THE EXOTIC $K(2)$-LOCAL PICARD GROUP AT THE PRIME 2

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Abstract. We calculate the group κ_2 of exotic elements in the $K(2)$-local Picard group at the prime 2 and find it is a group of order 2^9 isomorphic to $(\mathbb{Z}/8)^2 \times (\mathbb{Z}/2)^3$. In order to do this we must define and exploit a variety of different ways of constructing elements in the Picard group, and this requires a significant exploration of the theory. The most innovative technique, which so far has worked best at the prime 2, is the use of a J-homomorphism from the group of real representations of finite quotients of the Morava stabilizer group to the $K(n)$-local Picard group.

Contents

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1. Introduction

The focus of this paper is a computation and analysis of the subgroup κ_n of exotic elements in the Picard group of the $K(n)$-local stable homotopy category at a prime p. We give quite a few structural results and outline some significant techniques, all culminating in the main result Theorem 12.28, an isomorphism at $n = p = 2$

$$\kappa_2 \cong (\mathbb{Z}/8)^2 \times (\mathbb{Z}/2)^3.$$

We immediately note that this isomorphism, in itself, gives little insight into the importance of this group, where the elements arise, or what they mean. In fact, we might be better served writing

$$\kappa_2 \cong \left[\mathbb{Z}/8 \times (\mathbb{Z}/2)^2\right] \times \mathbb{Z}/2 \times \mathbb{Z}/8$$

reflecting the fact that the elements in this group arise in three different ways and reflect three fundamentally different aspects of the $K(2)$-local category. Making this clear and precise requires some explanation, and we begin there.

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category with unit object I. An object $X \in \mathcal{C}$ is invertible if there exists an object Y and an isomorphism $X \otimes Y \cong I$. If the collection of isomorphism classes of invertible objects is a set, then $\otimes$ defines a group structure on this set. This basic invariant of the category $\mathcal{C}$ is the *Picard group* $\mathrm{Pic}(\mathcal{C})$.

We are particularly interested in examples from stable homotopy theory. If we consider the stable homotopy category of spectra, the answer turns out to be simple, even if the proof is not: the only invertible spectra are the sphere spectra S^n, $n \in \mathbb{Z}$ and the map $\mathbb{Z} \to \mathrm{Pic}(\mathcal{S})$ sending n to S^n is an isomorphism. See [HMS94].

However, it is an insight due to Mike Hopkins [Str92] that we don't need to stop there and that, in fact, we can get a great deal of information if we pass to the localized stable homotopy categories which arise in chromatic stable homotopy theory. The basic example is then $K(n)$-local spectra, where $K(n)$ is the nth Morava K-theory at a fixed prime p.

Thus, the aim of this paper is to study Pic_n, the Picard group of the $K(n)$-local stable homotopy category at a fixed prime p. The first observation is that the $K(n)$-local category as a whole and its Picard group in particular can be accessed using Morava E-theory.

We fix a formal group law Γ_n of height n over the field $\mathbb{F}_{p^n}$ and let $E = E_n = E(\mathbb{F}_{p^n}, \Gamma_n)$ be the associated Lubin-Tate or Morava E-theory. (For a few more details, see Section 2.) For a spectrum X we then define

$$E_*X = \pi_* L_{K(n)}(E \wedge X).$$

This graded E_*-module has a continuous action by the Morava stabilizer group $\mathbb{G}_n = \mathrm{Aut}(\mathbb{F}_{p^n}, \Gamma_n)$, which gives it the structure of a Morava module in the sense of [BB20, Definition 5.2.30]. The completed tensor product over E_* endows the category of Morava modules with a symmetric monoidal structure with unit E_*S^0. Thus, the category of Morava modules also has a Picard group, which we write $(\mathrm{Pic}_n)_{\mathrm{alg}}$. It consists of isomorphism classes of Morava modules which are free of rank 1 over E_*. A basic fact from the foundational paper [HMS94] of Hopkins, Mahowald, and Sadofsky says that a $K(n)$-local spectrum X is in Pic_n if and only if E_*X is in $(\mathrm{Pic}_n)_{\mathrm{alg}}$. Thus we obtain a group homomorphism

$$\varepsilon : \mathrm{Pic}_n \to (\mathrm{Pic}_n)_{\mathrm{alg}} \; .$$

Calculating the target of this map is difficult, but fundamentally an algebraic problem, and can be reduced to questions in group cohomology.

If p is large with respect to n the map ε is an isomorphism; for example, if $n = 2$, then we need $p > 3$. This is a reflection of the fact that in those cases the homotopy theory of $K(n)$-local spectra is algebraic, in a sense which can be made completely precise. See [Pst22, Hea22, Pst21] for the state of the art. If p is small, however, very little is known. For example, ε is onto for $n = 1$ by [HMS94] and for $n = 2$ and $p \geq 3$ by [Lad13] and [Kar10], but we don't yet know if this is true in any other case, including $n = p = 2$.

This paper is focused on the group of *exotic* invertible $K(n)$-local spectra, denoted by κ_n, and defined as the kernel of ε. Thus $X \in \kappa_n$ if and only if

$$E_*X \cong E_*S^0 = E_*$$

as Morava modules. Again very little is known about κ_n. If $p = 2$, $\kappa_1 \cong \mathbb{Z}/2$ by [HMS94]. If $p = 3$, then $\kappa_2 \cong \mathbb{Z}/3 \times \mathbb{Z}/3$ by [GHMR15]. In earlier work [KS04], Kamiya and Shimomura had shown $\kappa_2 \neq 0$ at $p = 3$, and had given an upper bound as well. For all primes p, the group κ_{p-1} contains non-trivial elements of order p, by [BGHS22], and if $p = 2$ then $\kappa_n \neq 0$ by [HLS21], who also produce elements of 2-power order which grows with n. General considerations using [HS99a] and the vanishing line results of [DH04] show that there is an integer N so that $p^N\kappa_n = 0$, but we have no good bounds on N. It is possible to conjecture that κ_n is finite, just as it is possible to conjecture that $\pi_*L_{K(n)}S^0$ is topologically finitely generated in each degree. Both conjectures could be deduced from finiteness results for the cohomology of the group $\mathbb{G}_n$, which remains inaccessible at this point.

Our basic technique for studying κ_n uses the subgroup structure of $\mathbb{G}_n$. By Devinatz, Goerss, Hopkins, and Miller, the group $\mathbb{G}_n$ acts continuously on the spectrum E through $\mathscr{E}_\infty$-ring maps; thus if $H \subseteq \mathbb{G}_n$ is a closed subgroup, the homotopy fixed point spectrum E^{hH} is a ring spectrum with an associated category of modules, which has a Picard group $\mathrm{Pic}(E^{hH})$ and we get a homomorphism $\mathrm{Pic}_n \to \mathrm{Pic}(E^{hH})$. We would like to define $\kappa(H) \subseteq \kappa_n$ to be those elements in the kernel of this map; that is, those $X \in \kappa_n$ so that there is an equivalence $E^{hH} \simeq E^{hH} \wedge X$ of E^{hH}-modules. For various reasons, we need a more rigid definition of trivialization, which we call an H-orientation; see Definition 3.3. However, in every case we consider the weaker definition will suffice. See Lemma 3.8 and Example 3.9. In any case, if $H_1 \subset H_2 \subseteq \mathbb{G}_n$ is a sequence of subgroups will get inclusions $\kappa(H_2) \subseteq \kappa(H_1)$.

At $n = 2$ and $p = 2$ we will focus on two subgroups of $\mathbb{G}_2$. The first of these is G_{48}, a maximal finite subgroup with the property that $E^{hG_{48}}$ is the $K(2)$-localization of the Hopkins-Miller spectrum of topological modular forms. The other is $\mathbb{G}_n^1$ defined using the determinant map $\det : \mathbb{G}_n \to \mathbb{Z}_p^\times$. See Definition 4.1. Namely, if $\mu \subseteq \mathbb{Z}_p^\times$ is the finite subgroup, then we get a map

$$\mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p^\times/\mu \cong \mathbb{Z}_p$$

and $\mathbb{G}_n^1 \subseteq \mathbb{G}_n$ is the kernel of this map. This gives a filtration of κ_2 at $p = 2$

$$\kappa_2 \supseteq \kappa(G_{48}) \supseteq \kappa(\mathbb{G}_2^1) \tag{1.1}$$

and then in Theorem 8.13, Theorem 11.17, and Theorem 12.21 we compute the filtration quotients as

$$\begin{aligned} \kappa(\mathbb{G}_2^1) &\cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2 \\ \kappa(G_{48})/\kappa(\mathbb{G}_2^1) &\cong \mathbb{Z}/2 \\ \kappa_2/\kappa(G_{48}) &\cong \mathbb{Z}/8. \end{aligned} \tag{1.2}$$

We then show that this filtration splits in Corollary 11.23 and Theorem 12.28. To help with this, we compare this subgroup filtration with the descent filtration, the one arising from the Adams-Novikov spectral sequence. See Definition 3.27 for the details on this filtration. We remark immediately that this splitting is by no means canonical and our methods give no reason to expect anything comparable at a general height n. A roadmap and a more detailed synopsis is given after Definition 7.3, near the beginning of Part II and after we have all the definitions in place.

Echoing the three-part division of κ_2 displayed in (1.2), we have three ways of producing exotic elements in the Picard group. The first is a twisting construction, which appeared in [GHMR15] and was studied in some detail in [Wes17]. Let $\Lambda \subseteq \pi_0 E^{h\mathbb{G}_n^1}$ be in the kernel of the Hurewicz map to $E_0 E^{h\mathbb{G}_n^1}$. Then we can form the multiplicative subgroup $1 + \Lambda \subseteq (\pi_0 E^{h\mathbb{G}_n^1})^\times$. Note the quotient group $\mathbb{G}_n/\mathbb{G}_n^1 \cong \mathbb{Z}_p$ acts on $1 + \Lambda$. Now if $\alpha \in 1 + \Lambda$ and $\psi \in \mathbb{Z}_p$ is a topological generator we can form a fiber sequence

$$X(\alpha) \longrightarrow E^{h\mathbb{G}_n^1} \xrightarrow{\psi - \alpha} E^{h\mathbb{G}_n^1}.$$

In Theorem 4.19 we prove this assignment $\alpha \mapsto X(\alpha)$ defines an isomorphism

$$H^1(\mathbb{Z}_p, 1 + \Lambda) \cong \kappa(\mathbb{G}_n^1).$$

We then implement this isomorphism at $n = 2$ and $p = 2$. The group $\pi_0 E^{h\mathbb{G}_2^1}$ was computed in [BGH22]; the difficulty here is to compute the action of $\mathbb{G}_2/\mathbb{G}_2^1$ on this group. It turns out to be trivial, and the final calculation $\kappa(\mathbb{G}_2^1)$ is given in Theorem 8.13.

The second construction also uses the determinant map. At the prime 2, $\mu = \{\pm 1\} \subseteq \mathbb{Z}_2^\times$ is a cyclic group of order 2 and we have a homomorphism $\chi : \mathbb{G}_n \to C_2$ defined as the composition

$$\mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_2^\times \longrightarrow \mathbb{Z}_2^\times/(1 + 4\mathbb{Z}_2) \cong \mu = C_2.$$

If V is a virtual real representation of C_2 with one-point compactification S^V, $\mathbb{G}_n$ acts on S^V through this quotient, Giving $E \wedge S^V$ the diagonal $\mathbb{G}_n$-action, we can form

$$J(V) = (E \wedge S^V)^{h\mathbb{G}_n}.$$

If V has virtual dimension zero and C_2 acts trivially on $H_0 S^V \cong \mathbb{Z}$, then $J(V) \in \kappa_n$. Now let σ be the sign representation of C_2. We show in Theorem 11.17 and Theorem 11.21 that if $n = 2$

$$J(2\sigma - 2) \in \kappa(G_{48})$$

is an element of order 2 and generates the quotient group $\kappa(G_{48})/\kappa(\mathbb{G}_2^1) \cong \mathbb{Z}/2$. As a preliminary to all this, we devote Section 5 to discussing the general categorical properties of this J-construction. Before getting this far, we need to show $\kappa(G_{48})/\kappa(\mathbb{G}_2^1)$ is no larger than $\mathbb{Z}/2$. This requires a separate suite of ideas and uses the topological resolutions pioneered in [GHMR05] and made explicit at $p = 2$ in [BG18]. So far these techniques are very special to height 2.

Note that we might conjecture that $J(2\sigma - 2)$ is a non-trivial element of κ_n for all n at $p = 2$. This is true for $n = 1$ by a simple calculation and here we show it at $n = 2$. It is possible to prove this for n odd using [HLS21] and the fact that χ has a splitting. For larger even n, this would follow if we could show a certain cohomology class in $H^3(\mathbb{G}_n, E_2)$ was non-zero. The precise class at $n = 2$ is given in Proposition 10.9. This is quite plausible, but we make no attempt to prove it here.

The third technique uses the fact that the $K(n)$-local category has two dualities: Spanier-Whitehead duality D_n and Gross-Hopkins duality I_n. These are related by the formula

$$I_n X \cong D_n X \wedge I_n$$

where I_n is the Gross-Hopkins dual of the sphere. By [HG94b] and [Str00] we have an equation

$$I_n = \Sigma^{n^2-n} S\langle\det\rangle \wedge P$$

where $S\langle\det\rangle$ is a determinant sphere and $P \in \kappa_n$. See (12.17) and [BBGS22]. The technique then is to find an X so that we know $D_n X$ and $I_n X$, and then see that the equation for I_n forces $P \neq 0$. This is by now a classical idea, and has been used in [Beh06, GHMR15, BGHS22, HLS21], and probably elsewhere. In our case we take $X = E^{hG_{48}}$ where we can use [Pha21, MR99, Sto12, Sto14] for $I_2 E^{hG_{48}}$ and [Bob20, BGHS22] for $D_2 E^{hG_{48}}$. Then not only is $0 \neq P \in \kappa_2$, but P generates $\kappa_2/\kappa(G_{48}) \cong \mathbb{Z}/8$. See Theorem 12.21.

There is a drawback to this last technique. The spectrum I_n is the Brown-Comenetz dual of the nth monochromatic layer of the sphere and thus is a product of localization theory and Brown Representability. That we know $E_* I_n$ at all relies on a deep result in the algebraic geometry of formal groups: the identification of the dualizing sheaf of Lubin-Tate space, as in [HG94a]. This is all very indirect, and gives no detailed information on the homotopy type of I_n or, by extension, of the spectrum P. We only know that P is non-trivial because of its effects on much larger spectra. It is as if we've observed the perturbations of the orbit of Uranus, but have not yet discovered Pluto. Put another way, Gross-Hopkins duality remains a tantalizing mystery when the prime is low with respect to the height. For further thoughts on P, see Remark 12.24.

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Notation. This is a long paper, and concepts defined one place can reappear in others, often many pages distant. We collect here some of the less standard items, along with references to where they are defined.

Subgroups and cohomology classes of the Morava stabilizer group $\mathbb{G}_n$.

(1) $\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$ is defined in Definition 4.1
(2) $\zeta : \mathbb{G}_n \to \mathbb{Z}_p$ is defined in Definition 4.2
(3) $\chi : \mathbb{G}_n \to \mathbb{Z}/2$ is defined in Definition 4.4
(4) $S\mathbb{G}_n$, $\mathbb{G}_n^1$, and $\mathbb{G}_n^0$ are the kernels of det, ζ and χ respectively
(5) $\zeta \in H^1(\mathbb{G}_n, E_0)$ and $\widetilde{\chi} \in H^2(\mathbb{G}_n, E_0)$ are defined in Definition 4.6
(6) G_{48}, G_{24} and other finite subgroups of $\mathbb{G}_2$ are defined in Definition 7.1
(7) e and k in $H^*(\mathbb{G}_2^1, E_0)$ appear in Figure 1

Subgroups of κ_n.

(1) $\kappa(K)$ and orientations are discussed in Definition 3.3 and Remark 3.4
(2) $\kappa_{n,r}$ is defined in Definition 3.27
(3) $\phi_r\colon \kappa_{n,r} \longrightarrow E_r^{r,r-1}(S^0)$ is defined in Definition 3.27
(4) $\kappa_r(K) = \kappa(K) \cap \kappa_{n,r}$
(5) ϕ_r^1 (we need only ϕ_3^1) is defined in (10.1)

Various constructions and concepts.

(1) algebraic maps of spectra are defined in Definition 3.21
(2) $\Lambda \subseteq \pi_0 E^{h\mathbb{G}_n^1}$ is defined in (4.8)
(3) Λ_s is defined in Definition 4.20
(4) $X(\alpha) \in \kappa(\mathbb{G}_n^1)$ is defined in Definition 4.9
(5) $J(q, f, K)$ and $J(V)$ are defined in (5.1) and Example 5.2
(6) $\mathbf{X}_{\leq M}$ and $\mathbf{X}_K^M$ are defined in Definition 6.5 and Definition 6.7

(7) $\mathbb{RP}^n_k$ is defined in (11.4)

Part I. $K(n)$-Local Results for General n and p

This first part of the paper focuses on machinery and results used to study the $K(n)$-local category, its Picard group, and κ_n for general heights n and prime p. We will specialize to the case $n = p = 2$ in the second part.

2. The $K(n)$-local category

Detailed introductions to the $K(n)$-local category can be found in many places; for example, much of the foundations can be found in [HS99b]. A precise summary of what is needed here can be found in Section 2 of [BGH22] and we will use the language and notation of that reference. Here we give a short summary to establish the context.

We begin with the selection of a formal group Γ_n of height n. We will assume Γ_n is defined over $\mathbb{F}_p$ and that for any extension $\mathbb{F}_{p^n} \subseteq \mathbb{F}_q$ of finite fields the inclusion of automorphism groups

$$\mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \subseteq \mathrm{Aut}(\Gamma_n/\mathbb{F}_q)$$

is an equality. Examples include the Honda formal group of height n and the formal group arising from the standard supersingular elliptic curves at $p = 2$ and $p = 3$. See [Str00] and [Hen19].

We will write $\mathbb{G}_n$ for the automorphisms of the pair $(\mathbb{F}_{p^n}, \Gamma_n)$; there is a semi-direct product decomposition

$$\mathbb{G}_n \cong \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

It is customary to define $\mathbb{S}_n = \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n})$ and we will write $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$.

We define $K(n)$ to be a 2-periodic complex orientable homology theory with $K(n)_* \cong \mathbb{F}_{p^n}[u^{\pm 1}]$ with u in degree -2 and associated formal group Γ_n. This homology theory has the same Bousfield localization functor as the $2(p^n - 1)$-periodic version historically labelled $K(n)$. The 2-periodic version used here is better related to Morava E-theory. See (2.1).

The $K(n)$-local category has a symmetric monoidal structure with product given by

$$X \wedge Y = L_{K(n)}(X \wedge Y).$$

As this equation indicates, we adorn the smash product with the localization only when emphasis is needed; normally, we will leave it understood.

The most important and most basic algebraic invariant of the $K(n)$-local category is given by Lubin-Tate or Morava E-theory. In fact, E-theory is a functor that associates a $K(n)$-local $\mathcal{E}_\infty$-ring spectrum to a pair of a perfect field and a formal group law over it [GH04]. We will write

$$E = E_n = E(\mathbb{F}_{p^n}, \Gamma_n). \tag{2.1}$$

There is a non-canonical isomorphism

$$E_* \cong \mathbb{W}[[u_1 \ldots, u_{n-1}]][u^{\pm 1}],$$

where $\mathbb{W} = W(\mathbb{F}_{p^n})$ is the Witt vectors on $\mathbb{F}_{p^n}$. The power series rings is in degree zero and u is in degree -2. This is a Landweber exact complex orientable theory with formal group given by a universal deformation of Γ_n. Our version of Morava K-theory is chosen so that there is a map $E \to K(n)$; on coefficients this map is given by the quotient by the maximal ideal $\mathfrak{m} = (p, u_1, \ldots, u_{n-1})$.

We define

$$E_*X = \pi_* L_{K(n)}(E \wedge X).$$

Since the formal group for E is the universal deformation of Γ_n, the group $\mathbb{G}_n$ acts on E_* and, in fact, this lifts to an action on E through maps of $\mathcal{E}_\infty$-ring spectra. Thus E_*X is a continuous E_*-module with a continuous action by $\mathbb{G}_n$; more precisely, it is a Morava module in the sense of Definition 5.3.20 of [BB20]. We will write $\mathfrak{Mor}$ for the category of Morava modules.

A fundamental fact is that the action of $\mathbb{G}_n$ on the right factor of $E \wedge E = L_{K(n)}(E \wedge E)$ defines an isomorphism of Morava modules

$$E_*E \cong \mathrm{map}(\mathbb{G}_n, E_*) \tag{2.2}$$

to the ring of continuous functions. See [Str00], [DH04], or [Hen19]. From this it follows, as in [DH04], that for many reasonable spectra – including all the spectra of this paper – the $K(n)$-local Adams-Novikov Spectral Sequence has the form

$$E_2^{s,t}(X) = H^s(\mathbb{G}_n, E_tX) \Longrightarrow \pi_{t-s} L_{K(n)}X. \tag{2.3}$$

Note well: Here, and throughout the paper, group cohomology will be understood to be *continuous* cohomology whenever this makes sense. Likewise, maps of sets will be understood to be continuous whenever that makes sense. We will have more to say about this spectral sequence and its construction in Example 6.2.

Remark 2.4. The Devinatz-Hopkins fixed point theory [DH04] supplies a functorial assignment $K \mapsto E^{hK}$ from closed subgroups of $\mathbb{G}_n$ to $K(n)$-local $\mathcal{E}_\infty$-ring spectra. We have that $L_{K(n)}S^0 \simeq E^{h\mathbb{G}_n}$ and if $K = \{e\}$ then $E^{hK} = E$. This construction has the following properties. First, there is an isomorphism of Morava modules

$$E_*E^{hK} \cong \mathrm{map}(\mathbb{G}_n/K, E_*) \tag{2.5}$$

and, second, for dualizable $K(n)$-local spectra X, there is a homotopy fixed point or Adams-Novikov spectral sequence

$$E_2^{s,t}(K, X) = H^s(K, E_tX) \Longrightarrow \pi_{t-s}(E^{hK} \wedge X). \tag{2.6}$$

The spectral sequence of (2.3) is the case of $K = \mathbb{G}_n$. Further, the spectral sequence (2.5) for any X is a module over the analogous spectral sequence for $X = S^0$, a fact which will be exploited several times in this paper.

3. Orientations and Filtrations

Let $K \subseteq \mathbb{G}_n$ be a closed subgroup. In this section we introduce the concept of an E^{hK}-orientation of an exotic element in the Picard group. This allows us to introduce a decomposition on κ_n, which reflects the subgroup lattice of $\mathbb{G}_n$. We also introduce a second, more classical, descent filtration which, in essence, comes from the Adams–Novikov filtration.

3.1. **Orientations.** We begin with a basic definition.

Definition 3.1. Let $X \in \kappa_n$. Then a $\mathbb{G}_n$*-invariant generator* for X is a choice of a $\mathbb{G}_n$-invariant element ι_X in E_0X which generates E_*X as an E_*-module.

Remark 3.2. For $X \in \kappa_n$, a $\mathbb{G}_n$-invariant generator ι_X determines, and is determined by, a choice of isomorphism of Morava modules $\varphi : E_* \to E_*X$. This φ defines an isomorphism

$$H^*(\mathbb{G}_n, E_*) \cong H^*(\mathbb{G}_n, E_*X).$$

Two choices of $\mathbb{G}_n$-invariant generators differ by an element of $\mathbb{Z}_p^\times \cong H^0(\mathbb{G}_n, E_0)^\times$. The latter isomorphism was historically known to the experts; one proof is in [BG18, Lemma 1.33].

