trace map defined on p-simplices by the formula $\mathrm{tr}(a_1, \dots, a_p; b) = \mathrm{trace}(a_1 \dots a_p b)$ is not invariant under the face operator $d_\circ$ as claimed.

The bulk of the present paper, given the cyclotomic trace and calculus of functors, is the long and somewhat painful Section 4 below. It is a complete recast of [CCHG]. We finally note that Corollary 1.3 was conjectured in [CCH], without completion.

The above remarks explain the many authors of the present paper.

We offer our admiration to F. Waldhausen, who introduced $A(X)$ and discovered its close connection with differentiable pseudoisotopy theory. The paper is dedicated to him.

2. Proof of Theorem 1.2. For every functor with smash product we have stable maps (i.e., maps of spectra), cf. [BHM]:

$$
\begin{array}{ccc}
K(F) & \xrightarrow{\text{Trc}} & \mathrm{TC}(F,p) \\
 & \searrow^{\text{Tr}} \quad \swarrow_{\beta} & \\
 & \mathrm{THH}(F). &
\end{array}
\tag{2.1}
$$

If F is of the form $F = F_X$ with $F_X(A) = A \wedge \Omega X_+$ (say, the Moore loops of X, or in a simplicial setting the Kan loop group), then there are homotopy equivalences

$$
\begin{aligned}
K(F_X) &\simeq A(X) \\
\mathrm{TC}(F_X, p)_p^\wedge &\simeq \mathrm{TC}(X, p)_p^\wedge \\
\mathrm{THH}(F_X) &\simeq \Sigma_+^\infty(\mathscr{L}X).
\end{aligned}
\tag{2.2}
$$

Here the right-hand sides are Waldhausen's A-functor, the functor given by the pullback diagram (1.1) and the suspension spectrum of the free loop space, respectively. The decoration $(\ )_p^\wedge$ indicates p-adic completion in the sense of [BK].

Mostly in algebraic topology it does not really matter if maps like the above Trc or Tr are natural transformations or only natural transformations up to homotopy, i.e., natural transformations when the functors are projected into the homotopy category of spectra. However, this is not so for "calculus of functors," which is the theory we want to apply; the theorems from [G1], [G2] require "strict" natural transformations. We must argue that the cyclotomic trace with the right "models" of $K(F)$ and $\mathrm{TC}(F, p)$ is a strict natural transformation on the category of FSPs. In the end we are using only the FSPs of the form

$$F_X(Y) = Y \wedge \Omega X_+,$$

so we ask for strict transformations from spaces to spectra.

Fortunately, the machinery from [BHM] can be adapted to give strict transformations, but there are several steps, and we have to recall in part the definitions from [BHM, Section 5]. Readers who are not interested in these details can accept Proposition 2.6 below which summarizes the situation, and proceed from there with the main argument. Given F there are grouplike topological monoids $G_k = \mathrm{GL}_k(F)$. For monoids there are functors $G \mapsto G^\vee$ (free monoid) and $G \mapsto G^\wedge$ (group completion), and there are natural transformations $G \leftarrow G^\vee \rightarrow G^\wedge$, which induce the homotopy equivalences below when restricted to grouplike monoids:

$$|N_\cdot(G)| \xleftarrow{\sim} |N_\cdot(G^\vee)| \xrightarrow{\sim} |N_\cdot(G^\wedge)|$$

$$|sd_{p^n} N_\cdot^{cy}(G)|^{C_{p^n}} \xleftarrow{\sim} |sd_{p^n} N_\cdot^{cy}(G^\vee)|^{C_{p^n}} \xrightarrow{\sim} |sd_{p^n} N_\cdot^{cy}(G^\wedge)|^{C_{p^n}}.$$

Let $B'(G)$ be the pullback in

$$\begin{array}{ccc} B'(G) & \xrightarrow{I'} & (\operatorname{holim}_{\leftarrow} |sd_{p^n} N_{\cdot}^{cy}(G^\vee)|^{C_{p^n}})^{h\Phi} \\ \swarrow & & \downarrow \\ |N_{\cdot}(G^\vee)| \to |N_{\cdot}(G^\wedge)| & \xrightarrow{I} & (\operatorname{holim}_{\leftarrow} |sd_{p^n} N^{cy}(G^\wedge)|^{C_{p^n}})^{h\Phi} \end{array}$$

with I from [BHM], (2.13). By the above,

$$B'(G) \to |N_{\cdot}(G^\vee)| \to |N_{\cdot}(G)| \tag{2.3}$$

are homotopy equivalences. For $G_k = \mathrm{GL}_k(F)$ we have the transformations

$$|sd_{p^n} N_{\cdot}^{cy}(G_k^\vee)|^{C_{p^n}} \to |sd_{p^n} N_{\cdot}^{cy}(G_k)|^{C_{p^n}} \xrightarrow{S} |sd_{p^n} \mathrm{THH}(F_k)|^{C_{p^n}}$$

where $F_k = M_k(F)$ is the $k \times k$ matrix FSP associated with F (cf. [BHM, (5.11)]), so we obtain a strict transformation

$$t_k\colon B'(G_k) \to (\operatorname{holim}_{\leftarrow} |sd_{p^n} \mathrm{THH}_{\cdot}(F_k)|^{C_{p^n}})^{h\Phi},$$

one for each k. We next apply the infinite loop space machine in the formulation of [BHM, Section 4]. The transformations t_k induce a natural transformation of spectra

$$\mathrm{Trc}'\colon K'(F) \to \mathrm{TC}(F, p).$$

Indeed, in [BHM, (5.12)], $\mathrm{TC}(F, p)$ is defined to be the spectrum of the Γ-space associated to the target of t_k, cf. [BHM, (4.19)]. Moreover, the maps in (2.3) give a transformation

$$K'(F) \to K(F)$$

into the spectrum associated to the Γ-space build-up from the $|N_{\cdot}(G_k)|$, cf. [BHM, (5.6)]. This transformation is a homotopy equivalence of spectra. Restricting to the FSPs $F = F_X$ we can use $K'(F_X)$ as a model for $A(X)$. Indeed there is a strict natural transformation $K'(F_X) \to K(F_X)$ which is a homotopy equivalence, and $K(F_X)$ is one of the standard models for $A(X)$. In particular, $K(F_X)$ is a 1-analytic functor of X by [G2, (4.5)]. The same will then be the case for $K'(F_X)$.

We next discuss analyticity of $\mathrm{TC}(F_X, p)$ and some related functors. Given a group G and a G-space Y, let $\Sigma_G^\infty(Y_+)$ be the equivariant suspension spectrum; its nth space is

$$\Sigma_G^\infty(Y_+)_n = \lim_{\rightarrow} \mathrm{Map}(S^V, S^{V+\mathbb{R}^n} \wedge Y_+)$$

with V running over $\mathbb{R}G$-modules (or more precisely over all submodules of a complete universe in the language of [LMS]). For a normal subgroup Γ there is a transformation

$$\varphi\colon (\Sigma_G^\infty(Y_+))^\Gamma \to \Sigma_{N(\Gamma)/\Gamma}^\infty(Y_+^\Gamma)$$

which takes a map f from S^V to $S^{V+\mathbb{R}^n} \wedge Y_+$ to its induced map on Γ-fixed sets. We specialize to $G = C_{p^n}$, $\Gamma = C_p$, and $Y = \mathscr{L}X$. In this case we can compose with the inverse of the homeomorphism $\Delta_p\colon \mathscr{L}X \to (\mathscr{L}X)^{C_p}$ to get the transformation

$$\varphi\colon \Sigma_{C_{p^n}}^\infty(\mathscr{L}X_+)^{C_p} \to \Sigma_{C_{p^{n-1}}}^\infty(\mathscr{L}X_+).$$

It is $C_{p^{n-1}}$ equivariant when the $C_{p^{n-1}}$-action on the source is via the identification $C_{p^n}/C_p = C_{p^{n-1}}$. In particular, φ induces

$$\Phi\colon \Sigma_{C_{p^n}}^\infty(\mathscr{L}X_+)^{C_{p^n}} \to \Sigma_{C_{p^{n-1}}}^\infty(\mathscr{L}X_+)^{C_{p^{n-1}}}.$$