Definition 3.3. Let $K \subseteq \mathbb{G}_n$ be a closed subgroup. Then $\kappa(K) \subseteq \kappa_n$ is the subgroup of invertible spectra X such that there is a class $z \in \pi_0(E^{hK} \wedge X)$ which maps to a $\mathbb{G}_n$-invariant generator under the map

$$\pi_0(E^{hK} \wedge X) \to \pi_0(E \wedge X) = E_0X.$$

We call the class z an E^{hK}*-orientation* of X, or simply an *orientation* if K and X are understood.

Remark 3.4 (Subgroup filtrations)**.** If $K_1 \subseteq K_2$, then $\kappa(K_2) \subseteq \kappa(K_1)$, so the assignment $K \mapsto \kappa(K)$ defines a filtered diagram of subgroups of κ_n. In particular, any nested sequence of closed subgroups

$$\{e\} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_k = \mathbb{G}_n,$$

gives an associated filtration

$$\kappa_n = \kappa(\{e\}) \supseteq \kappa(K_1) \supseteq \cdots \supseteq \kappa(\mathbb{G}_n) = \{L_{K(n)}S^0\} \ ,$$

We call this a *subgroup filtration* of κ_n.

In order to compute the subgroups $\kappa(K)$, it is necessary to be able to find E^{hK}-orientations of elements X of κ_n. The goal of the remainder of this section is to give recognition principles for E^{hK}-orientations which use the homotopy fixed point spectral sequence (2.6).

Proposition 3.5. *Let $X \in \kappa_n$ and let $\iota_X \in E_0X$ be a $\mathbb{G}_n$-invariant generator. Then X is in $\kappa_n(K)$ if and only if ι_X is a permanent cycle in the homotopy fixed point spectral sequence*

$$H^s(K, E_tX) \Longrightarrow \pi_{t-s}(E^{hK} \wedge X).$$

Proof. First, suppose that ι_X is a permanent cycle detecting a class $z \in \pi_0(E^{hK} \wedge X)$. Then since ι_X is a $\mathbb{G}_n$-equivariant of E_*X, the class z is an E^{hK}-orientation.

Conversely, assume $X \in \kappa(K)$. Then we have $z \in \pi_0(E^{hK} \wedge X)$, whose image in E_0X is a $\mathbb{G}_n$-invariant generator, so it must be a unit multiple of ι_X. Thus z is represented by a permanent cycle equaling $a\iota_X$ for some $a \in \mathbb{Z}_p^\times$. This implies that ι_X itself is also a permanent cycle. □

We now examine some consequences of having an E^{hK}-orientation.

Proposition 3.6. *Let $z \in \pi_0(E^{hK} \wedge X)$ be an E^{hK}-orientation for X. Then the extension of z to an E^{hK}-module map*

$$j_z \colon E^{hK} \longrightarrow E^{hK} \wedge X$$

is an equivalence.

Proof. As mentioned in Remark 2.4, the homotopy fixed point spectral sequence (2.6)

$$E_2^{s,t}(K, S^0) = H^s(K, E_t) \Longrightarrow \pi_{t-s} E^{hK}$$

for S^0 acts on the spectral sequence $E_2^{s,t}(K, X)$, and calculates the action of E_*^{hK} on $E_*^{hK}(X) = \pi_*(E^{hK} \wedge X)$. Suppose $X \in \kappa_n(K)$, and suppose that we have chosen a $\mathbb{G}_n$-invariant generator $\iota_X \in E_0 X$. If $d_q(\iota_X) = 0$ for $q < r$, then the map $a \mapsto a\iota_X$ defines an isomorphism

$$E_r^{*,*}(K, S^0) \xrightarrow{\cong} E_r^{*,*}(K, X).$$

In particular, if ι_X is a permanent cycle detecting the class $z \in \pi_0(E^{hK} \wedge X)$, then the module map $E^{hK} \to E^{hK} \wedge X$ extending z is an equivalence. □

The equivalence of Proposition 3.6 also works well in E-homology.

Proposition 3.7. *Let $z \in \pi_0(E^{hK} \wedge X)$ be an E^{hK}-orientation for X and let $\varphi_* : E_* \to E_* X$ be the Morava module isomorphism induced by the $\mathbb{G}_n$-invariant generator determined by z. Then there is a commutative diagram*

$$\begin{tikzcd}
E_* E^{hK} \arrow[r, "\cong"] \arrow[d, "E_*(\varphi)"'] & \mathrm{map}(\mathbb{G}/K, E_*) \arrow[d, "\mathrm{map}(\mathbb{G}/K, \varphi_*)"] \\
E_*(E^{hK} \wedge X) \arrow[r, "\cong"'] & \mathrm{map}(\mathbb{G}/K, E_* X).
\end{tikzcd}$$

Proof. The horizontal equivalences are obtained as the adjoint of the composition

$$\mathbb{G}_n/K \times E_*(E^{hK} \wedge X) \to E_*(E \wedge X) \to E_* X$$

where the first map is given by the action map $\mathbb{G}_n/K \times E^{hK} \to E$ and the second by the multiplication $E \wedge E \to E$. Now it is a matter of chasing around the diagram. □

Proposition 3.6 suggests an alternative, potentially more natural, characterization of the elements of $\kappa(K)$ which only requires an equivalence of E^{hK}-modules $E^{hK} \to E^{hK} \wedge X$. We now give a criterion for when this is sufficient.

Lemma 3.8. *Suppose $X \in \kappa_n$ and that we have an E^{hK}-module equivalence*

$$f \colon E^{hK} \xrightarrow{\simeq} E^{hK} \wedge X,$$

not assumed to be induced by an E^{hK}-orientation of X. Suppose further that the edge homomorphism

$$\pi_0 E^{hK} \longrightarrow H^0(K, E_0)$$

is onto. Then X has an E^{hK}-orientation, and $X \in \kappa(K)$.

Proof. We need to find a class $z \in \pi_0(E^{hK} \wedge X)$ which maps to a $\mathbb{G}_n$-invariant generator of E_0X. The E^{hK}-module equivalence f determines a K-invariant generator $f_*(1) = y \in E_0X$. Since $X \in \kappa_n$, we also can choose a $\mathbb{G}_n$-invariant generator x of E_0X. There is an element $a \in (E_0)^\times$ so that $ay = x$, since both are E_0-module generators. Since x and y are both K-invariant generators of a free module, a is also K-invariant. Choose an element $\alpha \in \pi_0 E^{hK}$ which maps to a and let $\phi : E^{hK} \to E^{hK}$ be the E^{hK}-module map extending α. We then have a new E^{hK}-module equivalence $g = (\phi \wedge 1_X)f : E^{hK} \to E^{hK} \wedge X$ and $g_*(1) = x$ as needed. □

Example 3.9. The hypothesis on K in Lemma 3.8 is equivalent to the statement that $E_\infty^{0,0} = E_2^{0,0}$ in the homotopy fixed point spectral sequence

$$E_2^{s,t}(K, S^0) = H^s(K, E_t) \Longrightarrow \pi_{t-s} E^{hK}.$$

This happens in all the examples for which we have complete calculations, including all the examples at $n = p = 2$ we will consider in this paper.

The following result will allow for some flexibility in later sections, as it tells us that the Galois group will not cause complications when it comes to orientations.

Proposition 3.10. *Let $K \subseteq \mathbb{G}_n$ be a closed subgroup and $K_0 = \mathbb{S}_n \cap K$. Then the inclusion $\kappa_n(K) \subseteq \kappa_n(K_0)$ is an equality.*

Proof. Let $G = K/K_0 \subseteq \mathbb{G}_n/\mathbb{S}_n \cong \mathrm{Gal}$. For any dualizable spectrum X, which in particular includes invertible spectra, the same argument as for [BG18, Lemma 1.32] shows that

$$\mathbb{W} \otimes_{\mathbb{W}^G} E_2^{s,t}(K, X) \cong E_2^{s,t}(K_0, X).$$

But for such X, the differentials in $E_r^{s,t}(K_0, X)$ are automatically $\mathbb{W}$-linear since the spectral sequence is a module over the spectral sequence $E_r^{s,t}(K_0, S^0)$ and the latter has $\mathbb{W}$-linear differentials. This follows from Remark 1.35 of [BG18] and the fact that the unit $\mathbb{Z}_p \to \mathbb{W}$ is étale. Thus, for any dualizable spectrum X, we have

$$\mathbb{W} \otimes_{\mathbb{W}^G} E_r^{s,t}(K, X) \cong E_r^{s,t}(K_0, X)$$

and the isomorphism preserves differentials. Now apply Proposition 3.5. □

3.2. **The naturality of orientations.** The material in this section, specifically the notion of an algebraic map introduced in Definition 3.21 and their relationship with orientations presented in Proposition 3.22 will be used in an essential way in Section 10.1, specifically in Proposition 10.4.

For a closed subgroup K of $\mathbb{G}_n$, let $X \in \kappa(K)$, and let $z \in \pi_0(E^{hK} \wedge X)$ be a chosen E^{hK}-orientation of X. Let $\varphi_z : E^{hK} \to E^{hK} \wedge X$ be the equivalence of E^{hK}-modules obtained by extending z.

If $g \in \mathbb{G}_n$ and $H \subseteq gKg^{-1}$, the composite

$$E^{hK} \xrightarrow{g} E^{h(gKg^{-1})} \longrightarrow E^{hH}$$

gives a map which we also call $g : E^{hK} \to E^{hH}$. The following result is immediate from the definitions.

Lemma 3.11. *Let $g \in \mathbb{G}_n$ and $H \subseteq gKg^{-1}$. If $z \in \pi_0(E^{hK} \wedge X)$ is an E^{hK}-orientation then*
$$gz = (g \wedge 1)_*(z) \in \pi_0(E^{hH} \wedge X)$$
is an E^{hH}-orientation and the following diagram commutes

(3.12)
$$\begin{tikzcd} E^{hK} \arrow[rr, "g"] \arrow[d, "\varphi_z"', "\simeq"] & & E^{hH} \arrow[d, "\simeq"', "\varphi_{gz}"] \\ E^{hK} \wedge X \arrow[rr, "g \wedge 1_X"] & & E^{hH} \wedge X . \end{tikzcd}$$

In this result, the map g is induced from an element in the Morava stabilizer group. We would like to expand the class of maps for which we have a similar diagram. Our key result is Proposition 3.22, but to state and prove it, we first need to develop some language.

First, here is some material originally due to Devinatz–Hopkins [DH04]; it was also reviewed in [GHMR05] and [BBGS22]. For any dualizable X in the $K(n)$-local category, there is an isomorphism of Morava modules

$$E_*(E^{hK} \wedge X) \cong \mathrm{map}(\mathbb{G}_n/K, E_*X), \tag{3.13}$$

where $\mathbb{G}_n$ acts on the right-hand side by conjugation: $(g\phi)(hK) = g\phi(g^{-1}hK)$. We also have an isomorphism of Morava modules

$$\mathrm{map}(\mathbb{G}_n/K, E_*X) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\mathbb{G}_n/K]], E_*X), \tag{3.14}$$

where the right-hand side is the group of continuous $\mathbb{Z}_p$-module maps.

We let $X = S^0$. Given closed subgroups K_1, K_2 of $\mathbb{G}_n$, let $h : \mathbb{Z}_p[[\mathbb{G}_n/K_2]] \to \mathbb{Z}_p[[\mathbb{G}_n/K_1]]$ be a continuous $\mathbb{G}_n$-module map. Using the identification of (3.14) we obtain a map

$$\mathrm{Hom}(h, E_*)\colon\ \mathrm{map}(\mathbb{G}_n/K_1, E_*) \longrightarrow \mathrm{map}(\mathbb{G}_n/K_2, E_*)$$

of Morava modules and so we get a homomorphism

$$\begin{aligned} \Psi\colon\ \mathrm{Hom}_{\mathbb{Z}_p[[\mathbb{G}_n]]}(\mathbb{Z}_p[[\mathbb{G}_n/K_2]], \mathbb{Z}_p[[\mathbb{G}_n/K_1]]) \longrightarrow& \\ \mathrm{Hom}_{\mathfrak{Mor}}(\mathrm{map}(\mathbb{G}_n/K_1, E_*), \mathrm{map}(\mathbb{G}_n/K_2, E_*)).& \end{aligned} \tag{3.15}$$

The map Ψ is always an injection; however, it is not always surjective. For example, if $K_1 = \mathbb{G}_n$; then,

$$\mathrm{Hom}_{\mathbb{Z}_p[[\mathbb{G}_n]]}(\mathbb{Z}_p[[\mathbb{G}_n/K_2]], \mathbb{Z}_p) \cong \mathbb{Z}_p,$$

while

$$\mathrm{Hom}_{\mathfrak{Mor}}(E_*, \mathrm{map}(\mathbb{G}_n/K_2, E_*)) \cong H^0(K_2, E_0) \cong E_0^{K_2}.$$

If $n > 1$ and $K_2 = \{e\}$, then E_0 is a power series ring strictly containing $\mathbb{Z}_p$. More critically for us, if $p = 2$ and K_2 is finite, then $E_0^{K_2}$ is typically also a power series ring larger than $\mathbb{Z}_2$.

This discussion suggests that we single out a class of Morava module maps

$$E_*E^{hK_1} \cong \mathrm{map}(\mathbb{G}_n/K_1, E_*) \longrightarrow \mathrm{map}(\mathbb{G}_n/K_2, E_*) \cong E_*E^{hK_2}$$

that includes those the image of Ψ; that is, those that can be written $\mathrm{Hom}(h, E_*)$ for some continuous $\mathbb{G}_n$-map $h\colon \mathbb{Z}_p[[\mathbb{G}_n/K_2]] \to \mathbb{Z}_p[[\mathbb{G}_n/K_2]]$. We will also need to incorporate the periodicity of E^{hK} into the picture. Thus, we start by a brief discussion on the interplay of Morava modules and periodicities.

Definition 3.16. Let $K \subseteq \mathbb{G}_n$ be a closed subgroup. An *algebraic periodicity class* for K is a unit

$$d \in H^0(K, E_*)$$

in the ring of invariants. We say E^{hK} has *algebraic period* k if k is the smallest positive integer for which there is an algebraic periodicity class in degree k, i.e. $d \in H^0(K, E_k)$.

Let $K \subseteq \mathbb{G}_n$ be a closed subgroup, X be a spectrum, and suppose we have an element $d \in H^0(K, E_k)$. Then for all m we get an induced map of Morava modules

$$\begin{aligned} f_d^X \colon \operatorname{map}(\mathbb{G}_n/K, E_mX) &\longrightarrow \operatorname{map}(\mathbb{G}_n/K, E_{m+k}X) \\ \varphi &\longmapsto \psi \end{aligned}$$

with

$$\psi(gK) = (gd)\varphi(gK).$$

This is a map of $\operatorname{map}(\mathbb{G}_n/K, E_0)$-modules in the category of Morava modules. Definition 3.16 makes the following result obvious, since if d is invertible, then $f^X_{d^{-1}}$ is inverse to f^X_d.

Lemma 3.17. *Let $d \in H^0(K, E_k)$ be a periodicity class, then for all m and all X, the map*

$$f_d^X : \operatorname{map}(\mathbb{G}_n/K, E_mX) \longrightarrow \operatorname{map}(\mathbb{G}_n/K, E_{m+k}X)$$

is an isomorphism.

Remark 3.18. It could happen that the only algebraic periodicity classes are in $H^0(K, E_0)$, in which case E^{hK} doesn't have an algebraic period. This is the case for $K = \mathbb{G}_n$ itself, for example. We are more interested in this notion when K is a finite subgroup of $\mathbb{G}_n$. Indeed, if K is finite of order m then

$$d = \prod_{g \in K} gu^{-1} \in H^0(K, E_{2m})$$

is an algebraic periodicity class, although perhaps not one of minimal positive degree.

Remark 3.19. There is a corresponding notion of a topological periodicity class, namely a unit $x \in \pi_* E^{hK}$ in positive dimension k. Then the induced E^{hK}-module map $\Sigma^k E^{hK} \to E^{hK}$ is an equivalence of E^{hK}-module spectra. The corresponding notion of topological period is the minimal k for which such an x exists. An algebraic periodicity class determines a topological periodicity class if and only if it is a permanent cycle in the homotopy fixed point spectral sequence; hence, the algebraic and topological periods can and often do differ. For example, E^{hC_2} at height one (which is the 2-completion of the real K-theory spectrum KO) has algebraic period 4 and topological period 8. At height two and the prime 2, $E^{hG_{48}}$ has algebraic period 24 and topological period $192 = 8 \cdot 24$. (Details on the subgroup G_{48} can be found in Definition 7.1.)

For any spectrum X, an algebraic periodicity class $d \in H^0(K, E_k)$ determines an isomorphism

$$P_d^X : E_*(\Sigma^k E^{hK} \wedge X) \xrightarrow{\cong} \operatorname{map}(\mathbb{G}_n/K, E_*X)$$

as the composition

$$E_*(\Sigma^k E^{hK} \wedge X) \xrightarrow{\cong} \mathrm{map}(\mathbb{G}_n/K, E_{*-k}X) \xrightarrow[f_d^X]{\cong} \mathrm{map}(\mathbb{G}_n/K, E_*X).$$

Remark 3.20. We have isomorphisms

$$\begin{tikzcd}
E_*E^{hK} \otimes_{E_*} E_*X \arrow[r, "\cong"] \arrow[d, "\cong"] & E_*(E^{hK} \wedge X) \arrow[d, "\cong"] \\
\mathrm{map}(\mathbb{G}_n/K, E_*) \otimes_{E_*} E_*X \arrow[r, "\cong"] & \mathrm{map}(\mathbb{G}_n/K, E_*X)
\end{tikzcd}$$

Using these identifications and letting $f_d = f_d^{S^0}$ and $P_d = P_d^{S^0}$, we have that $f_d^X = f_d \otimes E_*X$ and $P_d^X \cong P_d \otimes E_*X$.

We now single out a particularly useful set of maps.

Definition 3.21. Let K_1, K_2 be closed subgroups of $\mathbb{G}_n$, and let $d \in H^0(K_2, E_k)$ be an algebraic periodicity class for K_2. A map $f : E^{hK_1} \to \Sigma^k E^{hK_2}$ of spectra is *algebraic* for d if the induced map g of Morava modules

$$\mathrm{map}(\mathbb{G}_n/K_1, E_*) \xrightarrow{g} \mathrm{map}(\mathbb{G}_n/K_2, E_*)$$

defined by the commutative square

$$\begin{tikzcd}
E_*E^{hK_1} \arrow[r, "E_*f"] \arrow[d, "\cong"'] & E_*\Sigma^k E^{hK_2} \arrow[d, "\cong"', "P_d"] \\
\mathrm{map}(\mathbb{G}_n/K_1, E_*) \arrow[r, "g"'] & \mathrm{map}(\mathbb{G}_n/K_2, E_*)
\end{tikzcd}$$

is in the image of the map Ψ of (3.15). Thus, $g = \mathrm{Hom}(h, E_*)$ for some continuous $\mathbb{G}_n$-module map

$$h\colon \mathbb{Z}_p[[\mathbb{G}_n/K_2]] \longrightarrow \mathbb{Z}_p[[\mathbb{G}_n/K_1]].$$

The next result mixes orientations and algebraic maps, so requires some set-up. We fix the following data

(1) a closed subgroup $K \subseteq \mathbb{G}_n$ and element $X \in \kappa(K)$;
(2) an E^{hK}-orientation $z \in \pi_0(E^{hK} \wedge X)$;
(3) two elements $g_i \in \mathbb{G}_n$, for $1 \le i \le 2$ and two closed subgroups $H_i \subseteq g_i K g_i^{-1}$; and,
(4) an algebraic periodicity class $d \in H^0(H_2, E_k)$.

Then by Lemma 3.11 we have induced orientations $g_i z \in \pi_0(E^{hH_i} \wedge X)$. Let

$$\varphi_i = \varphi_{g_i z}\colon E^{hH_i} \longrightarrow E^{hH_i} \wedge X$$

be the induced E^{hH_i}-module equivalences. In this context, we have the following result.

Proposition 3.22. *Let $f\colon E^{hH_1} \to \Sigma^k E^{hH_2}$ be any map which is algebraic for d. Then the following diagram of Morava modules commutes*

$$\begin{tikzcd} E_*E^{hH_1} \arrow[r, "E_*f"] \arrow[d, "E_*(\varphi_1)"'] & E_*\Sigma^k E^{hH_2} \arrow[d, "E_*(\Sigma^k\varphi_2)"] \\ E_*(E^{hH_1} \wedge X) \arrow[r, "E_*(f\wedge 1_X)"'] & E_*(\Sigma^k E^{hH_2} \wedge X). \end{tikzcd} \tag{3.23}$$

Proof. By Proposition 3.7, the induced H_i-orientations for X give us commutative diagrams

$$\begin{tikzcd} E_*E^{hH_i} \arrow[r, "\cong"] \arrow[d, "E_*(\varphi_{z_i})"'] & \mathrm{map}(\mathbb{G}/H_i, E_*) \arrow[d, "\mathrm{map}(\mathbb{G}/H_i,\varphi_*)"] \\ E_*(E^{hH_i} \wedge X) \arrow[r, "\cong"'] & \mathrm{map}(\mathbb{G}/H_i, E_*X). \end{tikzcd} \tag{3.24}$$

We now use the hypothesis that f is an algebraic map, as then Definition 3.21 gives a corresponding map g with

$$g = \mathrm{Hom}(h, E_*)$$

for some continuous $\mathbb{G}_n$-module map $h : \mathbb{Z}_p[[\mathbb{G}_n/H_2]] \to \mathbb{Z}_p[[\mathbb{G}_n/H_1]]$. Using Remark 3.20, we also have the following commutative diagram

$$\begin{tikzcd} E_*(E^{hH_1} \wedge X) \arrow[r, "E_*(f\wedge X)"] \arrow[d, "\cong"'] & E_*(\Sigma^k E^{hH_2} \wedge X) \arrow[d, "\cong"', "P_d^X"] \\ \mathrm{map}(\mathbb{G}_n/H_1, E_*X) \arrow[r, "g_X"'] & \mathrm{map}(\mathbb{G}_n/H_2, E_*X) \end{tikzcd}$$

with $g_X = \mathrm{Hom}(h, E_*X)$.

Thus, we have an isomorphism from the diagram of (3.23) to the diagram

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{Z}_p[[\mathbb{G}_n/H_1]], E_*) \arrow[r, "\mathrm{Hom}(h,E_*)"] \arrow[d, "\mathrm{Hom}(\mathbb{G}/H_1,\varphi_*)"'] & \mathrm{Hom}(\mathbb{Z}_p[[\mathbb{G}_n/H_2]], E_*) \arrow[d, "\mathrm{Hom}(\mathbb{G}/H_2,\varphi_*)"] \\ \mathrm{Hom}(\mathbb{Z}_p[[\mathbb{G}_n/H_1]], E_*X) \arrow[r, "\mathrm{Hom}(h,E_*X)"'] & \mathrm{Hom}(\mathbb{Z}_p[[\mathbb{G}_n/H_2]], E_*X). \end{tikzcd}$$

Note that we can use the same φ_* for both H_1 and H_2 because the E^{hH_i}-orientations $g_i z$ of $E^{hH_i} \wedge X$ map to the *same* $\mathbb{G}_n$-invariant generator of E_0X. This last diagram evidently commutes, finishing the argument. $\square$

3.3. **The descent filtration.**

Another commonly used filtration to study κ_n comes from the $K(n)$-local Adams–Novikov spectral sequence. See, for example [HS99a] and [KS04].