Let $\mathrm{TC}'(X, p)$ be the functor

$$\mathrm{TC}'(X, p) = (\operatorname{holim}_{\leftarrow} \Sigma_{C_{p^n}}^\infty(\mathscr{L}X_+)^{C_{p^n}})^{h\Phi}. \tag{2.4}$$

From [BHM] one gets a (noncomposable) sequence of strict transformations, all homotopy equivalences, which connect the functors $\mathrm{TC}(F_X, p)$ and $\mathrm{TC}'(X, p)$. See in particular the proofs of Proposition 3.7, Proposition 3.9, and Corollary 4.24 of [BHM]. It follows from the above that $\mathrm{TC}(F_X, p)$ is analytic if and only if $\mathrm{TC}'(X, p)$ is analytic, which we will now show. The cofibration

$$EC_{p^n+} \to S^0 \to \tilde{E}C_{p^n}$$

induces a cofibration of functors

$$\Sigma_{C_{p^n}}^\infty(EC_{p^n+} \wedge \mathscr{L}X_+)^{C_{p^n}} \to \Sigma_{C_{p^n}}^\infty(\mathscr{L}X_+)^{C_{p^n}} \to \Sigma_{C_{p^n}}^\infty(\tilde{E}C_{p^n} \wedge \mathscr{L}X_+)^{C_{p^n}}. \tag{2.5}$$

The mapping Φ gives a strict transformation

$$\Sigma_{C_{p^n}}^\infty(\tilde{E}C_{p^n} \wedge \mathscr{L}X_+)^{C_{p^n}} \xrightarrow{\simeq} \Sigma_{C_{p^{n-1}}}^\infty(\mathscr{L}X_+)^{C_{p^{n-1}}},$$

which is a homotopy equivalence; cf. [BHM, Section 5]. The transfer, in the formulation of [LMS, Chapter II, Section 7], is a transformation

$$\Sigma^\infty(EC_{p^n+} \wedge_{C_{p^n}} \mathscr{L}X_+) \xrightarrow{\simeq} \Sigma_{C_{p^n}}^\infty(EC_{p^n+} \wedge \mathscr{L}X_+)^{C_{p^n}},$$

which again is a homotopy equivalence, and such that the inclusions of fixed sets in the target corresponds to the transfers in the source.

Before we can proceed we must recall the definition from [G2, Section 4] of analyticity. Functors $\mathfrak{X}$ from the category of subsets of $\{1, \ldots, m\}$ to spaces are called m-cubes. The homotopy limit over all spaces in the diagram except the initial one, $\mathfrak{X}(\emptyset)$, is called $h(\mathfrak{X})$; there is a map $a(\mathfrak{X})$: $\mathfrak{X}(\emptyset) \to h(\mathfrak{X})$, and the diagram is called k-cartesian if $a(\mathfrak{X})$ is k-connected. A functor U has property $E_m(c, k)$ if for every strongly cocartesian $(m+1)$ cubical diagram $\mathfrak{X}$ with $\mathfrak{X}(\emptyset) \to \mathfrak{X}(s)$ k_s-connected and $k_s \geqslant k$, the diagram $U(\mathfrak{X})$ is $(\Sigma k_s - c)$-cartesian. U is called ρ-analytic if it satisfies $E_m(\rho m - c, \rho + 1)$ for all m and some constant c. We are interested in $\rho = 1$ where we just call the functor analytic. We will need to show that the functor $(\Sigma^\infty_{C_{p^n}}(EC_{p^n} \times \mathscr{L}(\mathfrak{X}))_+)^{C_{p^n}}$ is analytic. Our original argument used the equivalence

$$\tau_n\colon \Sigma^\infty_+(EC_{p^n} \times_{C_{p^n}} \mathscr{L}(X)_+) \xrightarrow{\simeq} \Sigma^\infty_{C_{p^n}}(EC_{p^n} \times \mathscr{L}(X)_+)^{C_{p^n}}$$

induced by the equivariant transfer. It is a sticky question if τ_n is a natural transformation in X. It seems to be when one unravels the definitions of [LMS], but the following argument, due to the referee, avoids the question.

The functor $\Sigma^\infty(X_+)$ is analytic according to a version of the Blakers-Massey theorem. The same is then the case for $\Sigma^\infty(\Omega X_+)$ and $\Sigma^\infty(\mathscr{L}X_+)$; the constant c decreases by 1. Precisely, if $\mathfrak{X}$ is a strongly cocartesian S-cube with $|S| = n$, then $\Omega(\mathfrak{X})$ and $\mathscr{L}(\mathfrak{X})$ are $(-n + \sum k_s)$-cartesian. According to [G2, (2.6)] it follows that $\mathscr{L}(\mathfrak{X})$ is $(-1 + \sum k_s)$-cocartesian. That is, the map

$$b\colon h_1(\mathscr{L}(\mathfrak{X})) \to (\mathscr{L}(\mathfrak{X}))$$

is $(-1 + \sum k_s)$-connected. Here $h_1(\mathscr{L}(\mathfrak{X}))$ is the homotopy colimit of $\mathscr{L}(\mathfrak{X})$ with the terminal vertex $\mathscr{L}(\mathfrak{X}(S))$ removed. Taking (equivariant) suspension spectra and smashing with a space commutes with homotopy colimits, so we have an equivalence

$$h_1(\Sigma^\infty_{C_{p^n}}(EC_{p^n} \times \mathscr{L}(\mathfrak{X}))_+) \cong \Sigma^\infty_{C_{p^n}}(EC_{p^n} \times h_1(\mathscr{L}(\mathfrak{X})))_+ .$$

The map $\Sigma^\infty_{C_{p^n}} b$ is a $(-1 + \sum k_s)$-connected map of free C_{p^n} spectra, so the induced map of C_{p^n}-fixed spectra is again $(-1 + \sum k_s)$-connected. Therefore, $(\Sigma^\infty_{C_{p^n}}(EC_{p^n} \times \mathscr{L}(\mathfrak{X}))_+)^{C_{p^n}}$ is $(-1 + \sum k_s)$-cocartesian, which for a spectrum is the same as $(-n + \sum k_s)$-cartesian. It follows that $(\Sigma^\infty_{C_{p^n}}(EC_{p^n} \times \mathscr{L}(\mathfrak{X}))_+)^{C_{p^n}}$ is 1-analytic with the constant $c = -1$.

It further follows that $\Sigma^\infty_+(EC_{p^n} \times_{C_{p^n}} \mathscr{L}X)$ is analytic. Using (2.5) and induction over n, we see that $\Sigma^\infty_{C_{p^n}}(\mathscr{L}X_+)^{C_{p^n}}$ is analytic for each n (with constant $c = 1$, independent of n). Given an inverse system

$$\cdots \to U_{n+1} \to U_n \to \cdots$$

of analytic functors with the same constant c, then the functor $U = \operatorname{holim}_{\leftarrow} U_n$ is analytic. Indeed, let $\mathfrak{X}$ be a strongly cocartesian $(m+1)$-cube. For each n, $U_n(\mathfrak{X})$

is $(\Sigma k_s - m + c)$-cartesian, and we must prove that the homotopy inverse limit of k-cartesian diagrams is k-cartesian. Since homotopy inverse limits commute among themselves, this reduces to the true statement that the homotopy inverse limit of k-connected spaces is k-connected.

Applied to $U_n(X) = \Sigma^{\infty}_{C_{p^n}}(\mathscr{L}X_+)^{C_{p^n}}$ we see that $\text{holim}_{\leftarrow} \Sigma^{\infty}_{C_{p^n}}(\mathscr{L}X_+)^{C_{p^n}}$ is analytic. Finally, for a transformation $\Phi: U \to U$, the homotopy fixed set is the homotopy inverse limit of

$$U \xrightarrow{(\Phi, \text{id})} U \times U \xleftarrow{\Delta} U.$$

It is analytic by the reasoning above. We have proved the following result.

PROPOSITION 2.6. *The functors* $\text{THH}(F_X)$ *and* $\text{TC}(F_X, p)$ *are analytic, and there is a model for* $A(X)$ *with a strict natural transformation* Trc *from* $A(X)$ *into* $\text{TC}(F_X, p)$. □

We next recall the part of "calculus of functors" needed for our main conclusion, Theorem 1.2. Recall from [G1] that the derivative at (X, x) of a functor Φ from based spaces to spectra is the spectrum $\partial_x \Phi(X)$ whose nth term is

$$(\partial_x \Phi)(X)_n = \text{fiber}(\Phi(X \vee S^n) \to \Phi(X)). \tag{2.7}$$

A natural transformation $t: \Phi \to \Psi$ between such functors induces a map of derivative spectra

$$\partial_x t(X): \partial_x \Phi(X) \to \partial_x \Psi(X).$$

It is, of course, a (weak) homotopy equivalence when t is, but for analytic functors there is a partial converse. This is Theorem 5.9 of [G2], which states that for a natural transformation between ρ-analytic functors, and for any $(\rho + 1)$-connected map $f: X \to B$,

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{t} & \Psi(X) \\ \big\downarrow{\scriptstyle \Phi(f)} & & \big\downarrow{\scriptstyle \Psi(f)} \\ \Phi(B) & \xrightarrow{t} & \Psi(B) \end{array}$$

is a fiber square, provided the induced map

$$\partial_x t: \partial_x \Phi(X) \to \partial_x \Psi(X)$$

is an equivalence for any based space (X, x).

We need one more fact about analytic functors, namely, the principle of "analytic continuation" from [G2]; cf. the proof of Proposition 5.1. It states in the special case of a natural transformation between 1-analytic functors $t\colon \Phi(X) \to \Psi(X)$ that t induces an equivalence of the reduced theories for 1-connected spaces if and only if it induces an equivalence on suspensions. Differently expressed, to check that $\partial_x t(X)$ is an equivalence for general 1-connected spaces it is enough to do the case $\partial_x t(\Sigma X)$. Moreover,

(2.8) $$\partial_x(\Phi \circ \Sigma)(X) = \Sigma \partial_x \Phi(\Sigma X).$$

We want to apply this theory to the functors at hand, and we must calculate derivatives. We know from [G1] that

$$\partial_x A(X) \simeq \Sigma^\infty_+(\Omega X)$$

$$\partial_x \Sigma^\infty_+(\mathscr{L}X) \simeq \mathscr{L}(\Sigma^\infty_+(\Omega X),$$

where ΩX denotes the loops at $x \in X$, and where $\mathscr{L}(\Sigma^\infty_+(\Omega X))$ is the functional spectrum $\mathrm{Map}(S^1, \Sigma^\infty_+(\Omega X))$.

We prove in Section 3 below that for the functor defined by (1.1),

(2.9) $$\partial_x \mathrm{TC}(X, p)^\wedge_p \simeq \Sigma^\infty_+(\Omega X)^\wedge_p .$$

More precisely, we show that the composition

(2.10) $$\partial_x \mathrm{TC}(X; p)^\wedge_p \xrightarrow{\partial_x \beta} \partial_x \mathrm{THH}(X)^\wedge_p \xrightarrow{e_1} \Sigma^\infty_+(\Omega X)^\wedge_p$$

is an equivalence. Here $\mathrm{THH}(X)^\wedge_p = \Sigma^\infty_+(\mathscr{L}X)^\wedge_p$, and

$$e_1\colon \mathscr{L}\Sigma^\infty_+(\Omega X) \to \Sigma^\infty_+(\Omega X)$$

is the evaluation at 1.

Let us assume that two homotopy functors F, G have the property that for all spaces X the two spaces $F(X)$, $G(X)$ are homotopy equivalent. It is obvious from the definition of a derivative that even if the two functors are not necessarily homotopy equivalent by a sequence of strict transformations, they do have homotopy equivalent derivatives. We have

(2.11) $$\begin{aligned} &\partial_x K(F_X) \simeq \partial_x K'(F_X) \simeq \Sigma^\infty_+(\Omega X) \\ &\partial_x \mathrm{TC}(F_X, p)^\wedge_p \simeq \partial_x \mathrm{TC}(X, p)^\wedge_p \simeq \Sigma^\infty_+(\Omega X)^\wedge_p , \end{aligned}$$

since by [BHM, Section 5] and [M, Section 4.4]

$$\mathrm{TC}(F_X, p)^\wedge_p \simeq \mathrm{TC}(X, p)^\wedge_p$$

and since taking derivatives commutes with completion.

To complete the proof of Theorem 1.2 we must argue that

$$\partial_x \mathrm{Trc}\colon \partial_x K'(F_X)^\wedge_p \to \partial_x \mathrm{TC}(F_X, p)^\wedge_p \tag{2.12}$$

is an equivalence. The source and target are abstractly homotopy equivalent. However, the calculation of $\partial_x K'(F_X) \simeq \partial_x K(F_X)$ from [G1] uses the manifold model for $A(X)$ and fits poorly with the above strictly homotopy-theoretic approach. We proceed therefore in a very roundabout way. First we notice from the analytic continuation (cf. (2.8)) that it suffices to show that

$$\partial_x K(F_{\Sigma X})^\wedge_p \xleftarrow{\simeq} \partial_x K'(F_{\Sigma X})^\wedge_p \xrightarrow{\partial_x \mathrm{Trc}} \partial_x \mathrm{TC}(F_{\Sigma X}, p)^\wedge_p$$

is an equivalence for connected X. Second, by (2.11), we are reduced to checking that

$$\partial_x K'(F_{\Sigma X})^\wedge_p \xrightarrow{\partial_x \mathrm{Trc}} \partial_x \mathrm{THH}(F_{\Sigma X})^\wedge_p \xrightarrow{\simeq} \partial_x(\Sigma^\infty_+(\mathscr{L}\Sigma X))^\wedge_p \xrightarrow{e_1} \Sigma^\infty_+(\Omega\Sigma X)^\wedge_p$$

is a homotopy equivalence. This is done in Section 4 below. Indeed, Corollary 4.15 implies that the composition induces a split surjection on spectrum homology with $\mathbb{F}_p$ coefficients, and hence an isomorphism because the homology is finitely generated in each degree, provided that X has finite type.

Remark 2.13. One would like to generalize Theorem 2.1 to include the statement that for a 2-connected map $f\colon X \to B$ there is a homotopy cartesian diagram

$$\begin{array}{ccc} A(X)^\wedge_p & \xrightarrow{\mathrm{Trc}} & \mathrm{TC}(X, p)^\wedge_p \\ \downarrow & & \downarrow \\ A(B)^\wedge_p & \xrightarrow{\mathrm{Trc}} & \mathrm{TC}(B, p)^\wedge_p . \end{array}$$

When $\pi_1 B$ is finite, [G2, Theorem 5.2] tells us that it suffices to check that $\partial_x \mathrm{Trc}$ is an equivalence for all (X, x). The above reasoning only gives this statement for X simply connected, but it is, of course, very likely true in general.

3. Derivative calculation. This section contains the derivative calculations used in the proofs in Section 2.

PROPOSITION 3.1. *The derivative $\partial_x \mathrm{TC}(X, p)^\wedge_p$ is naturally equivalent to $(\Sigma^\infty_+\Omega X)^\wedge_p$.*

Proof. The description of $\mathrm{TC}(X, p)^\wedge_p$ as a fiber product may be expressed as follows. The self-maps of the circle, in particular, the rotation group S^1 and the degree-p map $\Delta_p(z) = z^p$, act on $\mathscr{L}X$ and therefore on the spectrum $\Sigma^\infty_+\mathscr{L}X$. After

p-completion, we have a fiber square

$$(3.2)\qquad \begin{array}{ccc} \mathrm{TC}(X,p) & \longrightarrow & S^1 \wedge (ES^1_+ \wedge_{S^1} \Sigma^\infty_+ \mathscr{L}X) \\ \downarrow & & \downarrow \scriptstyle\mathrm{Trf} \\ \Sigma^\infty_+ \mathscr{L}X & \xrightarrow{\mathrm{id}-\Delta_p} & \Sigma^\infty_+ \mathscr{L}X. \end{array}$$

Here Trf is the S^1-transfer, a map of spectra

$$(3.3)\qquad S^1 \wedge (ES^1_+ \wedge_{S^1} \mathbf{Z}) \to \mathbf{Z}$$

defined whenever S^1 acts on a spectrum $\mathbf{Z}$. If $\mathbf{Z} = \{Z_i | i \geqslant 0\}$, then the left-hand side of (3.3) denotes the spectrum associated to the prespectrum $\{S^1 \wedge (ES^1_+ \wedge_{S^1} Z_i)\}$, and the map Trf is defined using the usual transfer maps

$$(3.4)\qquad \Sigma^\infty_+(S^1 \wedge (ES^1_+ \wedge_{S^1} Z_i)) \xrightarrow{\mathrm{Trf}} \Sigma^\infty_+ Z_i,$$

which in turn are defined whenever S^1 acts on a based space Z_i. Note that to define (3.3) all that is required is a spectrum $\mathbf{Z}$ with an S^1-action in the naive sense: S^1 acts on the spaces Z_i, and the structure maps $S^1 \wedge Z_i \to Z_{i+1}$ are S^1-maps.