Recall from Remark 2.4 that if X is a dualizable object in the $K(n)$-local category, then the E-based Adams–Novikov spectral sequence for $E^{hK} \wedge X$ is given by

$$E_2^{s,*}(K,X) \cong H^s(K, E_tX) \Longrightarrow \pi_{t-s}L_{K(n)}(E^{hK} \wedge X).$$

We will use the following key and deep property of this spectral sequence.

Remark 3.25. The $K(n)$-local E-based Adams-Novikov spectral sequence has a uniform and horizontal vanishing line at E_∞, i.e. there is an integer N, depending only on n, p, and K, so that in the Adams-Novikov spectral sequence we have

$$E_\infty^{s,*}(K,X) = 0, \qquad s > N.$$

This can be found in the literature in several guises; for example, it can be put together from the material in Section 5 of [DH04], especially Lemma 5.11. See also [BBGS22] for even further explanation. If $p-1 > n$, there is often a horizontal vanishing line at E_2, but we are decidedly not in that case in the second part of this paper.

Let $X \in \kappa_n$ and choose a $\mathbb{G}_n$-invariant generator $\iota_X \in E_0X$ as in Definition 3.1. The Adams–Novikov spectral sequence for the sphere acts on the Adams–Novikov spectral sequence X; thus, if $d_q(\iota_X) = 0$ for $r < q$, then we have an isomorphism of E_r-terms

$$\begin{aligned} E_r^{*,*}(\mathbb{G}_n, S^0) &\cong E_r^{*,*}(\mathbb{G}_n, X) \\ a &\longmapsto a\iota_X. \end{aligned} \tag{3.26}$$

Similar ideas were deployed in the proof of Proposition 3.6.

In Remark 3.2 we also observed that any two choices for a $\mathbb{G}_n$-invariant generator of E_*X differ by multiplication by a unit in $\mathbb{Z}_p^\times$. Thus the following definition is independent of the choices.

Definition 3.27 (Descent filtration)**.** For $r \geq 2$, let

$$\kappa_{n,r} = \{\ X \in \kappa_2 \mid d_k(\iota_X) = 0,\ 2 \leq k < r\ \},$$

where d_k denotes the differential in the homotopy fixed point spectral sequence

$$E_2^{s,t}(\mathbb{G}_n, X) = H^s(\mathbb{G}_n, E_tX) \Longrightarrow \pi_{t-s}X.$$

Define $\phi_r : \kappa_{n,r} \to E_r^{r,r-1}(S^0)$ by the equation

$$d_r(\iota_X) = \phi_r(X)\iota_X,$$

where the right-hand side uses the identification (3.26).

Lemma 3.28. *The subsets $\kappa_{n,r}$ of κ_n satisfy the following basic properties.*

1. *The subset $\kappa_{n,r}$ is a subgroup of κ_n and ϕ_r is a homomorphism.*
2. *The kernel of ϕ_r is identified with $\kappa_{n,r+1}$, and so we have an exact sequence*

$$0 \longrightarrow \kappa_{n,r+1} \longrightarrow \kappa_{n,r} \xrightarrow{\phi_r} E_r^{r,r-1}(\mathbb{G}_n, S^0)\ .$$

3. *There exists an integer N so that $\kappa_{n,N} = \{L_{K(n)}S^0\}$.*

Proof. Part (1) follows from the observation that $\iota_X \in E_0X$ and $\iota_Y \in E_0Y$ are $\mathbb{G}_n$-invariant generators then

$$\iota_X \wedge \iota_Y \in E_0(X \wedge Y)$$

is a $\mathbb{G}_n$-invariant generator. Part (2) is built into the definitions and part (3) follows from the horizontal vanishing line of Remark 3.25. □

Remark 3.29. The map $\phi_r : \kappa_{n,r} \to E_r^{r,r-1}(\mathbb{G}_n, S^0)$ need not be onto. In Remark 12.30 we note that the class $\eta^2 e$ is not in the image of ϕ_5 at $n = p = 2$.

4. The $\mathbb{G}_n^1$-orientable elements of the Picard group

For all n and all p, the group $\mathbb{G}_n$ has a closed subgroup $\mathbb{G}_n^1$ defined as the kernel of a reduced determinant map. In this section we give general results on $\kappa(\mathbb{G}_n^1) \subseteq \kappa_n$, the subgroup of exotic invertible elements which have an $E^{h\mathbb{G}_n^1}$-orientation, and give some remarks on the interaction between $\kappa(\mathbb{G}_n^1)$ and the filtration coming from the descent filtration of Definition 3.27. Most of the key ideas are already present in Section 5 of [GHMR15], and then adapted and generalized in [Wes17].

4.1. **The determinant, the subgroup $\mathbb{G}_n^1$ and the class ζ.** We briefly introduce the subgroup $\mathbb{G}_n^1$ and the closely related homotopy class ζ.

We have already defined $\mathbb{S}_n = \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \subseteq \mathbb{G}_n$ and indeed, we have a semi-direct product decomposition $\mathbb{S}_n \rtimes \mathrm{Gal} \cong \mathbb{G}_n$ where $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. The group $\mathbb{S}_n$ is the group of units in the endomorphism ring $\mathcal{O}_n$ of Γ_n over $\mathbb{F}_{p^n}$. The inclusion $\mathbb{Z}_p \to \mathcal{O}_n$ sending n to the multiplication by n extends to an inclusion of the Witt vectors $\mathbb{W} \to \mathcal{O}_n$ and $\mathcal{O}_n$ is a left $\mathbb{W}$ module of rank n. The right action of $\mathbb{S}_n$ on $\mathcal{O}_n$ then defines a map

$$\mathbb{S}_n \longrightarrow \mathrm{Gl}_n(\mathbb{W}).$$

The image of this map has enough symmetry that the determinant restricts to a map

$$\det\colon \mathbb{S}_n \to \mathbb{Z}_p^\times \subseteq \mathbb{W}^\times.$$

We then extend the determinant to all of $\mathbb{G}_n$ as follows.

Definition 4.1. The *determinant* $\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$ is defined to be the composite

$$\mathbb{G}_n \cong \mathbb{S}_n \rtimes \mathrm{Gal} \xrightarrow{\det \times \mathrm{Gal}} \mathbb{Z}_p^\times \times \mathrm{Gal} \longrightarrow \mathbb{Z}_p^\times$$

where the second map is the projection. The kernel of det is denoted by $S\mathbb{G}_n$.

We can now define one of the key players in this story, the reduced determinant ζ.

Definition 4.2. The homomorphism $\zeta = \zeta_n \colon \mathbb{G}_n \to \mathbb{Z}_p$ is the composite

$$\mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times/\mu \cong \mathbb{Z}_p$$

where the second map is the quotient by the subgroup μ of roots of unity. The kernel of ζ is denoted by $\mathbb{G}_n^1$.

Remark 4.3. We note that ζ is always a split surjection so that $\mathbb{G}_n \cong \mathbb{G}_n^1 \rtimes \mathbb{Z}_p$. Throughout, we will let

$$\psi \in \mathbb{G}_n/\mathbb{G}_n^1 \cong \mathbb{Z}_p$$

be a topological generator.

When $p = 2$, we can define another important homomorphism derived from the determinant, which will not be used in this section but later in Part II to study the case $n = p = 2$.

Definition 4.4. Let $p = 2$. The homomorphism $\chi\colon \mathbb{G}_n \to \mathbb{Z}/2$ is defined as the composite

$$\mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_2^\times \longrightarrow \mathbb{Z}_2^\times/(1+4\mathbb{Z}_2) \cong (\mathbb{Z}_2/4\mathbb{Z}_2)^\times \cong \mathbb{Z}/2.$$

We define $\mathbb{G}_n^0$ to be the kernel of χ.

Remark 4.5. The fixed point spectrum $E^{h\mathbb{G}_n^0}$ is a mysterious object, unusual even at $p = 2$ and $n = 1$. However, the other fixed point spectrum $E^{h\mathbb{G}_n^1}$ is much more familiar. If $n = 1$, the determinant $\mathbb{G}_1 \to \mathbb{Z}_2^\times$ is the identity, so $\mathbb{G}_1^1 = \{\pm 1\}$ and $E^{h\mathbb{G}_1^1} \simeq KO$, the 2-complete real K-theory spectrum.

We use the homomorphisms ζ and χ to define some of our key cohomology classes.

Definition 4.6. The homomorphisms

$$\zeta\colon \mathbb{G}_n \longrightarrow \mathbb{Z}_p$$
$$\chi\colon \mathbb{G}_n \longrightarrow \mathbb{Z}/2$$

define classes

$$\zeta \in H^1(\mathbb{G}_n, \mathbb{Z}_p) \qquad \text{and} \qquad \chi \in H^1(\mathbb{G}_n, \mathbb{Z}/2)$$

in cohomology with trivial coefficients. Since the inclusion $\mathbb{Z}_p \to E_0$ of the sub-module generated by the unit is $\mathbb{G}_n$-invariant we can also write

$$\zeta \in H^1(\mathbb{G}_n, E_0) \qquad \text{and} \qquad \chi \in H^1(\mathbb{G}_n, E_0/2)$$

for the image of these classes under the inclusion map. Finally, let

$$\widetilde{\chi} \in H^2(\mathbb{G}_2, E_0)$$

be the Bockstein on χ. Note that χ and $\widetilde{\chi}$ are only defined if $p = 2$.

We now can record a standard result for recovering $L_{K(n)}S^0$ from $E^{h\mathbb{G}_n^1}$. See, for example, [DH04, Proposition 8.1]. It follows easily from the isomorphism of Morava modules $\operatorname{map}(\mathbb{G}_n/\mathbb{G}_n^1, E_*) \cong E_*E^{h\mathbb{G}_n^1}$ of Remark 2.4.

Proposition 4.7. *Let ψ be a topological generator of $\mathbb{G}_n/\mathbb{G}_n^1$. Then there is a fiber sequence*

$$L_{K(n)}S^0 \xrightarrow{i} E^{h\mathbb{G}_n^1} \xrightarrow{\psi - 1} E^{h\mathbb{G}_n^1}$$

that gives a short exact of Morava modules

$$E_*S^0 \xrightarrow{i_*} E_*E^{h\mathbb{G}_n^1} \xrightarrow{(\psi-1)_*} E_*E^{h\mathbb{G}_n^1}.$$

The map i_* is an injection onto the sub-Morava module of rank one generated by the unit of the ring $E_*E^{h\mathbb{G}_n^1}$.

As noted above, the homomorphism $\zeta : \mathbb{G}_n \to \mathbb{Z}_p$ defines a cohomology class

$$\zeta = \zeta_n \in H^1(\mathbb{G}_n, \mathbb{Z}_p).$$

We will confuse ζ with its image under the map

$$H^1(\mathbb{G}_n, \mathbb{Z}_p) \longrightarrow H^1(\mathbb{G}_n, E_0)$$

defined by the inclusion of rings $\mathbb{Z}_p \to E_0$. As is customary, we will then further confuse the element with the homotopy class in

$$\zeta \in \pi_{-1} L_{K(n)} S^0$$

defined as the image of the unit $i\colon S^0 \to E^{h\mathbb{G}_n^1}$ under the boundary homomorphism

$$\partial\colon \pi_0 E^{h\mathbb{G}_n^1} \longrightarrow \pi_{-1} L_{K(n)} S^0.$$

The homotopy class ζ is detected by the cohomology class ζ in the Adams–Novikov Spectral Sequence

$$E_2^{s,t} = H^s(\mathbb{G}_n E_t) \Longrightarrow \pi_{t-s} L_{K(n)} S^0.$$

This follows from Proposition 4.7, the long exact sequence in cohomology

$$\cdots \longrightarrow H^s(\mathbb{G}_n^1, E_t) \xrightarrow{\psi-1} H^s(\mathbb{G}_n^1, E_t) \xrightarrow{\partial_2} H^{s+1}(\mathbb{G}_n, E_t) \longrightarrow \cdots ,$$

and the Geometric Boundary Theorem. This is all due to Hopkins and Miller, see Theorem 6 in [DH04].

We immediately have

$$\pi_k L_{K(n)} S^0 \xrightarrow{i_*} \pi_k E^{h\mathbb{G}_n^1} \xrightarrow{\partial} \pi_{k-1} L_{K(n)} S^0$$

is multiplication by ζ. This extends to the maps on Adams–Novikov Spectral Sequences: the composition

$$E_r^{s,t}(\mathbb{G}_n, S^0) \xrightarrow{i_*} E_r^{s,t}(\mathbb{G}_n^1, S^0) \xrightarrow{\partial_r} E_r^{s+1,t}(\mathbb{G}_n, S^0)$$

is multiplication by $\zeta \in E_r^{1,0}(\mathbb{G}_n, S^0)$.

4.2. **A description of $\kappa(\mathbb{G}_n^1)$ in terms of homotopy groups.** There is a standard way to produce spectra in $\kappa(\mathbb{G}_n^1)$, which we now review. The main result of this subsection is Theorem 4.19.

Define

$$\Lambda = \mathrm{Ker}\{\ \pi_0 E^{h\mathbb{G}_n^1} \longrightarrow E_0 E^{h\mathbb{G}_n^1}\ \} \tag{4.8}$$

to be the kernel of the E-Hurewicz homomorphism; this is the kernel of the edge homomorphism of the spectral sequence

$$E_2^{s,t}(\mathbb{G}_n, S^0) = H^s(\mathbb{G}_n^1, E_t) \Longrightarrow \pi_{t-s} E^{h\mathbb{G}_n^1}$$

when $t = s$. Since this spectral sequence has a horizontal vanishing line (see Remark 3.25) any element in Λ is nilpotent. We thus have a subgroup

$$1 + \Lambda \subseteq (\pi_0 E^{h\mathbb{G}_n^1})^\times$$

of the units in the ring $\pi_0 E^{h\mathbb{G}_n^1}$.

Definition 4.9 (Twisting Construction)**.** For $\psi \in \mathbb{G}_n/\mathbb{G}_n^1$ a topological generator and $\alpha \in 1+\Lambda$, we define $X(\alpha)$ by the fiber sequence

$$X(\alpha) \xrightarrow{i_\alpha} E^{h\mathbb{G}_n^1} \xrightarrow{\psi-\alpha} E^{h\mathbb{G}_n^1}.$$

Here, in an abuse of notation, we also denote by α the unique $E^{h\mathbb{G}_n^1}$-module map $\varphi_\alpha : E^{h\mathbb{G}_n^1} \to E^{h\mathbb{G}_n^1}$ obtained by extending $\alpha\colon S^0 \to E^{h\mathbb{G}_n^1}$.

By Proposition 4.7,

$$L_{K(n)}S^0 \simeq X(1)$$

and there is a short exact sequence of Morava modules

$$E_*S^0 \xrightarrow{(i_1)_*} E_*E^{h\mathbb{G}_n^1} \xrightarrow{(\psi-1)_*} E_*E^{h\mathbb{G}_n^1}. \tag{4.10}$$

The map $(i_1)_*$ is an injection onto the sub-Morava module of rank one generated by the unit of the ring $E_0E^{h\mathbb{G}_n^1} \cong \mathrm{map}(\mathbb{G}_n/\mathbb{G}_n^1, E_0)$. This observation is extended in the following result.

Proposition 4.11. *Let $\alpha \in 1+\Lambda$. Then the map*

$$(i_\alpha)_* : E_*X(\alpha) \longrightarrow E_*E^{h\mathbb{G}_n^1}$$

is an injection onto to the sub-Morava module of rank one generated by the unit. The assignment $\alpha \mapsto X(\alpha)$ defines a homomorphism

$$X(-) : 1+\Lambda \longrightarrow \kappa(\mathbb{G}_n^1).$$

Proof. Since $E_*\alpha = E_*(1)$ the first statement follows from the short exact sequence (4.10). It follows that the map $i_\alpha : X(\alpha) \to E^{h\mathbb{G}_n^1}$ extends to an equivalence

$$E^{h\mathbb{G}_n^1} \wedge X(\alpha) \xrightarrow{\simeq} E^{h\mathbb{G}_n^1}.$$

If we let $z \in \pi_0(E^{h\mathbb{G}_n^1} \wedge X(\alpha))$ be the element which maps to the unit in the ring $\pi_0E^{h\mathbb{G}_n^1}$ under this equivalence, then z is an $E^{h\mathbb{G}_n^1}$-orientation in the sense of Definition 3.3. Thus, $X(\alpha) \in \kappa(\mathbb{G}_n^1)$. Proposition 3.17 of [Wes17] shows that the function $\alpha \mapsto X(\alpha)$ has the property that there are canonical pairing maps

$$X(\alpha) \wedge X(\beta) \longrightarrow X(\alpha\beta).$$

and that this map is an equivalence. Thus we have a homomorphism. □

Remark 4.12. There's an omission in the statements of Propositions 3.15 and 3.17 of [Wes17]. They are stated for a general unit in $\pi_0E^{h\mathbb{G}_n^1}$; however, the proofs in [Wes17] work only if $\alpha \equiv 1$ modulo the maximal ideal in $\pi_0E^{h\mathbb{G}_n^1}$. In all our applications, and indeed in all the applications in [Wes17], this additional hypothesis holds.

Proposition 4.13. *Let $\alpha \in 1+\Lambda$ and let ψ be a topological generator of $\mathbb{G}_n/\mathbb{G}_n^1$. There is an equivalence $L_{K(n)}S^0 \to X(\alpha)$ if and only if*

$$\alpha\beta = \psi\beta$$

for some $\beta \in 1+\Lambda$.

Proof. First suppose $\psi\beta = \alpha\beta$. Then we have a factoring

$$\begin{tikzcd} & L_{K(n)}S^0 \arrow[dl, dashed, "f"'] \arrow[d, "\beta"] & \\ X(\alpha) \arrow[r, "i_\alpha"'] & E^{h\mathbb{G}_n^1} \arrow[r, "\psi-\alpha"] & E^{h\mathbb{G}_n^1}. \end{tikzcd}$$

Since $\beta \in 1+\Lambda$, the map $E_*\beta$ is injection onto the sub-Morava module of $E_*E^{h\mathbb{G}_n^1}$ generated by the unit. Then Proposition 4.11 implies the map f is an E_*-isomorphism.

Conversely, suppose we are given an equivalence $h\colon L_{K(n)}S^0 \to X(\alpha)$. We define γ to be the image of h in $\pi_0 E^{h\mathbb{G}_n^1}$ under the map $i_\alpha\colon X(\alpha) \to E^{h\mathbb{G}_n^1}$. Then we have a diagram

$$\begin{tikzcd} & E_*S^0 \arrow[dl, "h_*"', "\cong"] \arrow[d, "\gamma_*"] \\ E_*X(\alpha) \arrow[r, "(i_\alpha)_*"'] & E_*E^{h\mathbb{G}_n^1} \end{tikzcd}$$

Let $1 \in E_0S^0$ be the tautological generator. Since the ring of Morava module endomorphisms of E_*S^0 is isomorphic to $\mathbb{Z}_p$ and h_* is such an endomorphism, Proposition 4.11 implies that

$$\gamma_*(1) = a(i_\alpha)_*(1)$$

for some $a \in \mathbb{Z}_p^\times$. Define $\beta = a^{-1}\gamma$. Then $\beta \in 1+\Lambda$ and β is in the kernel of $\psi - \alpha$. □

The quotient group $\mathbb{Z}_p \cong \mathbb{G}_n/\mathbb{G}_n^1$ acts on $1+\Lambda$ and we have an exact sequence

$$1+\Lambda \xrightarrow{\partial} 1+\Lambda \longrightarrow H^1(\mathbb{Z}_p, 1+\Lambda) \longrightarrow 0$$

where $\partial(\beta) = \beta^{-1}\psi(\beta)$. Thus Proposition 4.13 implies that $X(\alpha)$ is trivial in κ_n if and only if α is a coboundary in $1+\Lambda$. Thus we have an injection

$$H^1(\mathbb{G}_n/\mathbb{G}_n^1, 1+\Lambda) \longrightarrow \kappa(\mathbb{G}_n^1)$$

sending the coset of α to $X(\alpha)$. We will show in Theorem 4.19 that this is an isomorphism, but we need some preliminaries.

Let $\Lambda^\infty \subseteq [E^{h\mathbb{G}_n^1}, E^{h\mathbb{G}_n^1}]$ be the set of maps f such that $E_*f = 0$; note that Λ^∞ is an ideal in the endomorphism ring. The map $f \mapsto f(1_E)$ defines a split surjection

$$\begin{tikzcd} \Lambda^\infty \arrow[r, "\epsilon"] & \Lambda \arrow[l, bend left, dotted, "s"] \end{tikzcd} \tag{4.14}$$

The splitting sends $a \in \Lambda$ to the $E^{h\mathbb{G}_n^1}$-module map defined by extending a.

Now let $g \in 1+\Lambda^\infty$ and form the cofiber sequence

$$X(g) \xrightarrow{i_g} E^{h\mathbb{G}_n^1} \xrightarrow{\psi-g} E^{h\mathbb{G}_n^1}.$$

The following is a generalization of Proposition 4.11 and of Proposition 4.13.

Proposition 4.15. *Let $g \in 1 + \Lambda^\infty$. Then the map*

$$(i_g)_* : E_*X(g) \longrightarrow E_*E^{h\mathbb{G}_n^1}$$

is an injection onto to the sub-Morava module of rank one generated by the unit. The assignment $g \mapsto X(g)$ defines a homomorphism

$$X(-) : 1 + \Lambda^\infty \longrightarrow \kappa(\mathbb{G}_n^1).$$

For $g \in 1 + \Lambda^\infty$ and ψ the chosen topological generator of $\mathbb{G}_n/\mathbb{G}_n^1$, there is an equivalence $L_{K(n)}S^0 \to X(g)$ if and only if

$$g\beta = \psi\beta$$

for some $\beta \in 1 + \Lambda$.

Proof. The proof of Proposition 4.11 goes through without change for the first part, and that of Proposition 4.13 for the second part. □

Remark 4.16. As in Proposition 4.15, we have in particular that if $g(1_E) = 1_E$, then $X(g)$ is trivial.

If we write $g \in 1 + \Lambda^\infty$ as $g = 1 + f$ with $f \in \Lambda^\infty$, then $g(1_E) = 1_E$ if and only if $f(1_E) = 0$. We then have a diagram

$$\begin{tikzcd} 1 + \Lambda^\infty \arrow[r] \arrow[d, "1+\epsilon"'] & \kappa(\mathbb{G}_n^1) \\ 1 + \Lambda \arrow[ur] & \end{tikzcd} \tag{4.17}$$

where the vertical map sends g to $g(1_E)$. Thus we have the following result.

Lemma 4.18. *Suppose $g \colon E^{h\mathbb{G}_n^1} \to E^{h\mathbb{G}_n^1}$ is any self map so that E_*g is the identity and let $\alpha = g(1_E) \in \pi_0 E^{h\mathbb{G}_n^1}$. Then*

$$X(g) = X(\alpha) \in \kappa(\mathbb{G}_n^1).$$

Proof. Let s be the splitting of (4.14), then $X(s(\alpha)) = X(\alpha)$ by definition. Therefore, $X(s(\alpha^{-1})g) = X(s(\alpha^{-1})) \wedge X(g) = X(\alpha^{-1}) \wedge X(g)$. But Proposition 4.15 implies that $X(s(\alpha^{-1})g)$ is trivial since $(s(\alpha^{-1})g)(1_E) = 1_E$. □

We can now prove the following result.

Theorem 4.19. *The assignment $\alpha \mapsto X(\alpha)$ defines an isomorphism*

$$H^1(\mathbb{G}_n/\mathbb{G}_n^1, 1 + \Lambda) \xrightarrow{\cong} \kappa(\mathbb{G}_n^1).$$

In particular, if $\mathbb{G}_n/\mathbb{G}_n^1$ acts trivially on Λ, then we have an isomorphism $1 + \Lambda \cong \kappa(\mathbb{G}_n^1)$.

Proof. Proposition 4.13 shows that the map is well-defined and an injection; thus we must show it is onto.

Let $X \in \kappa(\mathbb{G}_n^1)$. Choose an $E^{h\mathbb{G}_n^1}$-orientation $z \in \pi_0(E^{h\mathbb{G}_n^1} \wedge X)$; see Definition 3.3. Let $\varphi_1 \colon E^{h\mathbb{G}_n^1} \to E^{h\mathbb{G}_n^1} \wedge X$ and $\varphi \colon E \to E \wedge X$ be the equivalences induced by extending z. The image of z in E_0X is denoted ι_X; it is a $\mathbb{G}_n$-invariant generator of E_0X.

For any spectrum A we have an isomorphism ϕ_A of Morava modules

$$E_*A \xrightarrow{\cong} E_*A \otimes_{E_0} E_0X \xrightarrow{\cong} E_*(A \wedge X).$$

The first isomorphism sends a to $a \otimes \iota_X$, and the second is the Künneth isomorphism. Both maps are natural is A; thus, so is ϕ_A. If $A = E^{h\mathbb{G}_n^1}$ we will simply write $\phi = \phi_A$. The composition

$$E_*E^{h\mathbb{G}_n^1} \xrightarrow{(\varphi_1)_*} E_*(E^{h\mathbb{G}_n^1} \wedge X) \xrightarrow{\phi^{-1}} E_*E^{h\mathbb{G}_n^1}.$$

is then the identity. Let $\psi \in \mathbb{G}_n\mathbb{G}_n^1$ be a topological generator. (See Proposition 4.7.) We next define a self map $\widetilde{\psi}\colon E^{h\mathbb{G}_n^1} \to E^{h\mathbb{G}_n^1}$ by requiring the following diagram to commute

$$\begin{tikzcd}
E^{h\mathbb{G}_n^1} \arrow[r, "\widetilde{\psi}"] \arrow[d, "\varphi_1"', "\simeq"] & E^{h\mathbb{G}_n^1} \arrow[d, "\varphi_1"', "\simeq"] \\
E^{h\mathbb{G}_n^1} \wedge X \arrow[r, "\psi \wedge X"'] & E^{h\mathbb{G}_n^1} \wedge X.
\end{tikzcd}$$

Applying E-homology and prolonging with ϕ^{-1}, we obtain the commutative diagram

$$\begin{tikzcd}
E_*E^{h\mathbb{G}_n^1} \arrow[r, "\widetilde{\psi}_*"] \arrow[d, "(\varphi_1)_*"'] & E_*E^{h\mathbb{G}_n^1} \arrow[d, "(\varphi_1)_*"] \\
E_*(E^{h\mathbb{G}_n^1} \wedge X) \arrow[r, "(\psi \wedge X)_*"'] \arrow[d, "\phi^{-1}"'] & E_*(E^{h\mathbb{G}_n^1} \wedge X) \arrow[d, "\phi^{-1}"] \\
E_*E^{h\mathbb{G}_n^1} \arrow[r, "\psi_*"] & E_*E^{h\mathbb{G}_n^1},
\end{tikzcd}$$

where the vertical composites are the identity. From this we conclude that

$$\widetilde{\psi}_* = \psi_*\colon E_*E^{h\mathbb{G}_n^1} \longrightarrow E_*E^{h\mathbb{G}_n^1}.$$

Defining F to be the fiber of $\widetilde{\psi} - 1$, we get a diagram of fiber sequences

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\simeq"'] & E^{h\mathbb{G}_n^1} \arrow[r, "\widetilde{\psi}-1"] \arrow[d, "\varphi_1"', "\simeq"] & E^{h\mathbb{G}_n^1} \arrow[d, "\varphi_1"', "\simeq"] \\
X \arrow[r] & E^{h\mathbb{G}_n^1} \wedge X \arrow[r, "(\psi-1)\wedge X"'] & E^{h\mathbb{G}_n^1} \wedge X.
\end{tikzcd}$$

Define $f = \psi - \widetilde{\psi}\colon E^{h\mathbb{G}_n^1} \to E^{h\mathbb{G}_n^1}$. By construction $\widetilde{\psi} - 1 = \psi - (1+f)$. Furthermore, $E_*f = 0$. Hence $f \in \Lambda^\infty$ and $F \simeq X(1+f)$. The result now follows from Lemma 4.18. $\square$

4.3. **A comparison of filtrations.** In Definition 3.27 we defined a filtration on κ_n using the Adams-Novikov spectral sequence; specifically $\kappa_{n,s} \subseteq \kappa_n$ is the subgroup of elements so that $d_r(\iota_X) = 0$ for $r < s$ for any choice of $\mathbb{G}_n$-invariant generator ι_X of E_0X. We also discussed homomorphisms

$$\phi_s\colon \kappa_{n,s} \to E_r^{s,s-1}(\mathbb{G}_n, S^0)$$

determined by the formula

$$d_s(\iota_X) = \phi_s(X)\iota_X.$$

Define

$$\kappa_s(\mathbb{G}_n^1) = \kappa_{n,s} \cap \kappa(\mathbb{G}_n^1) \subseteq \kappa(\mathbb{G}_n^1) \cong H^1(\mathbb{G}_n/\mathbb{G}_n^1, 1+\Lambda).$$

These subgroups give a filtration of $\kappa(\mathbb{G}_n^1)$. In this section, we compare it with the Adams–Novikov filtration on Λ. This does not appear to be formal; indeed, it is not clear how the two filtrations can be compared without some additional hypotheses. We provide such a result with hypotheses that will suffice for our purposes in Theorem 4.27.

Definition 4.20. Let $\Lambda_s \subseteq \pi_0 E^{h\mathbb{G}_n^1}$ be the subgroup of elements whose Adams–Novikov filtration is greater than or equal to s in the spectral sequence

$$E_2^{s,t}(\mathbb{G}_n^1, S^0) = H^s(\mathbb{G}_n^1, E_t) \Longrightarrow \pi_{t-s} E^{h\mathbb{G}_n^1}.$$

Note that by definition, $\Lambda = \Lambda_1$, and furthermore, we get a corresponding filtration $1+\Lambda_s \subseteq 1+\Lambda$. The first observation about this latter filtration will be the cause of much technical complication.

Lemma 4.21. *Let $\alpha \in 1+\Lambda_s$ and $\beta \in 1+\Lambda_{s'}$ and $s < s'$. Then $\alpha\beta \in 1+\Lambda_s$. If α has exact filtration s, then so does $\alpha\beta$.*

Proof. Write $\alpha = 1+x$ and $\beta = 1+y$ with x and y of filtration s and s' respectively in the Adams-Novikov spectral sequence for $E^{h\mathbb{G}_n^1}$. Then $\alpha\beta - 1$ is congruent to x modulo elements of filtration greater than s. $\square$

Remark 4.22. Beyond the complications implied by Lemma 4.21, the filtration $1+\Lambda_s$ of $1+\Lambda$ also does not fit particularly well with the cohomological description in Theorem 4.19. While $1+\Lambda_s$ is closed under the action of $\mathbb{G}_n/\mathbb{G}_n^1$, the map

$$H^1(\mathbb{G}_n/\mathbb{G}_n^1, 1+\Lambda_s) \longrightarrow H^1(\mathbb{G}_n/\mathbb{G}_n^1, 1+\Lambda)$$

induced by the inclusion is not obviously one-to-one. These facts complicate the analysis of the relationship between the Adams-Novikov filtration $\kappa_r(\mathbb{G}_n^1)$ and the Adams-Novikov filtration on $\Lambda \subseteq \pi_0 E^{h\mathbb{G}_n^1}$, at least in the absence of further hypotheses. This observation explains the excessive (even in the context of this paper) technicality of Theorem 4.27.

Remark 4.23. A first hypothesis will be to require that $\mathbb{G}_n/\mathbb{G}_n^1$ act trivially on $\pi_0 E^{h\mathbb{G}_n^1}$. Then Theorem 4.19 specifies an isomorphism $1+\Lambda \cong \kappa(\mathbb{G}_n^1)$, sending α to $X(\alpha)$. We then get a filtration

$$\kappa(\mathbb{G}_n^1) \cong 1+\Lambda = 1+\Lambda_1 \supseteq 1+\Lambda_2 \supseteq \ldots$$

of $\kappa(\mathbb{G}_n^1)$. In that case, we have two filtrations of $\kappa(\mathbb{G}_n^1)$, namely $1+\Lambda_s$ and $\kappa_s(\mathbb{G}_n^1)$ and we wish to compare them. This is the goal of the rest of the section. Note that if $\alpha = 1+x$ with $x \in \Lambda_{s-1}$, then we hope to have a formula of the form

$$\phi_s(X(\alpha)) = \zeta\overline{x} \in E_s^{s,s-1}(\mathbb{G}_n, S^0)$$

where $\overline{x} \in E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$ detects x. If this holds, then $X(\alpha) \in \kappa_s(\mathbb{G}_n^1)$. Thus we would be comparing $1+\Lambda_{s-1}$ and $\kappa_s(\mathbb{G}_n^1)$.

Lemma 4.24. *Suppose that $\mathbb{G}_n/\mathbb{G}_n^1$ acts trivially on $\pi_0 E^{h\mathbb{G}_n^1}$, and let $\alpha \in 1+\Lambda$. The boundary map $\pi_0 E^{h\mathbb{G}_n^1} \to \pi_{-1} X(\alpha)$ of the long exact sequence on homotopy for the fibration*

$$X(\alpha) \to E^{h\mathbb{G}_n^1} \xrightarrow{\psi-\alpha} E^{h\mathbb{G}_n^1} \tag{4.25}$$

induces an injection

$$\pi_0 E^{h\mathbb{G}_n^1}/(\alpha-1) \hookrightarrow \pi_{-1} X(\alpha).$$

Proof. For $a \in \pi_0 E^{h\mathbb{G}_n^1}$, $\psi(a) = a$, so

$$(\psi-\alpha)(a) = (1-\alpha)a.$$

The claim then follows from the long exact sequence on homotopy groups for the fiber sequence (4.25). ☐

We make two basic observations help organize the assumptions in the following results. First, recall from Definition 4.20 that $\Lambda_s \subseteq \pi_0 E^{h\mathbb{G}_n^1}$ is the subgroup of elements of filtration at least s in the $\mathbb{G}_n^1$ homotopy fixed point spectral sequence. Since $H^1(\mathbb{G}_n^1, E_1) = 0$, we have $\Lambda_2 = \Lambda_1 = \Lambda$. Thus many of statements begin with $s = 2$.

Second, because of the uniform horizontal vanishing of the homotopy fixed point spectral sequence, we know that there is an integer N so that for all $s \geq N$ we have that $1+\Lambda_s$ and $\kappa_s(\mathbb{G}_n^1) \subseteq \kappa_{n,s}$ are trivial. One of our goals is to show that, at least under certain hypotheses, the assignment $\alpha \mapsto X(\alpha)$ defines an isomorphism $1+\Lambda_{s-1} \cong \kappa_s(\mathbb{G}_n^1)$. If $s > N$, this is obvious, so we are free to concentrate on smaller s.

The first step for proving our comparison result is the following factorization of the homomorphism ϕ_s restricted to $\kappa_s(\mathbb{G}_n^1)$.

Lemma 4.26. *Assume that $\mathbb{G}_n/\mathbb{G}_n^1$ acts trivially on $\pi_0 E^{h\mathbb{G}_n^1}$, that Λ_{s-1} is trivial for $s > N$, and assume further that for all $2 \leq s \leq N$ the following conditions are satisfied:*

(i) $E_\infty^{s-1,s-1}(\mathbb{G}_n^1, S^0) \cong E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$; *and*
(ii) *there is an exact sequence*

$$0 \longrightarrow E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0) \xrightarrow{\zeta} E_s^{s,s-1}(\mathbb{G}_n, S^0) \xrightarrow{i_*} E_s^{s,s-1}(\mathbb{G}_n^1, S^0)$$

where i_* is the restriction.

Then, the map ϕ_s restricts to a homomorphism

$$\phi_s : \kappa_s(\mathbb{G}_n^1) \to \zeta E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$$

for all $2 \leq s \leq N$.

Proof. By Theorem 4.19, $\kappa(\mathbb{G}_n^1) \cong 1+\Lambda$ and from (i) and (ii), for $2 \leq s \leq N$ we have an injection

$$0 \longrightarrow \Lambda_{s-1}/\Lambda_s \cong E_\infty^{s-1,s-1}(\mathbb{G}_n^1, S^0) \xrightarrow{\zeta} E_s^{s,s-1}(\mathbb{G}_n, S^0).$$

If $X \in \kappa(\mathbb{G}_n^1)$ then we can choose a $\mathbb{G}_n$-invariant generator $\iota_X \in E_0X$ which is a permanent cycle in the $\mathbb{G}_n^1$-homotopy fixed point spectral sequence

$$H^s(\mathbb{G}_n^1, E_tX) \Longrightarrow \pi_{t-s}(E^{h\mathbb{G}_n^1} \wedge X).$$

For this reason, any differential on ι_X in the $\mathbb{G}_n$-homotopy fixed point spectral sequence must lie in the kernel of the restriction $i_* : E_s^{s,s-1}(\mathbb{G}_n, S^0) \to E_s^{s,s-1}(\mathbb{G}_n^1, S^0)$ for some s. We conclude from (ii) that the possible differentials are of the form

$$d_s(\iota_X) = (\zeta y)\iota_X$$

for some element $y \in E_\infty^{s-1,s-1}(\mathbb{G}_n^1, S^0)$. Rephrased, we have shown that the map ϕ_s factors as a map

$$\kappa_s(\mathbb{G}_n^1) \longrightarrow \zeta E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0) \xrightarrow{\subseteq} E_s^{s,s-1}(\mathbb{G}_n, S^0).$$

We will continue to write ϕ_s for the first of these maps, and this is the map from the statement of the lemma. □

The stage is now set for our main comparison result. Note that all hypotheses will be checked in Section 8 in the case $p = n = 2$.

Theorem 4.27. *Assume that $\mathbb{G}_n/\mathbb{G}_n^1$ acts trivially on $\pi_0 E^{h\mathbb{G}_n^1}$, that Λ_{s-1} is trivial for $s > N$, and assume further that for all $2 \le s \le N$ the following conditions are satisfied:*

(i) $E_\infty^{s-1,s-1}(\mathbb{G}_n^1, S^0) \cong E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$;

(ii) there is an exact sequence

$$0 \longrightarrow E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0) \xrightarrow{\zeta} E_s^{s,s-1}(\mathbb{G}_n, S^0) \xrightarrow{i_*} E_s^{s,s-1}(\mathbb{G}_n^1, S^0)$$

where i_* is the restriction; and

(iii) all elements in $E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$ are torsion.

Then, we conclude that for all $s \ge 2$,

(1) the homomorphism $\alpha \mapsto X(\alpha)$ defines an isomorphism $1 + \Lambda_{s-1} \cong \kappa_s(\mathbb{G}_n^1)$;

(2) the map ϕ_s induces an isomorphism

$$\kappa_s(\mathbb{G}_n^1)/\kappa_{s+1}(\mathbb{G}_n^1) \cong \zeta E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0);$$

(3) if $x \in \Lambda_{s-1}$, $\alpha = 1 + x$, and $\overline{x} \in E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$ is the equivalence class of x, then

$$d_s(\iota_{X(\alpha)}) = a\,\overline{x}\zeta$$

for some $a \in \mathbb{Z}_p^\times$.

Proof. We set up an inductive argument, with the following induction hypothesis. For a fixed integer $2 \le s \le N$ and all integers r with $2 \le r \le s$

(a) The homomorphism $\alpha \mapsto X(\alpha)$ defines an injection $1 + \Lambda_{r-1} \to \kappa_r(\mathbb{G}_n^1)$.

(b) Let $x \in \Lambda_{r-1}$ and $\alpha = 1 + x$. If $0 \ne \overline{x} \in E_r^{r-1,r-1}(\mathbb{G}_n^1, S^0)$ then

$$d_r(\iota_{X(\alpha)}) = a\,\overline{x}\zeta$$

for some $a \in \mathbb{Z}_p^\times$.

(c) The map ϕ_r induces an isomorphism

$$\kappa_r(\mathbb{G}_n^1)/\kappa_{r+1}(\mathbb{G}_n^1) \cong \zeta E_r^{r-1,r-1}(\mathbb{G}_n^1, S^0).$$

Furthermore, for any $0 \neq y \in E_r^{r-1,r-1}(\mathbb{G}_n^1, S^0)$ there is a class $\beta \in 1 + \Lambda_{r-1}$ so that $\phi_r(X(\beta)) = \zeta y$ and β is non-trivial modulo $1 + \Lambda_r$.

(d) Let $x \in \Lambda_r$ and $\alpha = 1 + x$. Then $d_r(\iota_{X(\alpha)}) = 0$.

To deduce the result from this, note that since $\Lambda_N = 0$ and $\kappa(\mathbb{G}_n^1) = 1 + \Lambda$, (a)–(d) imply that $\kappa_{N+1}(\mathbb{G}_n^1) = 0$. So for $s \geq N+1$, (1)–(3) are trivial statements. For $2 \leq s \leq N$, part (1) of the result can be deduced from (a) and (c), part (2) from (c) and, and part (3) from (b) and (d).

We now begin the induction argument. The base case is $s = 2$. The statements follow from the sparseness of the spectral sequence. Specifically, we have that

$$E_2^{1,1}(\mathbb{G}_n^1, S^0) \cong H^1(\mathbb{G}_n^1, E_1) = 0$$
$$E_2^{2,1}(\mathbb{G}_n, S^0) \cong H^2(\mathbb{G}_n, E_1) = 0.$$

We now proceed with the induction step. So assume the statements (a)–(d) hold for $s-1$.

We first show (a). Since the homomorphism

$$X(-)\colon 1 + \Lambda \longrightarrow \kappa(\mathbb{G}_n^1)$$

is an isomorphism, we need only show $X(F_{s-1}) \subseteq \kappa_s(\mathbb{G}_n^1)$. By (d) for $s-1$, we have that if $\alpha \in F_{s-1}$, then $d_r(\iota_{X(\alpha)}) = 0$ for all $r < s$, and the assertion follows.

We now prove (b). Let $X = X(\alpha)$ where $\alpha = 1 + x$ for $x \in \Lambda_{s-1}$ and $\overline{x} \neq 0 \in E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$. We have just shown that $X(\alpha) \in \kappa_s(\mathbb{G}_n^1)$. In particular ι_X survives to the E_s-page of the $\mathbb{G}_n^1$-homotopy fixed point spectral sequence. By (ii) in our hypotheses, $0 \neq \overline{x}\zeta\iota_X$. By Lemma 4.24, the element $\overline{x}\zeta\iota_X$ must be hit by a differential; otherwise, it would survive to a non-trivial element detecting the boundary of $\alpha - 1$. Since $H^0(\mathbb{G}_n^1, E_0) \cong \mathbb{Z}_p$ generated by ι_X, the only possibility is

$$d_s(\iota_X) = a\,\overline{x}\zeta\iota_X, \qquad a \in \mathbb{Z}_p^\times.$$

This proves (b) and shows

$$\phi_s(X(\alpha)) = a\,\overline{x}\zeta \neq 0.$$

We can now move to (c). By definition ϕ_s induces an injection

$$\kappa_s(\mathbb{G}_n^1)/\kappa_{s+1}(\mathbb{G}_n^1) \longrightarrow E_s^{s,s-1}(\mathbb{G}_n, S^0).$$

Since $\zeta E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0) \subseteq E_s^{s,s-1}(\mathbb{G}_n, S^0)$, the map of (c) remains an injection: we have only changed the target. So we need to show that it is onto. Let $0 \neq y \in E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$ and chose an $x \in \Lambda_{s-1}$ with $\overline{x} = y$. Let $\alpha = 1 + x$. Then we have just shown

$$\phi_s(X(\alpha)) = a\,\overline{x}\zeta \neq 0$$

for some $a \in \mathbb{Z}_p^\times$. Since we assumed in (iii) that every element of $E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$ has finite order, we can choose a positive integer b coprime to p such that $ba\overline{x} = \overline{x}$. Since ϕ_s is a homomorphism,

$$\phi_s(X(\alpha^b)) = ba\,\overline{x}\zeta = y\zeta$$

as needed. Note we have proved the final statement of (c) as well: the class $\alpha^b = (1+x)^b \in 1+\Lambda_{s-1}$ has the property that the residue class of $\alpha^b - 1$ in $E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$ is by and, hence, non-zero.

We are left with (d). The only case we need to prove is $r = s$. So suppose that $x \in \Lambda_s$ and $\alpha = 1 + x$. Then $d_r(\iota_X) = 0$ for $r < s$ and we have

$$\phi_s(X(\alpha)) = y\zeta$$

for some $y \in E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$. We will show $y = 0$ by contradiction. So assume $y \neq 0$.

By (c), there is a class $\beta = 1 + z \in 1 + \Lambda_{s-1}$ so that the coset $\overline{z}$ of z in $E_s^{s-1,s-1}(\mathbb{G}_n^1, S^0)$ is non-zero and

$$\phi_s(X(\beta)) = -y\zeta.$$

Then $\phi_s(X(\alpha\beta)) = 0$, $\alpha\beta \in 1+\Lambda_{s-1}$, and $\alpha\beta \equiv 1+z$ modulo Λ_s. This contradicts (b).

This finishes the induction step, and the result follows. $\square$

5. The J-construction

We now come to a fundamental construction which allows us to produce invertible $K(n)$-local spectra from virtual representations of quotients of $\mathbb{G}_n$.

Let $q : \mathbb{G}_n \to H$ be a continuous map to a finite group. This will usually be surjective. Suppose we are given an action of H on $L_{K(n)}S^k$ specified by a map of spaces

$$f : BH \to \{k\} \times B\mathrm{Gl}_1(L_{K(n)}S^0) \subseteq \mathbb{Z} \times B\mathrm{Gl}_1(L_{K(n)}S^0).$$

Here $\mathrm{Gl}_1(L_{K(n)}S^0)$ is simply the topological monoid of self-equivalences of $L_{K(n)}S^0$. We will write $S(f)$ for $L_{K(n)}S^k$ with the action defined by f. We can form the spectrum $E \wedge S(f)$ with the diagonal $\mathbb{G}_n$-action.

It is straightforward to check that this action has a continuous refinement in the sense of [BBGS22, Definition 2.5]. The requisite map

$$E \wedge S(f) \to F_c(\mathbb{G}_{n+}, E \wedge S(f))$$

to the continuous function spectrum [BBGS22, Definition 2.2] can be built from the analogous maps for E and $S(f)$: the first coming from [DH04], the second from the fact that H is finite. Then we can define

$$J(f) = J(q, f, \mathbb{G}_n) = (E \wedge S(f))^{h\mathbb{G}_n}$$

as the continuous homotopy fixed point spectrum, in the style of Devinatz-Hopkins; see [BBGS22, Definition 2.8].

The notation $J(f)$ is under-decorated since $J(f)$ depends on the map $q : \mathbb{G}_n \to H$ as well as the map the map f. In context, we hope that q is clear. More generally, if $K \subseteq \mathbb{G}_n$ is any closed subgroup we then define the E^{hK}-module spectrum

$$J(q, f, K) = (E \wedge S(f))^{hK}. \tag{5.1}$$

The following basic case explains the choice of J for this notation.