Differentiation of (analytic spectrum-valued) functors commute (up to natural equivalence) with p-completion, fiber product, S^1-transfer, and subtraction of stable maps. It follows that after p-completion there is a fiber square

$$(3.5)\qquad \begin{array}{ccc} \partial_x \mathrm{TC}(X,p) & \longrightarrow & S^1 \wedge (ES^1_+ \wedge_{S^1} \partial_x \Sigma^\infty_+ \mathscr{L}X) \\ \downarrow & & \downarrow \scriptstyle\mathrm{Trf} \\ \partial_x \Sigma^\infty_+ \mathscr{L}X & \xrightarrow{\mathrm{id}-\Delta_p} & \partial_x \Sigma^\infty_+ \mathscr{L}X, \end{array}$$

where S^1 and Δ_p act on $\partial_x \Sigma^\infty_+ \mathscr{L}X$ by functoriality of ∂_x.

According to [G1, Corollary 2.5] and [G2, Appendix], $\partial_x \Sigma^\infty_+ \mathscr{L}X$ is equivalent to $\mathscr{L}\Sigma^\infty_+ \Omega X$, that is, the spectrum $\{\mathscr{L}Q(S^i \wedge \Omega X_+) | i \geqslant 0\}$, with S^1 and Δ_p acting as follows: Δ_p acts on $\mathscr{L}\Sigma^\infty_+ \Omega X$ both through the "$\mathscr{L}$" and the "Ω" in the sense that

$$S^1 \xrightarrow{\lambda} Q(S^i \wedge \Omega X_+)$$

goes to

$$S^1 \xrightarrow{\Delta_p} S^1 \xrightarrow{\lambda} Q(S^i \wedge \Omega X_+) \xrightarrow{Q(S^i \wedge \Delta_+)} Q(S^i \wedge \Omega X_+),$$

where in the last arrow, $\Delta(\omega) = \omega \circ \Delta_p$; S^1 acts on $\mathscr{L}\Sigma^\infty_+ \Omega X$ through the "$\mathscr{L}$"

only. Rewrite (3.5) and combine it with another square:

$$
\begin{array}{ccc}
\partial_x \mathrm{TC}(X, p) & \longrightarrow & S^1 \wedge (ES^1_+ \wedge_{S^1} \mathscr{L}\Sigma^\infty_+ \Omega X) \\
\downarrow & & \downarrow \\
\mathscr{L}\Sigma^\infty_+ \Omega X & \xrightarrow{\mathrm{id}-\Delta_p} & \mathscr{L}\Sigma^\infty_+ \Omega X \\
\downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_1} \\
\Sigma^\infty_+ \Omega X & \xrightarrow{\mathrm{id}-\Delta_p} & \Sigma^\infty_+ \Omega X .
\end{array} \tag{3.6}
$$

Here e_1 is evaluation at $1 \in S^1$, and Δ_p acts on $\Sigma^\infty_+ \Omega X$ by $\Sigma^\infty_+ \Delta_p$.

LEMMA 3.7. *The composed map* $e_1 \circ \mathrm{Trf}$ *is an equivalence.*

LEMMA 3.8. *The lower square in (3.6) is a fiber square after p-completion.*

Proof of Lemma 3.7. More generally, for any spectrum $\mathbf{W}$, the composed map

$$S^1 \wedge (ES^1_+ \wedge_{S^1} \mathscr{L}\mathbf{W}) \xrightarrow{\mathrm{Trf}} \mathscr{L}\mathbf{W} \xrightarrow{e_1} \mathbf{W}$$

is an equivalence. This follows from the fact that for any spectrum $\mathbf{Y}$, the composed map

$$S^1 \wedge (ES^1_+ \wedge_{S^1} (S^1_+ \wedge \mathbf{Y})) \xrightarrow{\mathrm{Trf}} S^1_+ \wedge \mathbf{Y} \longrightarrow S^1 \wedge \mathbf{Y}$$

is an equivalence. (Put $\mathbf{Y} = \Omega \mathbf{W}$ and use the map $S^1_+ \wedge \Omega\mathbf{W} \to \mathscr{L}\mathbf{W}$, which is an S^1-map and a nonequivariant equivalence.) □

Proof of Lemma 3.8. Δ_p acts on $\Omega\Sigma^\infty_+\Omega X = \mathrm{fiber}(e_1)$ through both the inside and outside "Ω", as described above. In particular Δ_p acts like zero on the mod-p spectrum homotopy, and id-Δ_p induces an equivalence after p-completion. □

This completes the proof of Proposition 3.1. □

PROPOSITION 3.9. *If X is 1-connected, then the square*

$$
\begin{array}{ccc}
\Sigma^\infty_+ \mathscr{L}X & \xrightarrow{\mathrm{id}-\Delta_p} & \Sigma^\infty_+ \mathscr{L}X \\
\downarrow{\scriptstyle \Sigma^\infty_+(e_1)} & & \downarrow{\scriptstyle \Sigma^\infty_+(e_1)} \\
\Sigma^\infty_+ X & \xrightarrow{0} & \Sigma^\infty_+ X
\end{array}
$$

is a fiber square after p-completion, so that by (3.2), we have

$$\mathrm{TC}(X, p)^\wedge_p \simeq (\Sigma^\infty_+ X \times \mathrm{fiber}(\Sigma^\infty_+(e_1) \circ \mathrm{Trf}))^\wedge_p .$$

Proof. This is certainly true when X is a point. Since the functors are 1-analytic, it will be enough (by [G2]) if the square of derivatives is a fiber square.

This is the outer square in

$$\begin{array}{ccc} \mathscr{L}\Sigma^\infty_+\Omega X & \xrightarrow{\mathrm{id}-\Delta_p} & \mathscr{L}\Sigma^\infty_+\Omega X \\ \downarrow & & \downarrow \\ \Sigma^\infty_+\Omega X & \xrightarrow{\mathrm{id}-\Delta_p} & \Sigma^\infty_+\Omega X \\ \downarrow & & \downarrow \\ \Sigma^\infty_+ * & \xrightarrow{0} & \Sigma^\infty_+ *. \end{array}$$

The top half is a fiber square (after p-completion) by Lemma 3.8. The bottom half is, too. That is, the map

$$\Sigma^\infty\Omega X \xrightarrow{\mathrm{id}-\Delta_p} \Sigma^\infty\Omega X \tag{3.10}$$

is an equivalence after p-completion. To check this, consider the mapping telescope of

$$\Omega X \xrightarrow{\Delta_p} \Omega X \xrightarrow{\Delta_p} \Omega X \longrightarrow \cdots.$$

Because Δ_p acts like multiplication by p on homotopy groups of ΩX, the telescope is a simple space with homotopy groups $\pi_*(\Omega X)[1/p]$. This implies that its mod-p homology is zero, and that means that (3.10) induces an isomorphism on mod-p spectrum homology. □

4. The suspension case. This section studies the topological Dennis trace

$$\mathrm{Tr}\colon A(\Sigma X) \to \mathrm{THH}(\Sigma X)$$

for a simply connected space X, homotopy equivalent to a CW-complex. Since both functors are homotopy invariants and commute with filtered direct limits, it suffices to consider the case where X is a finite CW complex.

The simplifications which occur in the suspension case are twofold. On the one hand the spectrum $\mathrm{THH}(\Sigma X)$ simplifies to a wedge of much simpler functors, and on the other hand there are good embeddings

$$\theta\colon \Sigma^\infty(S^1_+ \wedge X^a) \to A(\Sigma X), \tag{4.1}$$

one for each natural number a, cf. [CCGH]. We shall examine the compositions $\mathrm{Tr} \circ \theta$, but first we need to recall some general theory about topological Hochschild homology. Let F be a functor with smash products. There are spectra $\Sigma^\infty F$ and $\mathrm{THH}(F)$ and a suspension $S^1_+ \wedge \Sigma^\infty F \to \mathrm{THH}(F)$. In our applications, F will be associated to a grouplike monoid G, i.e., $F(X) = \mathbb{G}(X) = X \wedge G_+$, and $\Sigma^\infty F$ and $\mathrm{THH}(F)$ are then equivalent to the spectra $\Sigma^\infty_+(G)$ and $\Sigma^\infty_+(\mathscr{L}BG)$.

The spectrum homology $H_*(\mathrm{THH}(F))$ can be calculated via a spectral sequence whose E^2-term is a Hochschild homology of $H_*(\Sigma^\infty F)$, provided we use field coefficients

$$\mathrm{HH}_p(H_*(\Sigma^\infty F)) \underset{p}{\Rightarrow} H_*(\mathrm{THH}(F)). \tag{4.2}$$

For (4.2) to work, we need to assume that $F(S^n)$ is $(n-c)$-connected, and that the stabilization map $F(S^n) \to \Omega F(S^{n+1})$ is $(2n-c)$-connected, for some constant c.

The spectral sequence has been used before, e.g., in connection with the calculation of $\mathrm{THH}(\mathbf{Z})$, but it has not been fully documented in the available literature. We take the opportunity to do so here.

Let I be the category whose objects are the ordered sets $n = (1, \ldots, n)$ and whose morphisms are the injective maps. Each morphism $f \in I(n, m)$ can be decomposed as $f = \sigma \circ i$ with i the standard inclusion and σ a permutation of m letters. We let

$$f_\#\colon \Omega^n F(S^n \wedge X) \to \Omega^m F(S^m \wedge X)$$

be equal to $\sigma_\# \circ i_\#$, with $i_\#$ the obvious suspension map and $\sigma_\#$ the (conjugation) operation on $\Omega^m F(S^m)$; cf. [BHM, Section 3]. Consider the simplicial spaces, which have p-simplices:

$$\begin{aligned} &\operatorname*{holim}_{\overrightarrow{I^{p+1}}} \mathrm{Map}(S^{i_0} \wedge \cdots \wedge S^{i_p}, F(S^{i_0}) \wedge \cdots \wedge F(S^{i_p})) \\ &\operatorname*{holim}_{\overrightarrow{I^{p+2}}} \mathrm{Map}(S^{i_{-1}+i_0} \wedge \cdots \wedge S^{i_p}, F(S^{i_{-1}} \wedge S^k \wedge S^{i_0}) \wedge \cdots \wedge F(S^{i_p})), \end{aligned} \tag{4.3}$$

respectively, equipped with the usual "Hochschild-type" face operators. The first space is $\mathrm{THH}_\cdot(F)$ with topological realization $\mathrm{THH}(F)$; the second space is denoted $T^k_\cdot(F)$ with topological realization $T^k(F)$. The spaces $\mathrm{THH}(F)$ and $T^k(F)$, $k \geqslant 0$ all have Γ-structures. This fact is based on Morita invariance, and is explained in [BHM], Section 4 for $\mathrm{THH}(F)$. The argument is the same for each of the functors $T^k(F)$.

There is an obvious map of Γ-spaces

$$T^0(F) \to \mathrm{THH}(F)$$

which is a homotopy equivalence by the *approximation lemma* from [B], which states that the homotopy limits in (4.3) are well behaved in the sense that the term corresponding to $(i_0, \ldots, i_p)$ approximates the limit as $i_0, \ldots, i_p \to \infty$. See [M, Lemma 2.3.7].

The space of p-simplices $T_p^k(F)$ form a spectrum, but more importantly, we have the next result.

LEMMA 4.4. *The spaces $T^k(F)$ form a spectrum $T(F)$.*

Proof. In each simplicial degree p, one can define

$$S^1 \wedge T_p^k(F) \to T_p^{k+1}(F)$$

by using the composition

$$\begin{aligned} S^1 \wedge F(S^{i_{-1}} \wedge S^k \wedge S^{i_0}) &\to F(S^1) \wedge F(S^{i_{-1}} \wedge S^k \wedge S^{i_0}) \\ &\to F(S^1 \wedge (S^{i_{-1}} \wedge S^k \wedge S^{i_0})) \\ &\to F(S^{i_{-1}} \wedge (S^1 \wedge S^k) \wedge S^{i_0}), \end{aligned}$$

where the last map is induced by the obvious permutation of coordinates. One checks that the face operators commute with this suspension map, so it induces

$$S^1 \wedge T^k(F) \to T^{k+1}(F).$$

Its adjoint is a homotopy equivalence, because $\Omega T^{k+1}(F)$ can be calculated by realizing the simplicial space whose p-simplices is the space $\Omega T_p^{k+1}(F)$. □

LEMMA 4.5. *The spectrum $\{T^k(F)\}$ is homotopy equivalent to the spectrum $\{B^k \mathrm{THH}(F)\}$ associated with the Γ-structure on* THH(F).

Proof. Let $B(-)$ denote the classifying space in the Γ-structure; then

$$T^k(F) \simeq \Omega B T^k(F) \simeq B\Omega T^k(F) \simeq BT^{k-1}(F),$$

where the second equivalence involves the fact, also used above, that the loop space can be calculated degreewise. □

The spectral sequence in (4.2) is associated to the skeleton filtration of the spectrum $T(F) = \{T^k(F)\}$. Indeed, if $H_*(-)$ denotes homology with field coefficients, then

$$\begin{aligned} &H_{*+k}(\mathrm{holim}_{\rightarrow} \mathrm{Map}(S^{i_{-1}+i_0} \wedge \cdots \wedge S^{i_p}, F(S^{i_{-1}} \wedge S^k \wedge S^{i_0}) \wedge F(S^{i_1}) \wedge \cdots \wedge F(S^{i_p}))) \\ &\quad \simeq \lim_{\rightarrow} H_{*+k}(F(S^{i_{-1}} \wedge S^k \wedge S^{i_0}) \wedge F(S^{i_1}) \wedge \cdots \wedge F(S^{i_p})) \\ &\quad \simeq H_*(\Sigma^\infty F) \otimes \cdots \otimes H_*(\Sigma^\infty F), \end{aligned}$$

where the limits run over the category I^{p+2}. This uses the approximation lemma, the Künneth formula, and the fact that $F(S^n)$ is roughly n-connected. It follows that the E^1-term of the spectral sequence is the standard cyclic construction applied to the algebra $H_*(\Sigma^\infty F)$, and hence that the E^2-term is the claimed Hochschild homology groups.

We will apply (4.2) for the functor with smash product $\mathbb{G}$ with $\mathbb{G}(A) = A \wedge \Omega\Sigma X_+$ associated to the monoid $G = \Omega\Sigma X$, X connected. It will be convenient to use the stable splitting of $G = \Omega\Sigma X$, and we briefly recall it.