Example 5.2. Suppose H is finite and V is a virtual representation of H of dimension k. Then the one-point compactification S^V of V has an H action and has underlying sphere S^k. The localization $L_{K(n)}S^V$ inherits this action and we obtain

$$J(V) = (E \wedge S^V)^{h\mathbb{G}_n}.$$

Write $RO(H)$ for the real representation ring and $RO(H)^\wedge$ for the completion at the augmentation ideal. Then the induced map

$$RO(H) \to [BH, \mathbb{Z} \times B\mathrm{Gl}_1(L_{K(n)}S^0)]$$

factors as

$$\begin{aligned} RO(H) \to RO(H)^\wedge &\cong [BH, \mathbb{Z} \times BO] \\ &\to [BH, \mathbb{Z} \times B\mathrm{Gl}_1(S^0)] \\ &\to [BH, \mathbb{Z} \times B\mathrm{Gl}_1(L_{K(n)}S^0)]. \end{aligned}$$

Then $J(V) = J(f)$ where f is the image of V under this map. This example is also discussed in [BBHS20, Section 3] and [BGHS22, Section 12.1].

Because of this example, we make the following definition.

Definition 5.3. We refer to $f\colon BH \to \mathbb{Z} \times B\mathrm{Gl}_1(L_{K(n)}S^0)$ as a $K(n)$-local spherical representation of H or simply as a *spherical representation.* The integer k obtained by projection onto the $\mathbb{Z}$ factor is the *virtual dimension* of this spherical representation.

In the rest of the section, we will want to study properties of this construction. But we first need a technical result in $K(n)$-local homotopy theory that allows us to untwist certain homotopy fixed points. The result uses the language of Devinatz–Hopkins [DH04] which is also reviewed in depth in [BBGS22]. See also Remark 2.4. The initial version is a slight generalization on Devinatz–Hopkins's determination of E_*E, and is as follows.

Lemma 5.4 ([BBGS22, Corollary 2.18])**.** *Let X be a spectrum with a $\mathbb{G}_n$-action, dualizable in the $K(n)$-local category. Give $E \wedge X$ the diagonal $\mathbb{G}_n$-action and suppose this action is continuous. Then there is an equivalence*

$$E \wedge (E \wedge X)^{h\mathbb{G}_n} \simeq E \wedge X,$$

inducing an isomorphism of Morava modules $E_*(E \wedge X)^{h\mathbb{G}_n} \cong E_*X$, where $\mathbb{G}_n$ acts on $E_*X \cong \pi_*L_{K(n)}(E \wedge X)$ diagonally.

We will use the following upgrade.

Proposition 5.5 (Untwisting Equivalence). *Let X be a spectrum with a $\mathbb{G}_n$-action, dualizable in the $K(n)$-local category. Give $E \wedge X$ the diagonal $\mathbb{G}_n$-action and suppose this action is continuous. Then for all closed subgroups $K \subseteq \mathbb{G}_n$ the natural map*

$$(E \wedge X)^{h\mathbb{G}_n} \longrightarrow (E \wedge X)^{hK}$$

extends to an equivalence of E^{hK}-module spectra

$$E^{hK} \wedge (E \wedge X)^{h\mathbb{G}_n} \simeq (E \wedge X)^{hK}.$$

Proof. Let A be any spectrum, Y any $K(n)$-locally dualizable spectrum with a $\mathbb{G}_n$-action such that the diagonal $\mathbb{G}_n$-action on $E \wedge Y$ is continuous, and let K be any closed subgroup of $\mathbb{G}_n$. Then by [BBGS22, Proposition 2.17], there is a $K(n)$-local equivalence

$$A \wedge (E \wedge Y)^{hK} \simeq (A \wedge E \wedge Y)^{hK}.$$

For example, we could take $A = (E \wedge X)^{h\mathbb{G}_n}$ and $Y = S^0$ to obtain

$$(E \wedge X)^{h\mathbb{G}_n} \wedge E^{hK} \simeq ((E \wedge X)^{h\mathbb{G}_n} \wedge E)^{hK}.$$

Now it suffices to show that $E \wedge X$ with its diagonal $\mathbb{G}_n$-action is equivalent to $(E \wedge X)^{h\mathbb{G}_n} \wedge E$, where the $\mathbb{G}_n$-action on the latter is from the right-hand factor. This is immediate from Lemma 5.4. □

We can now explore the consequences of this untwisting result for spherical representations.

Proposition 5.6. *Let $f : BH \to \mathbb{Z} \times B\mathrm{Gl}_1(L_{K(n)}S^0)$ be a spherical representation and $S(f)$ the associated representation sphere. Suppose we are given a map $q\colon \mathbb{G}_n \to H$ and $J(f) = J(q, f, \mathbb{G}_n)$. Then for all closed $K \subseteq \mathbb{G}_n$ there is a natural equivalence*

$$E^{hK} \wedge J(f) = E^{hK} \wedge J(q, f, \mathbb{G}_n) \xrightarrow{\simeq} J(q, f, K) = (E \wedge S(f))^{hK}$$

of E^{hK}-module spectra, where K acts diagonally on $E \wedge S(f)$. If, in addition K is in the kernel of q, we have

$$E^{hK} \wedge J(f) \simeq \Sigma^k E^{hK},$$

where k is the virtual dimension of f.

Proof. This follows from Proposition 5.5 and the equation $J(f) = (E \wedge S(f))^{h\mathbb{G}_n}$. If K is in the kernel of q, then K acts trivially on $S(f) \simeq S^k$, and the second statement follows. □

Remark 5.7. Given a spherical H-representation f, the action of H on the right factor of $E \wedge S(f)$ induces an action on the Morava module $E_*S(f)$. Note that, as Morava modules, $E_*S(f) \cong E_*S^k$. We thus get a map

$$\chi_f : H \longrightarrow \mathrm{Aut}_{\mathfrak{Mor}}(E_*S^k) \cong \mathrm{Aut}_{\mathfrak{Mor}}(E_*S^0) \cong \mathbb{Z}_p^\times$$

where the automorphism group is the Morava module automorphisms. We refer to χ_f as the *character defined by* f. Then, using the diagonal action of $\mathbb{G}_n$ on $E \wedge S(f)$ we have a $\mathbb{G}_n$-isomorphism

$$E_*S(f) = E_*S^k \otimes \mathbb{Z}_p\langle\chi_f\rangle = \Sigma^k E_*\langle\chi_f\rangle. \tag{5.8}$$

Note that if f is defined by a virtual real representation of H, as in Example 5.2, then χ_f is trivial or acts through $\{\pm 1\}$. We now have the following result.

Proposition 5.9. *Let $q\colon \mathbb{G}_n \to H$ be a homomorphism and let f be a spherical H-representation of virtual dimension 0. If χ_f is trivial, then $J(f) \in \kappa_n$.*

Proof. This follows from Lemma 5.4 and (5.8). □

Since $\mathbb{Z}\times B\mathrm{Gl}_1(L_{K(n)}S^0)$ is an infinite loop space, the set $[BH, \mathbb{Z}\times B\mathrm{Gl}_1(L_{K(n)}S^0)]$ has the structure of an abelian group and $S(f+g)$ is equivalent to $S(f)\wedge S(g)$ with the diagonal H action. This is compatible with the J-construction in the following sense.

Proposition 5.10. *Let $f, g\colon BH \to \mathbb{Z}\times B\mathrm{Gl}_1(L_{K(n)}S^0)$ be two maps and let $K \subseteq \mathbb{G}_n$ be a closed subgroup. Then the natural map*

$$(E\wedge S(f))^{hK}\wedge_{E^{hK}}(E\wedge S(g))^{hK}\to (E\wedge S(f+g))^{hK}$$

is an equivalence.

Proof. It is sufficient to show that the map induces an isomorphism on $K(n)_*$ homology. As a consequence of (3.13) we have that

$$K(n)_*(E\wedge S(f))^{hK}\cong \mathrm{map}(\mathbb{G}_n/K, K(n)_*S(f))$$

and, hence, $K(n)_*(E\wedge S(f))^{hK}$ is a free $K(n)_*E^{hK}$ module of rank 1. We can apply the spectral sequence

$$\mathrm{Tor}_p^{K(n)_*E^{hK}}(K(n)_*X, K(n)_*Y)_q \Longrightarrow K(n)_{p+q}(X\wedge_{E^{hK}} Y)$$

with $X=(E\wedge S(f))^{hK}$ and $Y=(E\wedge S(g))^{hK}$. Since the higher Tor terms vanish, the result follows. $\square$

Proposition 5.10 has the following immediate consequence.

Proposition 5.11. *Let $K\subseteq\mathbb{G}_n$ be a closed subgroup and let $q:\mathbb{G}_n\to H$ be a map to a finite group. The J-construction defines a homomorphism*

$$J(q,-,K):[BH, B\mathrm{Gl}_1(L_{K(n)}S^0)]\to \mathrm{Pic}(E^{hK}).$$

For later use we record the following result, which can be proved using a similar Tor-spectral sequence, as in Proposition 5.10.

Proposition 5.12. *Let $S(f)$ be a $K(n)$-local spherical representation. Then for any sequence of closed subgroups $K_1\subseteq K_2\subseteq \mathbb{G}$ the natural map*

$$E^{hK_1}\wedge_{E^{hK_2}}(E\wedge S(f))^{hK_2}\longrightarrow (E\wedge S(f))^{hK_1}$$

is an equivalence.

Remark 5.13. By construction, we have a map of $\mathbb{G}_n$-spectra $S(f)\to E\wedge S(f)$, where the $\mathbb{G}_n$-action on $S(f)$ is defined via the map $q:\mathbb{G}_n\to H$. The H-equivariant homotopy theory may be of interest in its own right, and we'd like to make a comparison.

For K a closed subgroup of $\mathbb{G}_n$, the map $S(f)\to E\wedge S(f)$ and the composition of the inclusion $K\subseteq\mathbb{G}_n$ with the map $q:\mathbb{G}_n\to H$ define a map of augmented cosimplicial $\mathbb{G}_n$-spectra

$$\begin{tikzcd} S(f) \arrow[r] \arrow[d] & F(H^{\bullet+1}, S(f)) \arrow[d] \\ E\wedge S(f) \arrow[r] & F_c(K^{\bullet+1}, E\wedge S(f)) \end{tikzcd}$$

and hence a diagram of spectral sequences

$$\begin{tikzcd} H^s(H, \pi_t S(f)) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s} S(f)^{hH} \arrow[d] \\ H^s(K, E_t S(f) \arrow[r, Rightarrow] & \pi_{t-s} J(q, f, K) \end{tikzcd}$$

Example 5.14. There are variants of these constructions, for example H doesn't have to be a finite group. Consider the case of the reduced determinant map $q = \zeta : \mathbb{G}_n \to \mathbb{G}_n/\mathbb{G}_n^1 \cong \mathbb{Z}_p$. We can take $B\mathbb{Z}_p$ to be the p-completion of S^1, as was done in [BBGS22] in constructing the determinant sphere. We set up notions of continuity of actions can as done in that paper. Then the above results will have analogous counterparts. For example, consider

$$f \in [B\mathbb{Z}_p, B\mathrm{Gl}_1(L_{K(n)}S^0)] \subseteq \pi_1 B\mathrm{Gl}_1(L_{K(n)}S^0) \cong (\pi_0 L_{K(n)}S^0)^\times.$$

If f maps to $1 \in E_0S^0$ under the Hurewicz map, then χ_f is the trivial character and Lemma 5.4 implies that

$$E_*J(f) = E_*(E \wedge S(f))^{h\mathbb{G}_n} \cong E_*S^0$$

as a Morava module. Hence $J(f) \in \kappa_n$. Using Proposition 5.12, we have an equivalence of $E^{h\mathbb{G}_n^1}$-modules

$$E^{h\mathbb{G}_n^1} \wedge J(f) \simeq (E \wedge S(f))^{h\mathbb{G}_n^1} \simeq E^{h\mathbb{G}_n^1} \wedge S(f) \simeq E^{h\mathbb{G}_n^1}$$

since $\mathbb{G}_n^1$ is the kernel of ζ. We now apply Lemma 3.8 and the fact that $H^0(\mathbb{G}_n^1, E_0) \cong \mathbb{Z}_p$ to conclude that $J(f) \in \kappa(\mathbb{G}_n^1)$.

We end with a result relating the J-construction and the subgroups $\kappa(K)$ discussed in Section 3.1, analogous to Example 5.14 but involving spherical representations of finite groups as set up at the beginning of this section.

Proposition 5.15. *Suppose that $K \subseteq \mathbb{G}_n$ is a closed subgroup with the property that the edge homomorphism*

$$\pi_0 E^{hK} \longrightarrow H^0(K, E_0)$$

is onto. Let H be a finite group, and suppose we are given a homomorphism $q : \mathbb{G}_n \to H$. Let $f \colon BH \to B\mathrm{Gl}_1(L_{K(n)}S^0)$ be a spherical representation of virtual dimension 0 with χ_f trivial. Further suppose that the composite $K \to \mathbb{G}_n \to H$ is trivial. Then

$$J(f) \in \kappa(K).$$

Proof. By Proposition 5.9 we have $J(f) \in \kappa_n$. We also have from Proposition 5.6 that

$$E^{hK} \wedge J(f) \simeq (E \wedge L_{K(n)}S^0)^{hK} \simeq E^{hK}.$$

Now apply Lemma 3.8. □

6. Truncating spectral sequences

Here we explain a technique for taking apart some spectral sequences. This will be used in Section 8 and Section 11, and any close reading could be postponed until (or even if) it is needed.

Let

$$\cdots \longrightarrow X_s \longrightarrow X_{s-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

be a tower of fibrations in spectra. Write $\mathbf{X} = \{X_s\}$ for this tower and $X = \operatorname{holim} X_s$ for the homotopy inverse limit. If we write F_s for the fiber of $X_s \to X_{s-1}$, with $X_{-1} = *$, then we have a spectral sequence with

$$E_1^{s,t}(\mathbf{X}) = \pi_{t-s}F_s \Longrightarrow \pi_{t-s}X.$$

Convergence is an issue which we won't address in this generality.

Example 6.1. Such towers often arise from a Reedy fibrant cosimplicial spectrum $A^\bullet = \{A^s\}$. Using the partial totalization functors Tot_s, this writes $X = \mathrm{Tot}(A^\bullet)$ as a homotopy inverse limit of a tower of fibrations $\mathbf{X} = \{X_s\}$ with $X_s = \mathrm{Tot}_s(A^\bullet)$. The resulting spectral sequence is the Bousfield-Kan spectral sequence. This begins with

$$E_1^{s,t}(A^\bullet) = E_1^{s,t}(\mathbf{X}) = \pi_{t-s}F_s \cong N^s\pi_t(A^\bullet) := N_t^s(A^\bullet),$$

where $N^\bullet$ is the normalization functor on cosimplicial abelian groups. Write

$$B_*^s(A^\bullet) \subseteq Z_*^s(A^\bullet) \subseteq N_*^s(A^\bullet)$$

for the coboundaries and cocyles of this cochain complex and

$$\pi^s\pi_t A^\bullet = Z_t^s(A^\bullet)/B_t^s(A^\bullet).$$

The next page of the Bousfield-Kan spectral sequence then reads

$$E_2^{s,t}(A^\bullet) \cong \pi^s\pi_t A^\bullet \Longrightarrow \pi_{t-s}\mathrm{Tot}(A^\bullet) = \pi_{t-s}X.$$

Example 6.2. An important sub-example of Example 6.1 is the Adams-Novikov spectral sequence and its variants. For us, the $K(n)$-local E-based Adams-Novikov spectral sequence is the most important, so we expand some details about it here. When working in the $K(n)$-local category we write $X \wedge Y$ for $L_{K(n)}(X \wedge Y)$.

If Y is any spectrum, let $E^{\wedge\bullet} \wedge Y$ denote the standard cobar construction; that is, the augmented cosimplicial spectrum in the $K(n)$-local category

$$Y \longrightarrow E \wedge Y \rightleftarrows E \wedge E \wedge Y \cdots \tag{6.3}$$

If $Y = E \wedge X$ for X dualizable in the $K(n)$-local category, this becomes

$$E \wedge X \longrightarrow F_c(\mathbb{G}_{n+}, E \wedge X) \rightleftarrows F_c(\mathbb{G}_{n+}^2, E \wedge X) \cdots , \tag{6.4}$$

where F_c denotes the continuous fixed points as defined in [BBGS22, Definition 2.2], for example. If $K \subseteq \mathbb{G}_n$ is any closed subgroup, the Devinatz-Hopkins definition of E^{hK} is

$$(E \wedge X)^{hK} = \mathrm{Tot}\left(F_c(\mathbb{G}_{n+}^{\bullet+1}, E \wedge X)^K\right) \simeq \mathrm{Tot}\left(F_c(\mathbb{G}_n{}^\bullet \times \mathbb{G}_n/K_+, E \wedge X)\right)$$

and the associated Bousfield-Kan spectral sequence is isomorphic to a spectral sequence

$$E_2^{s,t}(K, X) = H^s(K, E_tX) \Longrightarrow \pi_{t-s}(E^{hK} \wedge X).$$

If $K = \mathbb{G}_n$, then $E^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$ and this is the $K(n)$-local E-based Adams-Novikov spectral sequence. If K is finite, then the spectral sequence is isomorphic to the classical homotopy fixed point spectral sequence. A concise reference for all this is [BBGS22, Section 2], which is a synopsis of many sources, especially [DH04, Appendix A].

Definition 6.5. Let $\mathbf{X}$ be a tower of fibrations and fix an integer $0 \leq M \leq \infty$. Define a new tower $\mathbf{X}_{\leq M} = \{Y_s\}$ by

$$Y_s = \begin{cases} X_s, & s \leq M; \\ X_M, & s \geq M. \end{cases}$$

There is an evident map of towers $\mathbf{X} = \{X_s\} \to \mathbf{X}_{\leq M} = \{Y_s\}$ and the induced map on homotopy inverse limits is the projection map $X \to X_M$. We will call the tower $\mathbf{X}_{\leq M}$ the Mth *truncated-tower* and the resulting spectral sequence the Mth *truncated spectral sequence.*

Note that the fiber of $Y_s \to Y_{s-1}$ is F_s if $s \leq M$ and contractible if $s > M$. Write

$$E^{s,t}_{r,\leq M}(\mathbf{X}) = E^{*,*}_r(\mathbf{X}_{\leq M})$$

for the truncated spectral sequence. If $\mathbf{X}$ is the tower of a cosimplicial spectrum $A^\bullet$ we will write

$$E^{s,t}_{r,\leq M}(\mathbf{X}) = E^{s,t}_{r,\leq M}(A^\bullet).$$

The following calculation is immediate.

Lemma 6.6. *Let $A^\bullet$ be a Reedy fibrant cosimplical spectrum. For the Mth truncated spectral sequence we have*

$$E^{s,t}_{2,\leq M}(A^\bullet) \cong \begin{cases} \pi^s\pi_t A^\bullet, & s < M; \\ N^M_t(A^\bullet)/B^M_t(A^\bullet), & s = M; \\ 0, & s > M. \end{cases}$$

The induced map

$$E^{s,*}_2(A^\bullet) = \pi^s\pi_t A^\bullet \to E^{s,t}_{2,\leq M}(A^\bullet)$$

is an isomorphism if $s < M$, and zero for $s > M$. If $s = M$, this map is the standard injection. The cokernel is isomorphic to $N^M_*(A^\bullet)/Z^M_t(A^\bullet)$, which injects into $N^{M+1}_t(A^\bullet) \subseteq \pi_t A^{M+1}$.

We also have a relative version of this construction.

Definition 6.7. Suppose we have two integers $M > K \geq 0$. Then we have a map of towers $\mathbf{X}_{\leq M} \to \mathbf{X}_{\leq (K-1)}$ and we let $\mathbf{X}^M_K = \{X^s_K\}$ be the tower obtained by taking level-wise fibers. Then we have

$$X^s_K = \begin{cases} \text{fiber of } X_s \to X_{K-1}, & K \leq s \leq M; \\ *, & \text{otherwise.} \end{cases}$$

Note that

$$\text{fiber of } X^s_K \to X^{s-1}_K = \begin{cases} F_s, & K \leq s \leq M; \\ *, & \text{otherwise.} \end{cases}$$

Lemma 6.8. *Let $A^\bullet$ be a Reedy fibrant cosimplical spectrum. For the relative truncated spectral sequence we have*

$$E_2^{s,t}(\mathbf{X}_K^M) \cong \begin{cases} Z_t^K(A^\bullet) & s = K; \\ \pi^s \pi_t A^\bullet, & K < s < M; \\ N_t^M(A^\bullet)/B_t^M(A^\bullet), & s = M; \\ 0, & \text{otherwise.} \end{cases}$$

One practical application of these truncations is that they can be used to "break up" differentials and turn various r-cycles into permanent cycles. Specifically, we will use the following straightforward geometric boundary result.

Lemma 6.9. *Let $\mathbf{X}$ be a tower of fibrations, and let $0 \le K < M < \infty$ be two integers. Consider the cofiber sequence of towers*

$$\mathbf{X}_K^M \xrightarrow{f} \mathbf{X}_{\le M} \xrightarrow{g} \mathbf{X}_{\le K-1} \xrightarrow{\delta} \Sigma \mathbf{X}_K^M,$$

with a corresponding cofiber sequence in the limit

$$X_K^M \xrightarrow{f} X_M \xrightarrow{g} X_{K-1} \xrightarrow{\delta} \Sigma X_K^M.$$

Let $r \ge 1$, and suppose $x \in E_r^{s,t}(\mathbf{X}_{\le M})$ supports a differential

$$d_r(x) = y \in E_r^{s+r,t+r-1}(\mathbf{X}_{\le M}),$$

where $0 \le s < K \le s + r$. Then

(1) $g_* x \in E_r^{s,t}(\mathbf{X}_{\le K-1})$ is a permanent cycle representing a homotopy class $[g_* x] \in \pi_{t-s} X_{K-1}$, and
(2) $\delta_*[g_* x]$ is detected by $y \in E_r^{s+r,t+r-1}(\mathbf{X}_{\le M}) \cong E_r^{s+r,t+r-1}(\mathbf{X}_K^M)$.

Conversely, suppose $0 \le s < K$, and let $x \in E_r^{s,t}(\mathbf{X}_{\le M})$ be such that $g_* x \in E_r^{s,t}(\mathbf{X}_{\le K-1})$ is a permanent cycle representing $[g_* x] \in \pi_{t-s} X_{K-1}$. If $\delta_*[g_* x] = 0 \in \pi_{t-s-1} X_K^M$, then x is a permanent cycle.

Proof. The proof is a chase of diagrams and definitions, such as those from [GJ09, VI.2], which is a gloss on [BK72]. For simplicity of notation, in this proof we let F_s denote the homotopy fiber of $(\mathbf{X}_{\le M})_s \to (\mathbf{X}_{\le M})_{s-1}$.

So, suppose $x \in E_r^{s,t}(\mathbf{X}_{\le M})$ is given; by definition, this means x is represented by a map

$$x : S^{t-s} \to F_s$$

which has a lift $\tilde{x}$ making the following diagram commute

$$\begin{tikzcd} X_{s+r-1} \arrow[r] & X_s \\ S^{t-s} \arrow[u, "\tilde{x}"] \arrow[r, "x"] & F_s \arrow[u] \end{tikzcd}$$

In particular, since $K - 1 \le s + r - 1$, there are no obstructions to lifting x to the intermediate term $X_{K-1} = \lim \mathbf{X}_{\le K-1}$, proving item (1).

The formula $d_r(x) = y$ means that y is represented by the (desuspension of the) composite

$$S^{t-s} \xrightarrow{\tilde{x}} X_{s+r-1} \to \Sigma F_{s+r}.$$

Now note that we have a commutative diagram of fiber sequences

$$\begin{tikzcd} X_M \arrow[r] \arrow[d] & X_{K-1} \arrow[r] \arrow[d] & \Sigma X_K^M \arrow[d] \\ X_{s+r} \arrow[r] & X_{K-1} \arrow[r] & \Sigma X_K^{s+r}, \end{tikzcd}$$

and the right-hand bottom map gives the commutative diagram

$$\begin{tikzcd} X_{s+r-1} \arrow[r] \arrow[d] & X_{K-1} \arrow[d] \\ \Sigma F_{s+r} \arrow[r] & \Sigma X_K^{s+r}. \end{tikzcd}$$

Here, y is represented by the image of $\tilde{x}$ along the left vertical map. On the other hand, g_*x is represented by the image of $\tilde{x}$ along the top horizontal map. The image of either to the bottom corner is a class detecting $\delta_*[g_*x]$, proving (2).