Consider the inclusion $i\colon X \to \Omega\Sigma X$. It maps the base point of X to the constant loop at the base point and induces

$$\bar{i}\colon \Sigma_+^\infty(X) \to \Sigma_+^\infty(\Omega\Sigma X); \qquad \bar{i}(u) = \Sigma_+^\infty i(u) - e,$$

where $e\colon \Sigma_+^\infty X \to \Sigma_+^\infty(*) \to \Sigma_+^\infty(\Omega\Sigma X)$. Using the product, we get

$$\Sigma_+^\infty(X^a) \to \Sigma_+^\infty(\Omega\Sigma X), \qquad u \to \Pi_{v=1}^a \bar{i}(u_v),$$

where u_v is the vth projection. This map factors over $\Sigma^\infty(X^{(a)})$ and induces the James-Milnor decomposition

$$\bigvee_{a=0}^{\infty} \Sigma^\infty(X^{(a)}) \xrightarrow{\simeq} \Sigma_+^\infty(\Omega\Sigma X); \qquad X^{(0)} = S^0.$$

It follows that the homology of $\Sigma^\infty\mathbb{G}$ (with field coefficients) is the tensor algebra of $\tilde{H}_*(X)$,

$$H_*(\Sigma^\infty\mathbb{G}) = T(\tilde{H}_* X).$$

The Hochschild homology of a (graded) tensor algebra is concentrated in degrees 0 and 1. In fact,

$$\text{(4.6)} \qquad \begin{aligned} \mathrm{HH}_0(H_*(\mathbb{G})) &= \Sigma^\oplus[\tilde{H}_*(X)^{\otimes a}]/(1-\tau) \\ \mathrm{HH}_1(H_*(\mathbb{G})) &= \Sigma^\oplus[\tilde{H}_*(X)^{\otimes a}]^\tau, \end{aligned}$$

where τ acts by cyclic permuting the factors with the usual sign convention, i.e.,

$$\tau(\xi_1 \otimes \cdots \otimes \xi_a) = (-1)^{|\xi_1|(|\xi_2|+\cdots+|\xi_a|)}\xi_2 \otimes \cdots \otimes \xi_a \otimes \xi_1$$

(cf. [LQ, Section 4]).

The spectral sequence (4.2) is forced to collapse, being concentrated in filtration degrees $p = 0$ and $p = 1$, but let us spell it out by examining the abutment directly. The analogue of the James-Milnor splitting for the free loop space is the decomposition

$$\Sigma_+^\infty(\mathscr{L}\Sigma X) \simeq \vee_{a=0}^\infty \Sigma^\infty(S_+^1 \wedge_{C_a} X^{(a)})$$

(cf. [CC], [BM]). Its homology can be calculated from the chain complex

$$0 \to k[C_a]e_1 \otimes_{C_a} \tilde{H}_*(X)^{\otimes a} \xrightarrow{\partial} k[C_a]e_0 \otimes_{C_a} \tilde{H}_*(X)^{\otimes a} \to 0,$$

where

$$\partial(e_1 \otimes x_1 \otimes \cdots \otimes x_a) = e_0 \otimes x_1 \otimes \cdots \otimes x_a - (-1)^\alpha e_0 \otimes x_2 \otimes \cdots \otimes x_a \otimes x_1$$

and $\alpha = |x_1|(|x_2| + \cdots + |x_a|)$. Indeed, this can be seen by choosing a chain homotopy equivalence $H_*(X) \xrightarrow{\simeq} C_*(X)$, and using the Eilenberg-Zilber theorem. We get

(4.7) $H_*(\Sigma_+^\infty \mathscr{L}\Sigma X) = \Sigma_{a=0}^\infty \{e_0 \otimes [\tilde{H}_*(X)^{\otimes a}]/(1-\tau) \oplus e_1 \otimes [\tilde{H}_{*-1}(X)^{\otimes a}]^\tau\}$

in agreement with (4.3). The suspension mapping

$$\sigma\colon S_+^1 \wedge \Sigma^\infty(F) \to \mathrm{THH}(F)$$

is in the case at hand, $F = \mathbb{G}$, the map

$$\sigma\colon S_+^1 \wedge \Sigma_+^\infty \Omega\Sigma X \to \Sigma_+^\infty \mathscr{L}\Sigma X$$

induced from the S^1-action on $\mathscr{L}\Sigma X$, and the inclusion of $\Omega\Sigma X$. On homology it is given by

$$\sigma_*(\iota_1 \otimes x_1 \otimes \cdots \otimes x_a) = e_1 N \otimes_{C_a} (x_1 \otimes \cdots \otimes x_a),$$

with $N = 1 + T + \cdots + T^{a-1}$.

We next review the embedding θ from [CCGH]. Given any FSP F, we have the associated matrix FSP

$$M_a(F)(S^i) = \mathrm{Map}^*(\mathbf{a}, \mathbf{a} \wedge F(S^i)),$$

where $\mathbf{a} = \{0, \ldots, a\}$ with zero as base point. There are maps

$$E_{\nu,\mu}\colon F(S^i) \to M_a(F)(S^i)$$

which to $u \in F(S^i)$ associate $E_{\nu,\mu}(u)$, the map which sends $\nu \in \mathbf{a}$ to $(\mu, u) \in \mathbf{a} \wedge F(S^i)$ and any other element of $\mathbf{a}$ to the base point. It defines a morphism of spectra

$$E_{\nu,\mu}\colon \Sigma^\infty F \to \Sigma^\infty M_a(F).$$

When $F = \mathbb{G}$ as above, we can compose with the map $\bar{\imath}\colon \Sigma_+^\infty X \to \Sigma^\infty \mathbb{G}$. Let us consider

$$P\colon \Sigma_+^\infty(X^a) \to \Sigma^\infty M_a(\mathbb{G})$$

given by

$$P(x) = \sum_{\nu=1}^{a} E_{\nu-1,\nu}(\bar{i}(x_\nu)),$$

where $E_{a+1,a} = E_{1,a}$ and $x \to \bar{i}(x_\nu)$ was defined above in connection with the James-Milnor splitting. We adjoin to get

$$P\colon X^a \to \lim_{\to} \Omega^n M_a(\mathbb{G})(S^n). \tag{4.8}$$

It reduces to zero in $M_a(\mathbf{Z})$ under linearization, so

$$\theta(x_1, \ldots, x_a) = I - P(x_1, \ldots, x_a) \in \mathrm{GL}_a(\mathbb{G}),$$

Using the suspension $S^1 \wedge \mathrm{GL}_a(\mathbb{G}) \to \mathrm{BGL}_a(\mathbb{G})$ and the inclusion of $\mathrm{GL}_a(\mathbb{G})$ into $A(\Sigma X)$, we obtain a map

$$\theta\colon \Sigma^\infty(S^1 \wedge X^a) \to A(\Sigma X).$$

The restriction of the topological Dennis trace to $\mathrm{BGL}_a(\mathbb{G})$ is by definition induced by the composition

$$\mathrm{BGL}_a(\mathbb{G}) \xrightarrow{\iota} \mathrm{THH}(M_a(\mathbb{G})) \underset{\simeq}{\xrightarrow{\mu}} \mathrm{THH}(\mathbb{G}) \tag{4.9}$$

Here ι is the realization of the simplicial map, which takes a k-simplex $[A_1|\cdots|A_k]$ into the k-simplex

$$A_0 \wedge \cdots \wedge A_k \in \operatorname{holim}_{\to} \operatorname{Map}(S^{i_0} \wedge \cdots \wedge S^{i_k}, M_a(\mathbb{G})(S^{i_0}) \wedge \cdots \wedge M_a(\mathbb{G})(S^{i_k})),$$

where A_0 is the inverse of the product $A_1 \cdots A_k$ in $\mathrm{GL}_a(\mathbb{G})$. (Some consistency in choosing inverses is needed (cf. Section 2 above); here however we are only interested in what the map induces on homotopy groups and, more generally, on the functor $[S^1 \wedge X^a, -]$.)

The Morita equivalence μ is cumbersome to define and hard to calculate directly, because it involves the use of a double simplicial space; cf. [BHM, Section 3]. However, we shall only evaluate it in homology, where we can make use of the spectral sequences (4.2) for $F = \mathbb{G}$ and $F = M_a(\mathbb{G})$.

More precisely, μ defines a map of spectral sequences, and on the E^2-level,

$$\mu_*\colon \mathrm{HH}_p(M_a(H_*(G))) \to \mathrm{HH}_p(H_*(G))$$

is equal to the "usual trace operator," $\mu_* = \mathrm{tr}_*$. In the cyclic bar construction, whose homology is $\mathrm{HH}_p(-)$, tr_* is given by the formula

$$\mathrm{tr}_*(A_0 \otimes \cdots \otimes A_p) = \sum A_0^{i_0, i_1} \otimes \cdots \otimes A_p^{i_p, i_0}, \tag{4.10}$$

with A^{ij} denoting the ijth entry, and where the sum extends over all sets of indices.