The converse is straightforward. □

Part II. $K(2)$-Local Computations at $p = 2$

In this part of the paper, we get specific and work at $n = p = 2$. The main goal isto compute κ_2. To do this, we need to use the full arsenal of $K(2)$-local chromatic homotopy theory. Even this is not enough, as we will also develop new tools, and call an academic family of collaborators. Every step of the computation has required either new theory and deep computations.

So, dear reader, get ready for a long and strenuous hike through the forest of intricacies that is $K(2)$-local chromatic homotopy theory at the prime 2!

7. The subgroup filtration of κ_2

In this section, we discuss the various subgroups of $\mathbb{G}_2$ at $p = 2$ and specify which subgroup filtration we will use to compute κ_2.

We customarily choose Γ_2 to be the formal group a supersingular elliptic curve C defined over $\mathbb{F}_2$; this has the advantage of direct access to the geometry of elliptic curves, although this geometry will be tacit in this paper, subsumed into the body of work that has led to this paper (e.g. [Bea17, BG18, BGH22]). We then have an inclusion of the automorphism groups

$$\mathrm{Aut}(\mathbb{F}_4, C) \xrightarrow{\subseteq} \mathrm{Aut}(\mathbb{F}_4, \Gamma_2) = \mathbb{G}_2.$$

The structure of $\mathrm{Aut}(\mathbb{F}_4, C)$ is well understood. See, for example, [Sil86, Appendix A]. We have $\mathrm{Aut}(\mathbb{F}_4, C) \cong \mathrm{Aut}(C/\mathbb{F}_4) \rtimes \mathrm{Gal}$, where we abbreviate the Galois group of $\mathbb{F}_4/\mathbb{F}_2$ as Gal, and there is an isomorphism

$$Q_8 \rtimes C_3 \cong \mathrm{Aut}(C/\mathbb{F}_4),$$

where Q_8 is the quaternion group of order 8 and $C_3 = \mathrm{Aut}(Q_8)$ is cyclic of order 3. This semidirect product is the binary tetrahedral group.

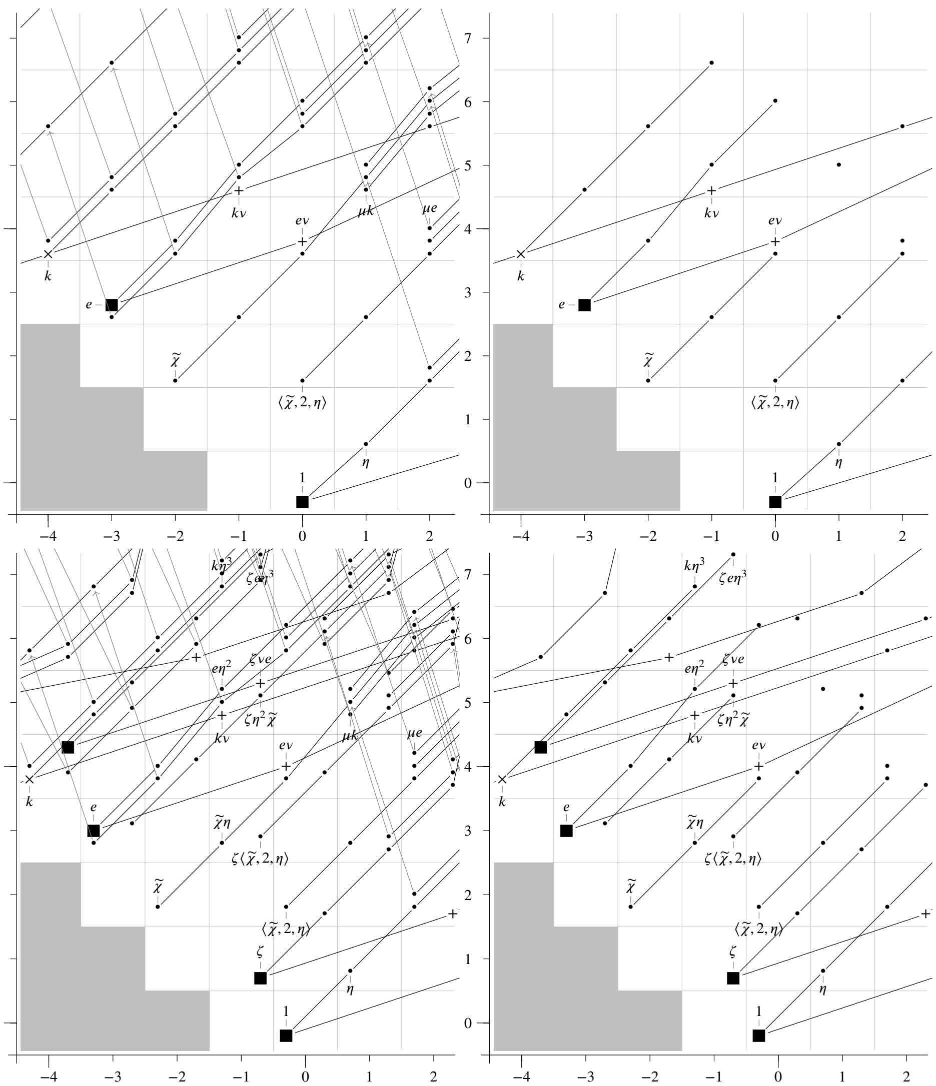

FIGURE 1. The E_3 and E_5-pages of the homotopy fixed point spectral sequences $H^s(\mathbb{G}_2^1, E_t) \Rightarrow \pi_{t-s}E^{h\mathbb{G}_2^1}$ (top) and $H^s(\mathbb{G}_2, E_t) \Rightarrow \pi_{t-s}E^{h\mathbb{G}_2}$ (bottom). Here, $\blacksquare = \mathbb{Z}_2$, $\bullet = \mathbb{Z}/2$, $+ = \mathbb{Z}/4$ and $\times = \mathbb{Z}/8$. If, instead, we let $\blacksquare = \mathbb{W}$, $\bullet = \mathbb{W}/2$, $+ = \mathbb{W}/4$ and $\times = \mathbb{W}/8$, then this is the E_3 and E_5-pages for $H^s(\mathbb{S}_2^1, E_t) \Rightarrow \pi_{t-s}E^{h\mathbb{S}_2^1}$ (top) and $H^s(\mathbb{S}_2, E_t) \Rightarrow \pi_{t-s}E^{h\mathbb{S}_2}$ (bottom). These computations were done in [BBG+22].

Definition 7.1 (The finite subgroups). Let $p = 2$ and $n = 2$. Define the following finite subgroups of $\mathbb{G}_2$

$$\begin{aligned} G_{24} &= Q_8 \rtimes C_3 \cong \mathrm{Aut}(C/\mathbb{F}_4) \\ G_{48} &= G_{24} \rtimes \mathrm{Gal} \cong \mathrm{Aut}(\mathbb{F}_4, C) \\ C_4 \subseteq G_{24} &= \text{any of the cyclic groups of order 4 in } Q_8 \\ C_2 \subseteq Q_8 &= \text{the center of } Q_8 \\ C_6 = C_2 \times C_3 &\subseteq G_{24}\ . \end{aligned}$$

The groups G_{48} and G_{24} are maximal finite subgroups in $\mathbb{G}_2$ and $\mathbb{S}_2$ respectively. See [Hen19] for more on the finite subgroups of $\mathbb{G}_2$.

All of the homotopy fixed point spectral sequences E^{hF} where F are the subgroups of Definition 7.1 have been heavily studied and, indeed, much of what follows depends on and is driven by that knowledge. See [BG18, Section 2] for a pithy summary. Note that all of the choices of C_4 are conjugate in G_{24}, so that the fixed point spectrum E^{hC_4} is independent of the choice.

We record here two additional facts about the subgroups of $\mathbb{G}_2$ which we use later.

First, there is an open subgroup $K \subseteq \mathbb{S}_2$ and a semi-direct product decomposition

$$K \rtimes G_{24} \cong \mathbb{S}_2. \tag{7.2}$$

In particular, G_{24} is a quotient of $\mathbb{S}_2$ as well as a subgroup. This can be found in many references; for example, see [Bea15].

As in Definition 4.2, we also a have the closed subgroup $\mathbb{G}_2^1$ which is the kernel of $\zeta\colon \mathbb{G}_2 \to \mathbb{Z}_2$. Similarly, $\mathbb{S}_2^1$ is the kernel of the restriction of ζ to $\mathbb{S}_2$. If we let $\pi \in \mathbb{S}_2$ be any class so that $\zeta(\pi) \in \mathbb{Z}_2$ is a topological generator, then π defines a section of $\zeta : \mathbb{S}_2 \to \mathbb{Z}_2$ and we get semi-direct product decompositions

$$\mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2 \quad \text{and} \quad \mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2,$$

where $\mathbb{S}_2^1 = \mathbb{S}_2 \cap \mathbb{G}_2^1$.

Any finite subgroup of $\mathbb{S}_2$ is automatically in $\mathbb{S}_2^1$; in particular, both G_{24} and $\pi G_{24} \pi^{-1}$ are in $\mathbb{S}_2^1$. They are evidently conjugate in $\mathbb{S}_2$, but not in $\mathbb{S}_2^1$. It is customary to write

$$G'_{24} = \pi G_{24} \pi^{-1}.$$

In what follows, we will use a filtration of κ_2 coming from a sequence of subgroups of $\mathbb{G}_2$ and the constructions of Remark 3.4.

Definition 7.3. The *subgroup filtration* of κ_2 is

$$\kappa(\mathbb{G}_2^1) \subseteq \kappa(G_{48}) \subseteq \kappa_2.$$

We can now give a road map for what is to come.

First, the associated graded of the subgroup filtration will be

$$\begin{aligned} \kappa(\mathbb{G}_2^1) &\cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2 \\ \kappa(G_{48})/\kappa(\mathbb{G}_2^1) &\cong \mathbb{Z}/2 \\ \kappa_2/\kappa(G_{48}) &\cong \mathbb{Z}/8. \end{aligned}$$

The results giving these filtration quotients can be found in Theorem 8.13, Theorem 11.17, and Theorem 12.21. Further, we show that this filtration splits by comparing the subgroup filtration with the descent filtration, namely the one arising from the Adams-Novikov Spectral Sequence. See Definition 3.27. Thus, $\kappa_{2,r} \subseteq \kappa_2$ is the subset of elements so that $d_q(\iota_X) = 0$ for $q < r$ in the Adams-Novikov Spectral Sequence for X and for any choice of $\mathbb{G}_2$-invariant generator ι_X of E_0X. If G is a closed subgroup of $\mathbb{G}_2$, let $\kappa_r(G)$ denote the intersection $\kappa(G) \cap \kappa_{2,r}$.

Thus, we will have a diagram, where all the arrows are inclusions

$$\begin{tikzcd}
\kappa_2 & \kappa(G_{48}) \arrow[l] & \kappa(\mathbb{G}_2^1) \arrow[l] \\
\kappa_{2,5} \arrow[u] & \kappa_5(G_{48}) \arrow[l] \arrow[u] & \kappa_5(\mathbb{G}_2^1) \arrow[l] \arrow[u] \\
\kappa_{2,7} \arrow[u] & \kappa_7(G_{48}) \arrow[l] \arrow[u] & \kappa_7(\mathbb{G}_2^1). \arrow[l] \arrow[u]
\end{tikzcd} \tag{7.4}$$

In the end we will show that the diagram of (7.4) maps isomorphically to following diagram of groups, again with all arrows inclusions.

$$\begin{tikzcd}
(\mathbb{Z}/8)^2 \oplus (\mathbb{Z}/2)^3 & \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^3 \arrow[l] & \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2 \arrow[l] \\
(\mathbb{Z}/8)^2 \oplus \mathbb{Z}/2 \arrow[u] & \mathbb{Z}/8 \oplus \mathbb{Z}/2 \arrow[l] \arrow[u] & \mathbb{Z}/8 \oplus \mathbb{Z}/2 \arrow[l, "="'] \arrow[u] \\
(\mathbb{Z}/2)^2 \arrow[u] & \mathbb{Z}/2 \arrow[l] \arrow[u] & \mathbb{Z}/2\ . \arrow[l, "="'] \arrow[u]
\end{tikzcd} \tag{7.5}$$

We can then filter this diagram first vertically and then horizontally. Using Theorem 11.24 and Theorem 12.29 the associated graded becomes

s			
3		$\mathbb{Z}/2\{\widetilde{\chi}\eta\}$	$\mathbb{Z}/2\{\zeta\langle\widetilde{\chi}, 2, \eta\rangle\}$
5	$\mathbb{Z}/4\{k\nu\}$		$\mathbb{Z}/4\{\zeta e\nu\} \oplus \mathbb{Z}/2\{\zeta\widetilde{\chi}\eta^2\}$
7	$\mathbb{Z}/2\{k\eta^3\}$		$\mathbb{Z}/2\{\zeta e\eta^3\}$

We have labelled the various generators here, using the following convention. Suppose $X \in \kappa_{2,r}$ and $\iota_X \in H^0(\mathbb{G}_2, E_0X) \cong \mathbb{Z}_2$ is a $\mathbb{G}_2$-invariant generator. Then note that Definition 3.27 implies an inclusion

$$\kappa_{2,r}/\kappa_{2,r+1} \longrightarrow E_r^{r,r-1}(S^0),$$

given by the formula $d_r(\iota_X) = \phi_r(X)\iota_X$.

Thus we have labelled the elements of the table by their names in $E_r^{r,r-1}(S^0)$, which in turn have names inherited from $E_2^{r,r-1}(S^0) \cong H^r(\mathbb{G}_2, E_{r-1})$. The relevant part of the group cohomology $E_2^{r,r-1}(S^0) \cong H^r(\mathbb{G}_2, E_{r-1})$, the d_3-differentials, and the resulting page $E_5^{r,r-1}(S^0)$ were computed in [BBG+22], and are summarized in Figure 1.

Remark 7.6. It is instructive to compare these results with what happened at $p = 3$ and height $n = 2$. See [GHMR15]. There the analogous subgroup filtration collapses to

$$\kappa(\mathbb{G}_2^1) = \kappa(G_{24}) \subseteq \kappa_2$$

and we had $\kappa(\mathbb{G}_2^1) \cong \mathbb{Z}/3$ and $\kappa_2/\kappa(\mathbb{G}_2^1) \cong \mathbb{Z}/3$. The descent filtration was particularly simple

$$0 = \kappa_{2,6} \subseteq \kappa_{2,5} = \kappa_2$$

and

$$\phi_5 : \kappa_2 \longrightarrow H^5(\mathbb{G}_2, E_4) \cong \mathbb{Z}/3 \times \mathbb{Z}/3$$

was an isomorphism. This gave the splitting.

The two summands of $\mathbb{Z}/8$ at $p = n = 2$ are completely analogous to the two summands of $\mathbb{Z}/3$ at $p = 3$; in both cases generators are detected in the decent filtration by $k\nu$ and $\zeta e\nu$, once we realize the name of ν at $p = 3$ is α.

The existence of the extra summands of $\mathbb{Z}/2$ at $p = 2$ is related to the fact that there is an extra homomorphism from $\mathbb{S}_2$ to $\mathbb{Z}/2$ at the prime 2, see [Hen17, Propositions 5.2, 5.3]. This is the same complication that gave rise to the modification of the Chromatic Splitting Conjecture at $p = 2$; see [BGH22]. Finally, the class in $\kappa(G_{48})/\kappa(\mathbb{G}_2^1) \cong \mathbb{Z}/2$ arises from the J-construction and, as far as we know, cannot be produced in any other way.

8. The $E^{h\mathbb{G}_2^1}$-orientable elements of the exotic Picard group

In this section we start with the calculation of $\kappa(\mathbb{G}_2^1) \subseteq \kappa_2$, the subgroup of exotic invertible elements which have an $E^{h\mathbb{G}_2^1}$-orientation. The complexity of $H^*(\mathbb{G}_2, E_*)$ makes this section technically forbidding. The key ideas are all in Section 4, and the key computational input comes from several sources. We refer to [BBG+22] for results about the cohomology $H^*(\mathbb{G}_2, E_*)$ and $H^*(\mathbb{G}_2^1, E_*)$. We also refer to [BGH22] for the computation of $\pi_* E^{h\mathbb{G}_2^1}$ in a range. We freely use the notation established in these sources.

Recall that in Theorem 4.19, we proved that

$$\kappa(\mathbb{G}_2^1) \cong H^1(\mathbb{G}_2/\mathbb{G}_2^1, 1 + \Lambda)$$

where Λ is the subgroup of $\pi_0 E^{h\mathbb{G}_2^1}$ of elements of positive Adams–Novikov filtration. We calculate the group $1+\Lambda$ in Proposition 8.5 and show that the action of $\mathbb{G}_2/\mathbb{G}_2^1 \cong \mathbb{Z}_2$ is trivial in Proposition 8.6 and Proposition 8.9. This allows us to determine $\kappa(\mathbb{G}_2^1)$ in Theorem 8.13. In Theorem 8.16, we compute the descent filtration of $\kappa(\mathbb{G}_2^1)$.

Lemma 8.1. *The unique non-zero class*

$$w \in \pi_{-2} E^{h\mathbb{G}_2^1} \cong \mathbb{Z}/2$$

is detected by the non-zero class $\widetilde{\chi} \in H^2(\mathbb{G}_2^1, E_0) \cong \mathbb{Z}/2$. *The class* w *is the image of a class* $w_0 \in \pi_{-2} L_{K(2)} S^0$ *of order* 2 *under the map*

$$\pi_{-2} L_{K(2)} S^0 \longrightarrow \pi_{-2} E^{h\mathbb{G}_2^1}.$$

Furthermore,

$$\pi_{-2}L_{K(2)}S^0 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

generated by w_0 and $\zeta w_0 \eta$.

Proof. By [BGH22, Corollary 9.1.8] we have that

$$\pi_{-1}E^{h\mathbb{G}_2^1} \cong \mathbb{Z}/2$$
$$\pi_{-2}E^{h\mathbb{G}_2^1} \cong \mathbb{Z}/2$$

generated by elements detected by $\widetilde{\chi}\eta \in H^3(\mathbb{G}_2^1, E_2)$ and $\widetilde{\chi} \in H^2(\mathbb{G}_2^1, E_0)$ respectively. The latter class detects w. It then follows from the fiber sequence

$$L_{K(2)}S^0 \to E^{h\mathbb{G}_2^1} \to E^{h\mathbb{G}_2^1}$$

that we have an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{-2}L_{K(2)}S^0 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

The generator of the subgroup in this extension is detected by $\zeta\widetilde{\chi}\eta \in H^4(\mathbb{G}_2, E_2)$, and there is a class $w_0 \in \pi_{-2}L_{K(2)}S^0$ detected by $\widetilde{\chi} \in H^2(\mathbb{G}_2, E_0)$ which generates the quotient. To prove the lemma, it suffices to show this sequence splits, i.e. that w_0 has order 2.

If the sequence is not split then the class in the kernel becomes trivial when passing to $\pi_{-2}L_{K(2)}V(0)$. But, by [BGH22, Theorem 8.2.6] both classes are nonzero under the map

$$H^*(\mathbb{G}_2, E_*) \longrightarrow v_1^{-1}H^*(\mathbb{G}_2, E_*V(0)).$$

By [BGH22, Theorem 8.3.5] they detect non-zero classes in $\pi_*L_{K(1)}L_{K(2)}V(0)$. Thus the sequence is split as needed. □

Lemma 8.2 ([BGH22, Corollary 9.1.8, 9.1.9])**.** *There is an isomorphism*

$$\pi_0 E^{h\mathbb{G}_2^1} \cong \mathbb{Z}_2\{1\} \oplus \mathbb{Z}/8\{x\} \oplus \mathbb{Z}/4\{y\},$$

where the generators y and x have the following description.

1) The class $y \in \pi_0 E^{h\mathbb{G}_2^1}$ can be chosen to be either of the two elements in the Toda bracket $\langle w, 2, \eta\rangle$, and is detected by the Massey product

$$\langle \widetilde{\chi}, 2, \eta\rangle \in H^2(\mathbb{G}_2^1, E_2).$$

The class $2y$ is then detected by $\widetilde{\chi}\eta^2 \in H^4(\mathbb{G}_2^1, E_4)$.

2) The class $x \in \pi_0 E^{h\mathbb{G}_2^1}$ is detected by $e\nu \in H^4(\mathbb{G}_2^1, E_4)$. The class $4x$ is detected by $e\eta^3 \in H^6(\mathbb{G}_2^1, E_6)$.

In particular,

$$\Lambda \cong \mathbb{Z}/8\{x\} \oplus \mathbb{Z}/4\{y\}. \tag{8.3}$$

Corollary 8.4. *In the spectral sequence*

$$E_2^{s,t}(\mathbb{G}_2^1) = H^s(\mathbb{G}_2^1, E_t) \Rightarrow \pi_{t-s}E^{h\mathbb{G}_2^1},$$

we have

$$E_\infty^{s,s}(\mathbb{G}_2^1) = \begin{cases} \mathbb{Z}_2\{1\} & s=0 \\ \mathbb{Z}/2\{\langle \widetilde{\chi}, 2, \eta\rangle\} & s=2 \\ \mathbb{Z}/2\{\eta^2\widetilde{\chi}\} \oplus \mathbb{Z}/4\{\nu e\} & s=4 \\ \mathbb{Z}/2\{\eta^3 e\} & s=6 \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore, $E_2^{0,0}(\mathbb{G}_2^1) = E_\infty^{0,0}(\mathbb{G}_2^1)$*, and, for* $0 < s \le 6$*, we have* $E_{s+1}^{s,s} = E_\infty^{s,s}(\mathbb{G}_2^1)$*.*

Proof. The E_5-page of the spectral sequence was computed in [BBG+22, Section 8]. See Figure 1 for a figure illustrating that computation. The classes listed are permanent cycles by Lemma 8.2. For degree reasons, they cannot be hit by differentials. Therefore, they survive to the E_∞-page and account for all of the classes in the homotopy group $\pi_0 E^{h\mathbb{G}_2^1}$. It follows that there can be no non-zero classes in higher filtration at E_∞. The claim about the stabilization of $E_{s+1}^{s,s}(\mathbb{G}_2^1)$ is immediate from the computation in [BBG+22]. □

Proposition 8.5. *There is an isomorphism*

$$1 + \Lambda \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2.$$

Proof. In [BGH22, Corollary 9.1.9] it is shown that $x^2 = 0$ and

$$y^2 = 2y + 2lx$$

for some integer l which only depends on its congruence class modulo 4. Further, xy has Adams-Novikov filtration at least 6, so we have

$$xy = 4\varepsilon x$$

for some $\varepsilon = 0$ or 1. Then

$$(1+x)^n = 1 + nx$$
$$(1 + y - lx)^2 = 1.$$

Thus $1 + \Lambda \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2$ generated by $1+x$, $1+y-lx$, and $1+2y$. □

The next step is to show that the $\mathbb{Z}_2 \cong \mathbb{G}_2/\mathbb{G}_2^1$ action on $1+\Lambda$ is trivial; this is equivalent to showing that the action on Λ is trivial.

Proposition 8.6. *The class* $y \in \pi_0 E^{h\mathbb{G}_2^1}$ *is invariant under the action of* $\mathbb{G}_2/\mathbb{G}_2^1$*.*

Proof. The class w of Lemma 8.1 is the image of a class w_0 of order 2 under the map $\pi_{-2} L_{K(2)} S^0 \longrightarrow \pi_{-2} E^{h\mathbb{G}_2^1}$. Once we have w_0 we can form the Toda bracket $\langle w_0, 2, \eta\rangle$. So, by Part (1) of Lemma 8.2, we see that y is in the image of

$$\pi_0 L_{K(2)} S^0 \longrightarrow \pi_0 E^{h\mathbb{G}_2^1}.$$ □

Our next goal is to prove the invariance of x.

Remark 8.7. In [BBG+22, Lemma 7.2], we proved that the action of $\mathbb{G}_2/\mathbb{G}_2^1$ on $H^*(\mathbb{G}_2^1, E_t)$ is trivial in the range $0 \le t < 12$.

The triviality of the action of $\mathbb{G}_2/\mathbb{G}_2^1$ in cohomology implies that the action of $\mathbb{G}_2/\mathbb{G}_2^1$ on $E_\infty^{s,s}(\mathbb{G}_2^1)$ is trivial, but this is only the associated graded of $\pi_0 E^{h\mathbb{G}_2^1}$. In particular, this result is not enough to show the class x is invariant, because $4x$ is detected in higher filtration. So, we will use the technique of truncated spectral sequences, from Section 6 to discuss the invariant of x.

As in Example 6.2 and [DH04], the spectral sequence

$$E_2^{s,t} = H^s(\mathbb{G}_2^1, E_t) \Longrightarrow \pi_{t-s} E^{h\mathbb{G}_2^1}$$

is constructed as the $K(2)$-local E-based Adams-Novikov Spectral Sequence; that is, as the Bousfield-Kan Spectral Sequence of the augmented cosimplicial $K(2)$-local spectrum

$$E^{h\mathbb{G}_2^1} \to E^{\bullet+1} \wedge E^{h\mathbb{G}_2^1} = A^\bullet.$$

Using the partial totalization functors Tot_n, this writes $E^{h\mathbb{G}_2^1}$ as a homotopy inverse limit of a tower of fibrations $\mathbf{X} = \{X_s\}$

$$\cdots \longrightarrow X_s \longrightarrow X_{s-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

with $X_s = \mathrm{Tot}_s(E^{\bullet+1} \wedge E^{h\mathbb{G}_2^1})$. The limit X is then $E^{h\mathbb{G}_2^1}$. We will work with the truncated tower $\mathbf{X}_{\leq 6}$ with $X_6 = \operatorname{holim} \mathbf{X}_{\leq 6}$. See Section 6 for the definitions and notation.

Lemma 8.8. *In the Adams-Novikov tower $\mathbf{X}$ under $X = E^{h\mathbb{G}_2^1}$, the map*

$$\pi_0 E^{h\mathbb{G}_2^1} \to \pi_0 X_6$$

is an injection.

Proof. In the Adams-Novikov spectral sequence

$$H^s(\mathbb{G}_2^1, E_t) \Longrightarrow \pi_{t-s} E^{h\mathbb{G}_2^1}$$

we have $E_\infty^{s,s} = 0$ for $s > 6$ by Corollary 8.4. Now use Lemma 6.6. $\square$

Proposition 8.9. *Let $x \in \pi_0 E^{h\mathbb{G}_2^1}$ be any class detected by $e\nu \in H^4(\mathbb{G}_2^1, E_4)$. Then x is invariant under the action of $\mathbb{G}_2/\mathbb{G}_2^1$.*

Proof. Truncation is natural in towers of fibrations and the cobar construction $X \to E^{\bullet+1} \wedge X$ is natural in X. Let $\psi\colon E^{h\mathbb{G}_2^1} \to E^{h\mathbb{G}_2^1}$ be the map induced by the action of a topological generator $\psi \in \mathbb{G}_2/\mathbb{G}_2^1$. This induces a map

$$\psi : \mathbf{X}_{\leq 6} \longrightarrow \mathbf{X}_{\leq 6}$$

of truncated towers and hence induces a map of truncated spectral sequences. By Lemma 8.8 it is sufficient to show that $x \in \pi_0 X_6$ is invariant; that is, $\psi(x) = x$.

Te class x is detected by

$$e\nu \in H^4(\mathbb{G}_2^1, E_4) \cong E_{2,\leq 6}^{4,4}(A^\bullet),$$

and by Part (2) of Lemma 8.2, we know that $4x$ is detected by

$$e\eta^3 \in H^6(\mathbb{G}_2^1, E_6) \subseteq E_{2,\leq 6}^{6,6}(A^\bullet).$$

The advantage of working with X_6 is that this class represented by $e\eta^3$ is in fact a product of η^3 and a class detected by e. Indeed, we now turn to the class $e \in H^3(\mathbb{G}_2^1, E_0)$, which is invariant by Remark 8.7. It was shown in [BBG+22, Proposition 8.3] that $d_3(e) = 0$ in the Adams-Novikov spectral sequence. By naturality, the same is true in the truncated spectral sequence, where longer differentials on e are not possible. Thus there is a class $z \in \pi_{-3}X_6$ detected by e. We do not know that z is invariant, but we may conclude

$$\psi(z) = z + b$$

where b has filtration 5.

Now we have that $z\nu$ and x are two classes of order 8 in $\pi_0 X_6$ which are detected by $e\nu \in H^4(\mathbb{G}_2^1, E_4)$. If we write

$$v = z\nu - x$$

then $v \in E_{2,\leq 6}^{6,6}(A^\bullet)$ is torsion. But by Lemma 6.6, we have an inclusion

$$H^6(\mathbb{G}_2^1, E_6) \subseteq E_{2,\leq 6}^{6,6}(A^\bullet),$$

whose cokernel is a summand of $\pi_*(A^7)$, so is torsion-free. Thus v must be detected in $H^6(\mathbb{G}_2^1, E_6)$. The group $H^6(\mathbb{G}_2^1, E_6)$ is invariant under the action of $\mathbb{G}_2/\mathbb{G}_2^1$, by [BBG+22, Lemma 7.2]. Since this is the top filtration of $\pi_0 X_6$, any class in $\pi_0 X_6$ detected by an element of $H^6(\mathbb{G}_2^1, E_6) \subseteq E_{2,\leq 6}^{6,6}(A^\bullet)$ must be invariant. We conclude that v is invariant, thus $z\nu$ is invariant if and only if x is.

So finally, we show that $z\nu$ is invariant to complete the proof. Since $\psi(z) = z + b$ for b detected by a class in $H^5(\mathbb{G}_2^1, E_2)$, and ν is invariant, we have that $\psi(z\nu) = z\nu + b\nu$. We computed in [BBG+22, Table 2] that all classes in $H^5(\mathbb{G}_2^1, E_2)$ are multiples of η, thus b must be a multiple of η. See also Figure 1. Since $\eta\nu = 0$, we get that $b\nu = 0$, and so $\psi(z\nu) = z\nu$ as needed. □

Now we combine the above results into the following consequence.

Theorem 8.10. *The action of $\mathbb{G}_2/\mathbb{G}_2^1$ on $\pi_0 E^{h\mathbb{G}_2^1}$ is trivial. In particular, the map $\pi_0 L_{K(2)}S^0 \to \pi_0 E^{h\mathbb{G}_2^1}$ defined by the unit of $\pi_0 E^{\mathbb{G}_2^1}$ is surjective.*

Proof. The first claim follows from Proposition 8.6 and Proposition 8.9 and the fact that the unit is fixed. The second claim follows from the fact that the image of the unit map on π_0 is the kernel of $\psi - 1$ for ψ a topological generator of $\mathbb{G}_2/\mathbb{G}_2^1$. □

As an aside, this also allows us to compute $\pi_{-1}L_{K(2)}S^0$.

Corollary 8.11. *There is an isomorphism*

$$\pi_{-1}L_{K(2)}S^0 \cong \mathbb{Z}_2\{\zeta\} \oplus \mathbb{Z}/8\{\zeta x\} \oplus \mathbb{Z}/4\{\zeta y\} \oplus \mathbb{Z}/2\{\eta w_0\}.$$

Proof. The fiber sequence

$$L_{K(2)}S^0 \longrightarrow E^{h\mathbb{G}_2^1} \xrightarrow{\psi-1} E^{h\mathbb{G}_2^1} \tag{8.12}$$

gives an exact sequence

$$\pi_0 E^{h\mathbb{G}_2^1} \xrightarrow{\partial} \pi_{-1}L_{K(2)}S^0 \longrightarrow \pi_{-1}E^{h\mathbb{G}_2^1}$$

which we prove is split short exact. For injectivity of ∂, we use that the kernel is the image of $\psi - 1$ acting on $\pi_0 E^{h\mathbb{G}_2^1}$, which is zero by Theorem 8.10. By [BGH22, Corollary 9.1.8], we know that $\pi_{-1}E^{h\mathbb{G}_2^1} \cong \mathbb{Z}/2$ generated by $w\eta$, which is the image of $w_0\eta$ by Lemma 8.1. Since $w_0\eta$ has order 2, the sequence is split. Note that the map ∂ is multiplication by ζ, giving the claim. $\square$

Combining the results proved above, we get the following explicit identification of the group of $E^{h\mathbb{G}_2^1}$-orientable elements of κ_2.

Theorem 8.13. *The twisting construction $\alpha \mapsto X(\alpha)$ of Definition 4.9 gives an isomorphism*

$$\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2 \cong 1 + \Lambda \cong \kappa(\mathbb{G}_2^1).$$

Proof. By Theorem 8.10, the action of $\mathbb{G}_2/\mathbb{G}_2^1$ on $1+\Lambda$ is trivial. Thus Theorem 4.19 gives the isomorphism between $\kappa(\mathbb{G}_2^1)$ and $1+\Lambda$. The identification of $1+\Lambda$ is Proposition 8.5. $\square$

Now we turn to identifying the descent filtration on $\kappa(\mathbb{G}_2^1)$. To do that, we will use Theorem 4.27, which requires us to check one more technical condition. We check that condition in Proposition 8.15, for which we'll use the following input.

Lemma 8.14. *In the homotopy fixed point spectral sequence for $G = \mathbb{G}_2$ and $\mathbb{G}_2^1$, the class*

$$\eta^3 k \in H^7(G, E_6)$$

is a d_5-cycle.

Proof. In [BBG+22, §8], we compute the E_5-page $E_5^{s,t}(G)$ of both homotopy fixed point spectral sequences in a range. See Figures 12 and 13 of that reference. In particular, we show that $k \in E_2^{4,0}(G)$ is a d_3-cycle. Therefore,

$$d_5(\eta^3 k) = \eta^3 d_5(k).$$

But

$$d_5(k) \in E_5^{9,4}(G) \cong \begin{cases} \mathbb{Z}/4\{\nu k^2\} \oplus \mathbb{Z}/2\{\eta^2 ek\} & G = \mathbb{G}_2^1 \\ \mathbb{Z}/4\{\nu k^2, \zeta\nu ek\} \oplus \mathbb{Z}/2\{\eta^2 ek\} & G = \mathbb{G}_2. \end{cases}$$

Since $\eta^3\nu = 0$ and we prove that $\eta^5 ek = d_3(\mu ek)$, it follows that

$$d_5(\eta^3 k) \in \eta^3(E_5^{9,4}(G)) = 0.$$ $\square$

Proposition 8.15. *For $2 \le s \le 7$, the sequence*

$$0 \to E_s^{s-1,s-1}(\mathbb{G}_2^1) \xrightarrow{\zeta} E_s^{s,s-1}(\mathbb{G}_2) \xrightarrow{i_*} E_s^{s,s-1}(\mathbb{G}_2^1) \to 0$$

is exact, where i_* is the restriction.

Proof. The triviality of the action of $\mathbb{Z}_2 \cong \mathbb{G}_2/\mathbb{G}_2^1$ (see Remark 8.7) gives such an exact sequence on E_2-pages. In [BBG+22, §8], we computed the d_3-differentials in a range. See Figure 1 above. From that explicit determination of $E_r^{*,t}(\mathbb{G}_2)$ and $E_r^{*,t}(\mathbb{G}_2^1)$ for $2 \le r \le 5$ and $0 \le t \le 10$, we conclude that we have an exact sequence

$$0 \to E_r^{s-1,t}(\mathbb{G}_2^1) \xrightarrow{\zeta} E_r^{s,t}(\mathbb{G}_2) \xrightarrow{i_*} E_r^{s,t}(\mathbb{G}_2^1) \to 0$$

in that same range.

In particular, our sequence is exact at the E_s-page for $2 \leq s \leq 5$. Since the E_6 and E_7 pages agree, it remains to check that this is also true at the E_7-page. At E_5, we have the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & E_5^{6,6}(\mathbb{G}_2^1) \arrow[r, "\zeta"] \arrow[d, equal] & E_5^{7,6}(\mathbb{G}_2) \arrow[r, "i_*"] \arrow[d, equal] & E_5^{7,6}(\mathbb{G}_2^1) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z}/2\{\eta^3 e\} \arrow[r] & \mathbb{Z}/2\{\zeta\eta^3 e, \eta^3 k\} \arrow[r] & \mathbb{Z}/2\{\eta^3 k\} \arrow[r] & 0. \end{tikzcd}$$

From Lemma 8.2, we know that $\eta^3 e$ is a permanent cycle in the $\mathbb{G}_2^1$-spectral sequence, and from Theorem 8.10, we get that it remains a permanent cycle in the $\mathbb{G}_2$-spectral sequence. Hence $\zeta\eta^3 e$ is also a permanent cycle. From Lemma 8.14, $\eta^3 k$ is a d_5-cycle, hence it survives to the E_7-page.

The d_5 differentials that have $E_5^{6,6}(\mathbb{G}_2^1)$, $E_5^{7,6}(\mathbb{G}_2)$, and $E_5^{7,6}(\mathbb{G}_2^1)$ as targets originate in

$$E_5^{1,2}(\mathbb{G}_2^1) = \mathbb{Z}/2\{\eta\}, \quad E_5^{2,2}(\mathbb{G}_2) = \mathbb{Z}/2\{\langle \widetilde{\chi}, 2, \eta\rangle, \eta\zeta\}, \quad E_5^{2,2}(\mathbb{G}_2^1) = \mathbb{Z}/2\{\langle \widetilde{\chi}, 2, \eta\rangle\},$$

respectively. However, η, $\eta\zeta$ and $\langle \widetilde{\chi}, 2, \eta\rangle$ are permanent cycles. For $\langle \widetilde{\chi}, 2, \eta\rangle$, see Lemma 8.2 above. Therefore, the exact sequence remains the same at the E_7-page. ☐

Finally, we apply Theorem 4.27 to study the filtration on $\kappa(\mathbb{G}_n^1)$.

Theorem 8.16. *In the filtration*

$$0 \subseteq \kappa_7(\mathbb{G}_2^1) \subseteq \kappa_5(\mathbb{G}_2^1) \subseteq \kappa_3(\mathbb{G}_2^1) = \kappa(\mathbb{G}_2^1) \cong \mathbb{Z}/8 \times (\mathbb{Z}/2)^2$$

we have isomorphisms

$$\kappa(\mathbb{G}_2^1)/\kappa_5(\mathbb{G}_2^1) \xrightarrow[\cong]{\phi_3} \mathbb{Z}/2\{\zeta\langle \widetilde{\chi}, 2, \eta\rangle\}$$

$$\kappa_5(\mathbb{G}_2^1)/\kappa_7(\mathbb{G}_2^1) \xrightarrow[\cong]{\phi_5} \mathbb{Z}/4\{\zeta e\nu\} \times \mathbb{Z}/2\{\zeta\widetilde{\chi}\eta^2\}$$

$$\kappa_7(\mathbb{G}_2^1) \xrightarrow[\cong]{\phi_7} \mathbb{Z}/2\{\zeta e\eta^3\}.$$

Proof. We apply Theorem 4.27 with $N = 7$. Conditions (i) and (ii) follow from Corollary 8.4, while condition (iii) is checked in Proposition 8.15. The identification of the filtration quotients follows from Corollary 8.4. ☐

9. Duality Resolutions

A crucial tool for analyzing the rest of κ_2 is the Duality Resolution of the half sphere $E^{h\mathbb{S}_2^1}$. In this section we recall the basic material.

The algebraic duality resolution, first introduced by Goerss–Henn–Mahowald–Rezk and developed in detail in [Bea15], is a device for isolating the contributions of finite subgroups F to the cohomology of $\mathbb{S}_2^1$; similarly the topological duality resolution of [BG18] isolates the contributions of π_*E^{hF} to $\pi_*E^{h\mathbb{S}_2^1}$. Some of the basic finite subgroups of $\mathbb{G}_2$ were defined and discussed in Section 7.

The *A*lgebraic Duality Resolution is an augmented exact complex of continuous $\mathbb{Z}_2[[\mathbb{S}_2^1]]$-modules

$$0 \to \mathbb{Z}_2[[\mathbb{S}_2^1/F_3]] \to \mathbb{Z}_2[[\mathbb{S}_2^1/F_2]] \to \mathbb{Z}_2[[\mathbb{S}_2^1/F_1]] \to \mathbb{Z}_2[[\mathbb{S}_2^1/F_0]] \to \mathbb{Z}_2[[\mathbb{S}_2^1/F_{-1}]] \to 0$$

where

$$F_{-1} = \mathbb{S}_2^1, \quad F_0 = G_{24}, \quad F_1 = F_2 = C_6, \quad F_3 = G'_{24}. \tag{9.1}$$

Note $\mathbb{Z}_2[[\mathbb{S}_2^1/F_{-1}]] \cong \mathbb{Z}_2$ is the trivial module.

We then induce up to a sequence of $\mathbb{Z}_2[[\mathbb{G}_2]]$-modules

$$\begin{aligned} 0 \to \mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \to &\mathbb{Z}_2[[\mathbb{G}_2/C_6]] \to \mathbb{Z}_2[[\mathbb{G}_2/C_6]] \\ &\to \mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \to \mathbb{Z}_2[[\mathbb{G}_2/\mathbb{S}_2^1]] \to 0. \end{aligned} \tag{9.2}$$

Since G'_{24} is conjugate to G_{24} in $\mathbb{G}_2$, we have dropped the distinction. We write

$$d = d_p : \mathbb{Z}_2[[\mathbb{G}_2/F_p]] \to \mathbb{Z}_2[[\mathbb{G}_2/F_{p-1}]], \qquad 0 \le p \le 3.$$

We next apply $\mathrm{Hom}_{\mathbb{Z}_2}(-, E_*)$ to the sequence (9.2) to obtain an exact sequence of Morava modules

$$\begin{aligned} 0 \to \mathrm{map}(\mathbb{G}_2/\mathbb{S}_2^1, E_*) \to &\mathrm{map}(\mathbb{G}_2/G_{24}, E_*) \to \mathrm{map}(\mathbb{G}_2/C_6, E_*) \\ &\to \mathrm{map}(\mathbb{G}_2/C_6, E_*) \to \mathrm{map}(\mathbb{G}_2/G_{24}, E_*) \to 0. \end{aligned}$$

By Remark 2.4 this is isomorphic to an exact sequence of Morava modules

$$0 \to E_*E^{h\mathbb{S}_2^1} \to E_*E^{hG_{24}} \to E_*E^{hC_6} \to E_*E^{hC_6} \to E_*E^{h\mathbb{G}_{24}} \to 0. \tag{9.3}$$

The project of the Topological Duality Resolutions from [GHMR05] or [BG18] is to realize such sequences as maps of spectra.

Remark 9.4. There is some ambiguity here; for example, the Morava module $\mathrm{map}(\mathbb{G}_2/G_{24}, E_*)$ is 24-periodic, but the spectrum $E^{hG_{24}}$ is only $8 \cdot 24 = 192$ periodic. In the language of Definition 3.16 we can produce an algebraic periodicity class

$$\Delta \in H^0(G_{24}, E_{24})$$

as the discriminant of a deformation of the unique supersingular elliptic curve over $\mathbb{F}_4$. Then Δ^8 is a permanent cycle and

$$\pi_{192}E^{hG_{24}} \longrightarrow H^0(G_{24}, E_{192})$$

is an isomorphism. Thus Δ^8 detects a topological periodicity class. We will use these classes in our constructions below.

The main theorem of [BG18] produces a sequence of spectra

$$E^{h\mathbb{S}_2^1} \to E^{hG_{24}} \to E^{hC_6} \to E^{hC_6} \to \Sigma^{48}E^{hG_{24}} \tag{9.5}$$

realizing the sequence (9.3) of Morava modules, and called the *T*opological Duality Resolution. All the maps in (9.5) are algebraic in the sense of Definition 3.21 for $d = \Delta$. Furthermore, all compositions are zero and all Toda brackets in (9.5) contain zero, so the sequence refines to a tower.

Definition 9.6 (**Duality Spectral Sequences).** For F_s as in (9.1), we get the following spectral sequences.

(1) If X is any spectrum, then the spectral sequence of the tower gives the Topological Duality spectral sequence for X

$$E_1^{s,t}(X) = \pi_t(E^{hF_s} \wedge X) \Longrightarrow \pi_{t-s}(E^{h\mathbb{S}_2^1} \wedge X).$$

(2) Let M be a Morava module. The original sequence of $\mathbb{Z}_2[[\mathbb{S}_2^1]]$-modules and the Schapiro Isomorphism gives the Algebraic Duality spectral sequence

$$E_1^{p,q}(M) = H^p(F_q, M) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M).$$

for any appropriate $\mathbb{Z}_2[[\mathbb{S}_2^1]]$-module M.

We now have two ways of calculating $\pi_* E^{h\mathbb{S}_2^1}$ from $H^*(F_p, E_*)$ encoded in the two ways around the following square

$$\begin{tikzcd}
H^s(F_p, E_t X) \arrow[r, Rightarrow, "HFPSS"] \arrow[d, Rightarrow, "ADSS"'] & \pi_{t-s}(E^{hF_p} \wedge X) \arrow[d, Rightarrow, "TDSS"] \\
H_c^{s+p}(\mathbb{S}_2^1, E_t X) \arrow[r, Rightarrow, "ANSS"'] & \pi_{t-s-p}(E^{h\mathbb{S}_2^1} \wedge X).
\end{tikzcd}$$

We hope the acronyms are clear. In general these two methods have a relatively complicated relationship, but under certain hypotheses we can deduce some information. The hypotheses of the following result are crafted to ensure there are no exotic jumps of filtration in related Adams-Novikov Spectral Sequences. The proof is exactly the same as for [BGH22, Lemma 2.5.11].

Lemma 9.7. *Let $x \in \pi_n E^{h\mathbb{S}_2^1}$. Suppose*

(1) the class x is detected by $\alpha \in H^p(\mathbb{S}_2^1, E_t X)$ in the ANSS, so $n = t - p$;
(2) the class x is detected by $y \in \pi_t(E^{hF_p} \wedge X)$ in the TDSS; and,
(3) the class y is detected by $\beta \in H^0(F_p, E_t X)$ in the HFPSS.

Then α is detected by β in the ADSS.

10. An upper bound for the G_{48}-orientable elements

We continue to examine the subgroup filtration

$$\kappa(\mathbb{G}_2^1) \subseteq \kappa(G_{48}) \subseteq \kappa_2.$$

of κ_2 given in Definition 7.3. In this section we give an upper bound on the filtration quotient $\kappa(G_{48})/\kappa(\mathbb{G}_2^1)$. This group is the group of elements in κ_2 which have an $E^{hG_{48}}$-orientation, but cannot be given an $E^{h\mathbb{G}_2^1}$-orientation. In this section we will show that there is an injective homomorphism $\kappa(G_{48})/\kappa(\mathbb{G}_2^1) \to \mathbb{Z}/2$, hence this group can be at most of order 2. In the next section we will show that this group has order 2, by constructing and examining a non-zero element in $\kappa(G_{48})/\kappa(\mathbb{G}_2^1)$. In some sense, these two sections contain some of the main innovations of this paper, as it is here we depart considerably from the program laid out in [GHMR15] at the prime 3.