We can now begin to examine $\mathrm{Tr} \circ \theta$ (in homology). In fact, we settle for less and just attempt to describe the map into filtration degree $p = 1$, or what is the same thing (see below), the quotient map

$$\mathrm{Tr} \circ \theta \colon \Sigma^\infty(S^1 \wedge X^a) \to \mathrm{THH}(\mathbb{G})/\mathrm{THH}_0(\mathbb{G}),$$

where $\mathrm{THH}_0(\mathbb{G}) \simeq \Sigma^\infty \mathbb{G}$ is the subspectrum of the zero-simplices in $|\mathrm{THH}.(\mathbb{G})|$

$$(4.11) \qquad \mathrm{THH}(\mathbb{G})/\mathrm{THH}_0(\mathbb{G}) \simeq \Sigma^\infty(\mathscr{L}\Sigma X/\Omega\Sigma X) \simeq \vee_{b=1}^{\infty} \Sigma^\infty(S^1/C_b \wedge X^{(b)}).$$

Its homology can be calculated from (4.7). We notice that

$$e_1 \otimes [\tilde{H}_*(X)^{\otimes b}]^\tau \subset H_*(\mathrm{THH}(\mathbb{G}))$$

injects into $H_*(\mathrm{THH}(\mathbb{G})/\mathrm{THH}_0(\mathbb{G}))$, and that the other summand $e_0 \otimes \tilde{H}_*(X)^{\otimes b}/(1 - \tau)$ is sent to zero. By definition,

$$[\imath \circ \theta] = [(I - P)^{-1} \wedge (I - P)] \in [\Sigma^\infty(S^1 \wedge X^a), \mathrm{THH}(M_a(\mathbb{G}))].$$

The advantage of dividing out the spectrum of zero-simplices $\mathrm{THH}_0(M_a(\mathbb{G}))$ is that we can make use of the decomposition

$$[(I - P)^{-1}] = [I + P + P^2 + \cdots]$$

as follows. Each term $P^k \wedge P$ (or $P^k \wedge I$) gives a map

$$X^a \to \mathrm{THH}_1(M_a(\mathbb{G}))$$

into the spectrum of 1-simplices, and hence a map

$$\Delta^1 \times X^a/\partial\Delta^1 \times X^a \to \Delta^1 \times \mathrm{THH}_1(M_a(\mathbb{G}))/\partial\Delta^1 \times \mathrm{THH}_1(M_a(\mathbb{G}))$$
$$\to \mathrm{THH}(M_a(\mathbb{G}))/\mathrm{THH}_0(M_a(\mathbb{G})).$$

This determines a well-defined homotopy class

$$[P^k \wedge P] \in [\Sigma^\infty(S^1 \wedge X^a), \mathrm{THH}(M_a(\mathbb{G}))/\mathrm{THH}_0(M_a(\mathbb{G}))].$$

It follows that there is a well-defined weak homotopy class:

$$[\imath \circ \theta] = \sum_{k=0}^{\infty} [P^k \wedge I] - \sum_{k=0}^{\infty} [P^k \wedge P] \in \left[\Sigma^\infty(S^1 \wedge X^{\wedge a}), \prod_{b=1}^{\infty} \Sigma^\infty(S^1/C_b \wedge X^{(b)})\right].$$

Since X is connected and $X^{(b)}$ is $(b-1)$-connected, the inclusion

$$\bigvee_{b=1}^{\infty} \Sigma^{\infty}(S^1/C_b \wedge X^{(b)}) \subset \prod_{b=1}^{\infty} \Sigma^{\infty}(S^1/C_b \wedge X^{(b)})$$

is a weak homotopy equivalence, so that the infinite sums also make sense in

$$\mathrm{THH}(M_a(\mathbb{G}))/\mathrm{THH}_0(M_a(\mathbb{G})).$$

The Morita equivalence map can be chosen so as to map $\mathrm{THH}_0(M_a(\mathbb{G}))$ into $\mathrm{THH}_0(\mathbb{G})$, inducing

$$\mu\colon \mathrm{THH}(M_a(\mathbb{G}))/\mathrm{THH}_0(M_a(\mathbb{G})) \to \mathrm{THH}(\mathbb{G})/\mathrm{THH}_0(\mathbb{G}).$$

This reduces us to calculating $\mu_*[P^k \wedge P]$ in homology. Before we do this, let us retreat to the algebraic situation and consider the *linear* Dennis trace

$$\mathrm{Tr}_1\colon K_1(R) \to \mathrm{HH}_1(R),$$

when R is a (completed) tensor algebra

$$R = \Pi V^{(a)}, \qquad V^{(a)} = V \otimes \cdots \otimes V.$$

Given elements $\xi_i \in V$, we want to determine the image of $I - P(\xi_1, \ldots, \xi_a)$ in $\mathrm{HH}_1(R)$. An easy calculation (in $R \otimes R$) gives

$$\mathrm{Tr}_1(I-P) = \Sigma\, \mathrm{tr}_*(P^i \otimes I) - \mathrm{tr}_*(P^i \otimes P)$$

$$\mathrm{tr}_*(P^i \otimes I) = 0$$

$$\mathrm{tr}_*(P^{i-1} \otimes P) = \begin{cases} 0 & \text{if } i \not\equiv 0 \pmod a \\ \sum_{j=0}^{a-1} \xi_{j+1} \cdots \xi_a \xi^{k-1} \xi_1 \cdots \xi_{j-1} \otimes \xi_j & \text{if } i = ka. \end{cases}$$

Here ξ is the product of the ξ_i. Hence we get in $\mathrm{HH}_1(R)$ the formula

$$\mathrm{Tr}_1(I - P(\xi_1, \ldots, \xi_a)) = \sum_{k=1}^{\infty} \sum_{i=0}^{a-1} T^i(\xi_1 \otimes \cdots \otimes \xi_a)^{\otimes k}. \tag{4.12}$$

Here T is the cyclic permutation of ka letters, and each term $T^i(\xi_1 \otimes \cdots \otimes \xi_a)^{\otimes k}$ in $V^{(ka)}$ is considered an element of $R \otimes R$ by rewriting $V^{(ka)} = V^{(ka-1)} \otimes V$. Let

$$\mathrm{proj}_b\colon \mathrm{THH}(\mathbb{G})/\mathrm{THH}_0\mathbb{G} \to \Sigma^{\infty}(S^1/C_b \wedge X^{(b)})$$

be the projection onto the bth factor in (4.11).

THEOREM 4.13. *For classes* $\xi_i \in \tilde{H}_*(X)$,

$$(\mathrm{proj}_b \circ \mathrm{Tr} \circ \theta)_*(\iota_1 \otimes (\xi_1 \otimes \cdots \otimes \xi_a)) = \begin{cases} 0 & \text{if } b < a \\ \sum_{i=0}^{a-1} e_1 \otimes T^i(\xi_1 \otimes \cdots \otimes \xi_a) & \text{if } b = a. \end{cases}$$

Proof. There are maps

$$\rho_i^k \colon X^a \to G \times G$$

$$e_i \colon \Sigma^\infty(G_+ \wedge G_+) \to \mathrm{THH}_1(M_a(\mathbb{G})),$$

with

$$\rho_i^k(x_1, \ldots, x_k) = (\bar{x}_{i+1} \cdots \bar{x}_a \bar{x}^{k-1} \bar{x}_1 \cdots \bar{x}_{i-1}, \bar{x}_i)$$

$$e_i(g, h) = E_{i,i+1}(g) \wedge E_{i+1,i}(h).$$

The notation is $\bar{x} = \Pi \bar{x}_v$ and $\bar{x}_v = \bar{\imath}(x_v)$; cf. the James-Milnor splitting above. Then

$$\Sigma_{i=0}^{a-1}[e_i \circ \Sigma^\infty \rho_i^k] = [P^{ka-1} \wedge P].$$

We are primarily interested in $k = 1$, where we see from (4.2) and (4.12) that

$$(\mu \circ e_i \circ \Sigma^\infty \rho_i)_*(\iota_1 \otimes (\xi_1 \otimes \cdots \otimes \xi_a)) = e_1 \otimes T^i(\xi_1 \otimes \cdots \otimes \xi_a).$$

For $k > 1$, $(\mu \circ e_i \circ \Sigma^\infty \rho_i^k)_*$ maps trivially under $(\mathrm{proj}_a)_*$, but does in general have a nontrivial image under $(\mathrm{proj}_{ka})_*$. The result, however, is cumbersome to describe because it involves the reduced diagonal in $\tilde{H}_*(X)$. Fortunately we will have no use for it. For $i \not\equiv 0 \pmod a$, $(\mu \circ (P^{i-1} \wedge P))_* = 0$ and $(\mu \circ (P^i \wedge I))_* = 0$, since the resulting matrices have no tr_*-invariant (P^i has no diagonal entries). □

The projection $X^a \to X^{(a)}$ of the cartesian product into the smash product has a stable splitting, and θ then induces

$$\tilde{\theta} \colon \Sigma^\infty(S^1 \wedge X^{(a)}) \to A(\Sigma X).$$

Suppose $X = Y \vee Z$. Then

$$S^1_+ \wedge_{C_b} (Y \vee Z)^{(b)} = S^1_+ \wedge_{C_b} Y^{(b)} \vee S^1_+ \wedge Y^{(b-1)} \wedge Z \vee \cdots.$$

We have the injection and the projection

$$j\colon \Sigma^\infty(S^1 \wedge Y^{(a-1)} \wedge Z) \to \Sigma^\infty(S^1 \wedge X^{(a)})$$

$$\pi\colon \Sigma^\infty(S^1_+ \wedge_{C_b} X^{(b)}) \to \Sigma^\infty(S^1_+ \wedge Y^{(b-1)} \wedge Z).$$

COROLLARY 4.14. *The composition* $\pi \circ \mathrm{proj}_b \circ \mathrm{Tr} \circ \tilde{\theta} \circ j$ *from* $\Sigma^\infty(S^1 \wedge Y^{(a-1)} \wedge Z)$ *into* $\Sigma^\infty(S^1 \wedge Y^{(b-1)} \wedge Z)$ *induces zero in homology for* $b < a$ *and isomorphism for* $b = a$.

Since for any functor $\partial_y(F \circ \Sigma)(Y) = \Sigma \partial_y F(\Sigma Y)$, (2.8) implies

$$\partial_y(\mathrm{THH} \circ \Sigma)(Y) \simeq \Sigma \mathscr{L} \Sigma^\infty_+(\Omega \Sigma Y) \simeq \vee_{a=1}^{\infty} \Sigma^\infty(S^1_+ \wedge Y^{(a-1)}).$$

This follows alternatively from the homotopy equivalence

$$\Sigma^\infty_+(\mathscr{L}\Sigma X) \simeq \vee_{a=0}^{\infty} \Sigma^\infty(S^1_+ \wedge_{C_a} X^{(a)}).$$

Since

$$S^1_+ \wedge_{C_a} (Y \vee \Sigma^k Z)^{(a)} \underset{2k-1}{\simeq} S^1_+ \wedge_{C_a} Y^{(a)} \vee S^1_+ \wedge_{C_a} Y^{(a-1)} \wedge \Sigma^k Z,$$

so that

$$\lim_{\to} \Omega^k \Sigma^\infty(S^1_+ \wedge_{C_a} (Y \vee \Sigma^k Z)^{(a)})/S^1_+ \wedge_{C_a} Y^{(a)} \simeq \Sigma^\infty(S^1_+ \wedge Y^{(a-1)} \wedge Z).$$

Similarly,

$$\partial_y \Sigma^\infty(S^1 \wedge Y^{(a)}) = \vee_1^a \Sigma^\infty(S^1 \wedge Y^{(a-1)}).$$

Consider now the diagram

$$\begin{array}{ccc}
\vee_{a=1}^{\infty} \Sigma^\infty(S^1 \wedge X^{(a)}) & \xrightarrow{\bar{\theta}} & A(\Sigma X) \\
 & & \big\downarrow \mathrm{Tr} \\
\big\downarrow & & \mathrm{THH}(\Sigma X) \\
 & & \big\downarrow \simeq \\
\vee_{a=1}^{\infty} \Sigma^\infty(S^1/C_a \wedge X^{(a)}) & \xleftarrow{e_1} & \vee_{a=1}^{\infty} \Sigma^\infty(S^1_+ \wedge_{C_a} X^{(a)}).
\end{array}$$

COROLLARY 4.15. *The map* $\partial_y(e_1 \circ \mathrm{Tr} \circ \tilde{\theta})$ *is a split surjection of spectra.*

Proof. It is directly from (4.14) and the definition of derivative that

$$\vee_{a=1}^{\infty} \Sigma^{\infty}(S^1 \wedge Y^{(a-1)}) \xrightarrow{j} \vee_{a=1}^{\infty} \vee_1^a \Sigma^{\infty}(S^1 \wedge Y^{(a-1)}) \xrightarrow{\partial_y(e_1 \circ \mathrm{Tr} \circ \tilde{\theta})} \vee_{a=1}^{\infty} \Sigma^{\infty}(S^1 \wedge Y^{(a-1)})$$

induces an isomorphism on homology (with field coefficients). Hence it induces isomorphism on homology with integral coefficients, and is an equivalence of spectra. □

References

[A] J. F. Adams, "Prerequisites (on equivariant theory) for Carlsson's lecture" in *Algebraic Topology, Aarhus, 1982*, Lecture Notes in Math. **1051**, Springer-Verlag, Berlin, 1984, 483–532.

[BM] C. F. Bödigleimer and I. Madsen, *Homotopy quotients of mapping spaces and their stable splitting*, Quart. J. Math. Oxford Ser. (2) **39** (1988), 401–409.

[B] M. Bökstedt, *Topological Hochschild homology*, to appear in Topology.

[BHM] M. Bökstedt, W. C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), 865–940.

[BK] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions, and Localizations*, Lecture Notes in Math. **304**, Springer-Verlag, 1987.

[Bu] D. Burghelea, "Cyclic homology and algebraic K-theory of spaces I" in *Applications of Algebraic K-theory to Algebraic Geometry and Number Theory*, Contemp. Math. **55**, Amer. Math. Soc., Providence, 1986, 89–115.

[CC] G. Carlsson and R. Cohen, *The cyclic groups and the free loop space*, Comment. Math. Helv. **62** (1987), 423–449.

[CCGH] G. Carlsson, R. Cohen, T. Goodwillie, and W. C. Hsiang, *The free loop space and the algebraic K-theory of spaces*, K-Theory **1** (1987), 53–82.

[CCH] G. Carlsson, R. Cohen, and W. C. Hsiang, *The homotopy type of the space of pseudo-isotopies*, preprint.

[G1] T. Goodwillie, *Calculus I. The first derivative of pseudoisotopy*, K-Theory **4** (1990), 1–27.

[G2] ———, *Calculus II. Analytic functors*, K-Theory **5** (1991), 295–332.

[G3] ———, "The differential calculus of homotopy functors" in *Proceedings of the International Congress of Mathematicians (Kyoto, 1990)*, Springer-Verlag, Berlin, 1991, 621–630.

[I] K. Igusa, *The stability theorem for smooth pseudoisotopies*, K-Theory **2** (1988), 1–355.

[LQ] J. Loday and D. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Math. Helv. **59** (1984), 565–591.

[M] I. Madsen, "Algebraic K-theory and traces" in *Current Developments in Mathematics*, International Press, Boston, 1995, 191–321.

[S] D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. (2) **100** (1974), 1–79.

[W] F. Waldhausen, "Algebraic K-theory of spaces: A manifold approach" in *Current Trends in Algebraic Topology*, CMS Conf. Proc. **2**, Amer. Math. Soc., Providence, 1982, 141–184.

[W2] ———, "Algebraic K-theory of spaces, concordance, and stable homotopy theory" in *Algebraic Topology and Algebraic K-Theory*, Ann. of Math. Stud. **113**, Princeton Univ. Press, Princeton, 1987, 392–417.

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