Recall from Definition 3.3 that $\kappa(K) \subseteq \kappa_2$ is the subgroup of invertible spectra X with an orientation class $z \in \pi_0(E^{hK} \wedge X)$; then z extends to an equivalence

$\varphi_z : E^{hK} \simeq E^{hK} \wedge X$ of E^{hK}-modules and defines a $\mathbb{G}_2$-invariant generator $\iota_X \in E_0X$. As in Definition 3.27, we introduce a homomorphism

$$\phi_3^1 : \kappa_2 \to H^3(\mathbb{G}_2^1, E_2) \cong \mathbb{Z}/2\{\widetilde{\chi}\eta\} \tag{10.1}$$

by the equation

$$d_3(\iota_X) = \phi_3^1(X)\iota_X$$

where d_3 is the differential in the homotopy fixed point spectral sequence

$$H^s(\mathbb{G}_2^1, E_tX) \Longrightarrow \pi_{t-s}(E^{h\mathbb{G}_2^1} \wedge X).$$

We will prove that $\kappa(\mathbb{G}_2^1)$ is the kernel of the composition

$$\kappa(G_{48}) \xrightarrow{\subseteq} \kappa_2 \xrightarrow{\phi_3^1} H^3(\mathbb{G}_2^1, E_2) \cong \mathbb{Z}/2\{\widetilde{\chi}\eta\}$$

and thus obtain an injective group homomorphism $\kappa(G_{48})/\kappa(\mathbb{G}_2^1) \to \mathbb{Z}/2$.

The main idea is that given $X \in \kappa(G_{48})$, if we can factor the orientation class

$$\begin{tikzcd} & E^{h\mathbb{G}_2^1} \wedge X \arrow[d] \\ S^0 \arrow[r, "z"'] \arrow[ur, dashed] & E^{hG_{48}} \wedge X \end{tikzcd}$$

then we would have $X \in \kappa(\mathbb{G}_2^1)$. Note that $\kappa(\mathbb{G}_2^1) \cong \kappa(\mathbb{S}_2^1)$ by Proposition 3.10, so it is equivalent to lift to $E^{h\mathbb{S}_2^1}$. We will examine the obstructions to this latter lifting the using the Topological Duality resolution.

10.1. **Untwisting the Topological Duality Resolution.**

A key observation for this section is that if $X \in \kappa(G_{48})$ then, the first two pages of the Topological Duality spectral sequences for S^0 and for X agree. The claim for the first page is clear, and the point of this subsection is to prove the statement for the second pages.

The Topological Duality resolution was defined and discussed in Section 9. In the next result, the groups $F_i \subseteq \mathbb{G}_2$ are the finite subgroups that appear in the Duality resolution: $F_0 = F_3 = G_{24}$ and $F_1 = F_2 = C_6$. The resolution itself is

$$E^{h\mathbb{S}_2^1} \longrightarrow E^{hG_{24}} \xrightarrow{d} E^{hC_6} \xrightarrow{d} E^{hC_6} \xrightarrow{d} \Sigma^{48}E^{hG_{24}}\ .$$

To prove our untwisting result, we need some facts about the Hurewicz homomorphism.

Lemma 10.2. *Let $H_1, H_2 \in \{C_6, G_{24}\}$. The Hurewicz map*

$$\pi_tF(E^{hH_1}, E^{hH_2}) \to \mathrm{Hom}_{\mathfrak{Mor}}(E_0E^{hH_1}, E_tE^{hH_2})$$

is

1. *injective if $t \equiv 0$ modulo 48, and*
2. *bijective if $t = 0$.*

Proof. By [GHMR05, Proposition 2.7] we have a commutative diagram

$$\begin{tikzcd}
\pi_t E[\![\mathbb{G}_2/H_1]\!]^{hH_2} \arrow[r] \arrow[d, "\cong"] & (E_t[\![\mathbb{G}_2/H_1]\!])^{H_2} \arrow[d, "\cong"] \\
\pi_t F(E^{hH_1}, E^{hH_2}) \arrow[r] & \mathrm{Hom}_{\mathfrak{Mor}}(E_0 E^{hH_1}, E_t E^{hH_2})
\end{tikzcd}$$

where the horizontal maps are the Hurewicz homomorphism. The top arrow is a limit of Hurewicz maps

$$\pi_t E^{hF} \to H^0(F, E_t) \tag{10.3}$$

for various subgroups $F \subseteq G_{24}$ (see [BG18, Sec. 2], [BBHS20], [Bau08], [DFHH14] and [MR09] for computation of $\pi_* E^{hF}$). The maps (10.3) are all injective under our condition that $t \equiv 0 \mod 48$. Indeed, any class in $\pi_t E^{hF}$ has Adams–Novikov filtration 0 in these cases. Furthermore, when $t = 0$, the Hurewicz maps (10.3) are isomorphisms for $F \subseteq G_{24}$. This proves the claims. □

Now let $X \in \kappa(G_{48})$ and let

$$z \in \pi_0(E^{hG_{48}} \wedge X)$$

be a choice of $E^{hG_{48}}$-orientation. For $0 \le i \le 3$ we get an induced orientation

$$z \in \pi_0(E^{hF_i} \wedge X)$$

and hence an equivalence $\varphi_i : E^{hF_i} \to E^{hF_i} \wedge X$ of E^{hF_i}-modules.

Proposition 10.4. *The following diagram commutes up to homotopy*

$$\begin{tikzcd}
E^{hG_{24}} \arrow[r, "d"] \arrow[d, "\varphi_0"'] & E^{hC_6} \arrow[r, "d"] \arrow[d, "\varphi_1"'] & E^{hC_6} \arrow[r, "d"] \arrow[d, "\varphi_2"] & \Sigma^{48} E^{hG_{24}} \arrow[d, "\Sigma^{48}\varphi_3"] \\
E^{hG_{24}} \wedge X \arrow[r, "d\wedge 1_X"] & E^{hC_6} \wedge X \arrow[r, "d\wedge 1_X"] & E^{hC_6} \wedge X \arrow[r, "d\wedge 1_X"] & \Sigma^{48} E^{hG_{24}} \wedge X.
\end{tikzcd}$$

Proof. By Lemma 10.2 and the fact that $E^{hF} \wedge X \simeq E^{hF}$ for $F = C_6, G_{24}$, we need only check that the diagram commutes after we apply $E_*(-)$. This follows from Proposition 3.22. □

Note we have made no claim about extending Proposition 10.4 to the augmentation from $E^{h\mathbb{S}_2^1} \to E^{hG_{24}}$; indeed, the existence of a map completing the diagram

$$\begin{tikzcd}
E^{h\mathbb{S}_2^1} \arrow[r] \arrow[d, dashed] & E^{hG_{24}} \arrow[d, "\phi_1"] \\
E^{h\mathbb{S}_2^1} \wedge X \arrow[r] & E^{hG_{24}} \wedge X
\end{tikzcd}$$

is equivalent to the assertion that $X \in \kappa(\mathbb{S}_2^1)$.

As an immediate consequence of Proposition 10.4, we have the following conclusion.

Corollary 10.5. *Let $X \in \kappa(G_{48})$ and let $z \in \pi_0(E^{hG_{48}} \wedge X)$ be a choice of an $E^{hG_{48}}$-orientation. Then z determines an isomorphism between the E_2-term of the Topological Duality spectral sequence for S^0 and the E_2-term of the Topological Duality spectral sequence for X.*

10.2. **Analyzing the obstruction.** Let $X \in \kappa(G_{48})$ and let $z \in \pi_0(E^{hG_{48}} \wedge X)$ be an $E^{hG_{48}}$-orientation. If $F \subseteq G_{48}$ is any subgroup, we write $z \in \pi_0(E^{hF} \wedge X)$ for the induced E^{hF}- orientation. If F is the trivial subgroup we will also write $\iota_X \in E_0X$ for the induced $\mathbb{G}_2$-invariant generator.

We now examine the obstructions to lifting $z : S^0 \to E^{hG_{24}} \wedge X$ up the duality resolution tower

$$\begin{tikzcd}
 & E^{h\mathbb{S}_2^1} \wedge X \arrow[d] & \\
 & X_2 \arrow[r] \arrow[d] & \Sigma^{46} E^{hG_{24}} \\
 & X_1 \arrow[r] \arrow[d] & \Sigma^{-1} E^{hC_6} \wedge X \\
S^0 \arrow[r, "z"'] \arrow[ur, dashed] \arrow[uur, dashed] \arrow[uuur, dashed] & E^{hG_{24}} \wedge X \arrow[r] & E^{hC_6} \wedge X \ .
\end{tikzcd}$$

This is equivalent to computing $d_r(z)$ in the Topological Duality spectral sequence

$$E_1^{s,t}(X) = \pi_t E^{hF_s} \wedge X \Longrightarrow \pi_{t-s} E^{h\mathbb{S}_2^1} \wedge X.$$

Note that this is a spectral sequence of $\mathbb{W}$-modules as $\pi_0 E^{h\mathbb{S}_2^1}$ acts on the spectral sequence.

We first record some results of [BGH22] about the Topological Duality spectral sequence for S^0. See Remark 9.1.5, Theorem 9.1.7, and especially Figure 6 of [BGH22], which we also included in Figure 2.

Lemma 10.6 ([BGH22, §9])**.** *In the Topological Duality spectral sequence for S^0, we have*

(1) $E_2^{0,0}(S^0) \cong \mathbb{W}$ generated by a class detecting the unit element of $\pi_0 E^{h\mathbb{S}_2^1}$.
(2) $E_2^{2,0}(S^0) \cong \mathbb{W}/2$ generated by a class $\overline{b}_0$ detecting the class $\widetilde{\chi}$.
(3) $E_2^{2,1}(S^0) \cong \mathbb{W}/2$ generated by class $\overline{b}_0\eta$ detecting the class $\widetilde{\chi}\eta$.
(4) $E_2^{3,2}(S^0) = 0$.

Proposition 10.7. *Let $X \in \kappa(G_{48})$ and let $E_r^{s,t}(X)$ be the Topological Duality spectral sequence*

$$E_1^{s,t}(X) = \pi_t(E^{hF_s} \wedge X) \Longrightarrow \pi_{t-s}(E^{h\mathbb{S}_2^1} \wedge X).$$

Let $z \in \pi_0(E^{hG_{24}} \wedge X) = E_1^{0,0}$ be the induced $E^{hG_{24}}$-orientation. Then

(1) $d_1(z) = 0$,
(2) $d_2(z) \in E_2^{2,1} = \mathbb{W}/2\{\eta\overline{b}_0 z\}$,
(3) if $d_2(z) = 0$, then z is a permanent cycle and $X \in \kappa(\mathbb{G}_2^1) \cong \kappa(\mathbb{S}_2^1)$, and
(4) if $d_2(z) \neq 0$, then $\pi_{-1}(E^{h\mathbb{S}_2^1} \wedge X) = 0$.

Proof. Proposition 10.4 implies that we have an isomorphism

$$E_2^{*,*}(S^0) \cong E_2^{*,*}(X)$$

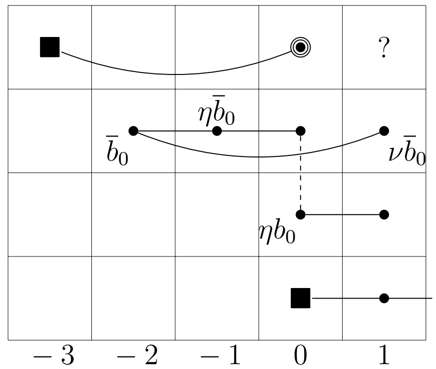

FIGURE 2. The E_2-page of the Topological Duality spectral sequence for S^0. The vertical axis is the Topological Duality spectral sequence filtration s and the horizontal axis is $t-s$. Further, $\blacksquare = \mathbb{W}$, $\bullet = \mathbb{F}_4$, and the circled bullet represents $E_0^{3,3} \cong \mathbb{W}/8$. Horizontal lines are η-multiplications and curved lines are ν-multiplications.

sending a to az. Then (1)–(3) follow from Lemma 10.6 and Figure 2. For point (4) note that $\pi_{-1}(E^{h\mathbb{S}_2^1} \wedge X)$ is a $\pi_0 E^{h\mathbb{S}_2^1}$-module and hence a $\mathbb{W}$-module. The smallest non-zero $\mathbb{W}$ module is $\mathbb{F}_4$; hence if $d_2(z) \neq 0$, we must have $\pi_{-1}(E^{h\mathbb{S}_2^1} \wedge X) = 0$. □

We also have that following result, which we will use in the proof of Proposition 10.9.

Lemma 10.8. *Let $X \in \kappa(G_{48})$. Let $z \in \pi_0(E^{hG_{24}} \wedge X) = E_1^{0,0}$ be the induced $E^{hG_{24}}$-orientation and $\iota_X \in E_0 X$ the induced $\mathbb{G}_2$-invariant generator. Then we have*

$$\pi_{-2} E^{h\mathbb{S}_2^1} \wedge X \cong \mathbb{W}/2.$$

The generator y is detected by

$$\overline{b}_0\, z \in E_2^{0,2}(X)$$

in the Topological Duality spectral sequence and by

$$\widetilde{\chi}\, \iota_X \in H^2(\mathbb{S}_2^1, E_0 X)$$

in the Adams-Novikov spectral sequence.

Proof. All but the last statement follow from Lemma 10.6 and Figure 2. The last statement follows from Lemma 9.7 with $\beta = \overline{b}_0\, z$ and the fact that $\overline{b}_0$ detects $\widetilde{\chi}$ in the Algebraic Duality spectral sequence, as shown in [BGH22, Lemma 5.2.10]. □

Proposition 10.7 defines a function

$$\kappa(G_{48}) \longrightarrow \mathbb{W}/2$$

sending X to $d_2(z)$. Because the Topological Duality spectral sequence does not have good multiplicative properties it is not clear if this function is a homomorphism. We would also like to cut down the image to $\mathbb{Z}/2$. To do all this, we relate it to a differential in the Adams–Novikov spectral sequence.

Proposition 10.9. *Let $X \in \kappa(G_{48})$ and let $z \in \pi_0(E^{hG_{24}} \wedge X)$ be an $E^{hG_{24}}$-orientation induced by an $E^{hG_{48}}$-orientation of X. Let*

$$\iota_X \in H^0(\mathbb{G}_2, E_0X) \subseteq H^0(\mathbb{G}_2^1, E_0X) \subseteq E_0X$$

be the $\mathbb{G}_2$-invariant generator determined by z.

Then $d_2(z) \neq 0$ in the Topological Duality spectral sequence computing $\pi_*(E^{h\mathbb{S}_2^1} \wedge X)$ if and only if

$$d_3(\iota_X) = \widetilde{\chi}\eta\,\iota_X \in H^3(\mathbb{G}_2^1, E_2) \cong \mathbb{Z}/2\{\widetilde{\chi}\eta\iota_X\}$$

in the Adams-Novikov spectral sequence computing $\pi_*(E^{h\mathbb{G}_2^1} \wedge X)$.

Proof. We use Lemma 10.8. Let $y \in \pi_{-2}(E^{h\mathbb{S}_2^1} \wedge X)$ be the class detected by $\bar{b}_0 z$ in the Topological Duality spectral sequence and $\widetilde{\chi}\iota_X$ in the Adams-Novikov spectral sequence. Suppose

$$d_2(z) = \alpha\,\bar{b}_0\eta z \in E_2^{2,1}(X) \cong \mathbb{W}/2\{\bar{b}_0\eta z\}, \qquad 0 \neq \alpha \in \mathbb{W}/2.$$

Then point (4) of Proposition 10.7 gives that

$$y\eta = 0 \in \pi_{-1}(E^{h\mathbb{S}_2^1} \wedge X).$$

However,

$$0 \neq \widetilde{\chi}\eta\iota_X \in H^3(\mathbb{S}_2^1, E_2X) \cong H^3(\mathbb{S}_2^1, E_2).$$

See Figure 1. Because $\widetilde{\chi}\eta\iota_X$ detects $y\eta = 0$, we must have (for degree reasons) that $d_3(\iota_X)$ is a non-zero multiple of $\widetilde{\chi}\eta\iota_X$.

Since we have a map of spectral sequences

$$\begin{tikzcd} H^s(\mathbb{G}_2^1, E_tX) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}E^{h\mathbb{G}_2^1} \wedge X \arrow[d] \\ H^s(\mathbb{S}_2^1, E_tX) \arrow[r, Rightarrow] & \pi_{t-s}E^{h\mathbb{S}_2^1} \wedge X. \end{tikzcd} \tag{10.10}$$

and the map on E_2-terms is an injection onto the Galois invariants we exactly have

$$d_3(\iota_X) = \widetilde{\chi}\eta\,\iota_X$$

as needed.

For the converse, if $d_2(z) = 0$, then $X \in \kappa(\mathbb{G}_2^1)$ by Proposition 10.7 so ι_X is a permanent cycle in the Adams-Novikov spectral sequence for $E^{h\mathbb{G}_2^1} \wedge X$ by Proposition 3.5. □

For the next result, we recall that the homomorphism ϕ_3^1 was defined in (10.1) (*c.f.* Definition 3.27). It is given by the formula

$$d_3(\iota_X) = \phi_3^1(X)\iota_X$$

where $\iota_X \in E_0X$ is any $\mathbb{G}_2$-invariant generator and d_3 is the differential in the spectral sequence

$$H^s(\mathbb{G}_2^1, E_tX) \Longrightarrow \pi_{t-s}E^{h\mathbb{G}_2^1} \wedge X. \tag{10.11}$$

Theorem 10.12. *The composite*

$$\kappa(G_{48}) \to \kappa_2 \xrightarrow{\phi_3^1} H^3(\mathbb{G}_2^1, E_2) \cong \mathbb{Z}/2\{\eta\widetilde{\chi}\}$$

induces an injective homomorphism

$$\kappa(G_{48})/\kappa(\mathbb{G}_2^1) \to \mathbb{Z}/2\ .$$

Proof. Let $X \in \kappa(\mathbb{G}_2^1)$. Then by Proposition 3.5 the class ι_X is a permanent cycle in the spectral sequence of (10.11). Therefore, $\phi_3^1(\iota_X) = 0$ and so $\kappa(\mathbb{G}_2^1)$ is in the kernel of ϕ_3^1.

Now, suppose that $X \in \kappa(G_{48})$ is in the kernel of the composite. Then $d_3(\iota_X) = 0$ in the spectral sequence of (10.11). Then Proposition 10.9 implies that $d_2(z) = 0$ in the Topological Duality spectral sequence for X. By Proposition 10.7, $X \in \kappa(\mathbb{G}_2^1)$. □

11. The G_{48}-orientable elements of the Picard group

In this section we compute $\kappa(G_{48})$, the group of $E^{hG_{48}}$-orientable elements in κ_2. The main result is Corollary 11.23. We begin by finding and analyzing a non-trivial element in $\kappa(G_{48})/\kappa(\mathbb{G}_2^1)$; that is, we will find a spectrum Q which has an $E^{hG_{48}}$-orientation that cannot be refined to an $E^{h\mathbb{G}_2^1}$-orientation. We will use Theorem 10.12; thus we need to find $Q \in \kappa(G_{48})$ so that $d_3(\iota_Q) \neq 0$ in the spectral sequence

$$E_2(\mathbb{G}_2^1, Q) = H^s(\mathbb{G}_2^1, E_tQ) \Longrightarrow \pi_{t-s}(E^{h\mathbb{G}_2^1} \wedge Q).$$

To do this we will use the J-construction as in Section 5 and some classical C_2-equivariant homotopy theory.

Let σ be the one dimensional real sign representation of C_2. We have an isomorphism

$$\mathbb{Z}[\sigma]/(\sigma^2 - 1) \cong RO(C_2)$$

describing the real representation ring of C_2. The augmentation $RO(C_2) \to \mathbb{Z}$ sends a representation to its virtual dimension; the augmentation ideal $I(C_2)$ is of rank 1 over $\mathbb{Z}$ generated by $\sigma - 1$.

Now take $\chi\colon \mathbb{G}_2 \to (\mathbb{Z}_2/4)^\times \cong C_2$ to be the surjective homomorphism of Definition 4.4, whose kernel we called $\mathbb{G}_2^0$. Given any virtual representation $V \in RO(C_2)$, we get an action of $\mathbb{G}_2$ on S^V by restriction along χ. Recall from Example 5.2 and Proposition 5.11 that we get a homomorphism

$$J : RO(C_2) \to \mathrm{Pic}_2,$$

defined by the formula

$$J(V) = (E \wedge S^V)^{h\mathbb{G}_2}.$$

This uses the diagonal action of $\mathbb{G}_2$ on the smash product.

Applying Proposition 5.6, we have that if $K \subseteq \mathbb{G}_2$ is a closed subgroup, then

$$E^{hK} \wedge J(V) = E^{hK} \wedge (E \wedge S^V)^{h\mathbb{G}_2} \simeq (E \wedge S^V)^{hK},$$

so that in particular, if K is in the kernel of χ, then we obtain an equivalence

$$E^{hK} \wedge J(V) \simeq \Sigma^{\dim V} E^{hK}.$$

Let $V \in I(C_2)$ and let $j = j_V \in \pi_0 S^V \cong \mathbb{Z}$ a chosen generator. Define $\iota = \iota_V \in E_0 S^V$ to the the image of j under the Hurewicz map. We have an isomorphism of Morava modules

$$\pi_*(E \wedge J(V)) \cong \pi_*(E \wedge S^V) \cong E_* \otimes_{E_0} E_0 S^V$$

where $\mathbb{G}_2$ acts diagonally on $E_* \otimes_{E_0} E_0 S^V$. In particular, if $V \in I(C_2)^2 = (2(\sigma - 1))$, ι_V is a $\mathbb{G}_2$-invariant generator and $J(V) \in \kappa_2$. Compare Remark 5.7.

Lemma 11.1. *If $V \in I(C_2)$, then there is an equivalence of $E^{hG_{48}}$-modules*

$$E^{hG_{48}} \simeq E^{hG_{48}} \wedge J(V).$$

If $V \in I(C_2)^2$, then $J(V) \in \kappa(G_{48})$.

Proof. Note that $G_{24} \subseteq \mathbb{S}_2$ is in the kernel of the determinant (Definition 4.1), since there are no non-trivial group homomorphisms $G_{24} \to \mathbb{Z}_2^\times$. (See, for example, Remark 5.1.6 of [BGH22] for this last fact.) Then G_{48} is in the kernel of χ, and the first statement follows. The second statement follows from the first by Lemma 3.8, since ι_V gives $E_* J(V)$ a $\mathbb{G}_2$-invariant generator by the above discussion. □

Now we come to the star of this section.

Definition 11.2. Let $Q \in \kappa(\mathbb{G}_2^1)$ be the spectrum

$$Q = J(2\sigma - 2) = (E \wedge S^{2\sigma - 2})^{h\mathbb{G}_2} .$$

By Lemma 11.1, we know that Q is an element of $\kappa(G_{48})$, and we will show that its class in the quotient $\kappa(G_{48})/\kappa(\mathbb{G}_2^1)$ is non-trivial. We will also make an analysis of $2Q \simeq J(4\sigma - 4)$ for use in solving extension problems.

For any virtual C_2 representation V, there is a C_2-equivariant map

$$S^V \longrightarrow E^{h\mathbb{G}_2^0} \wedge S^V,$$

hence a map

$$(S^V)^{hC_2} \longrightarrow (E^{h\mathbb{G}_2^0} \wedge S^V)^{hC_2} \simeq J(V).$$

In the next subsection, we will use standard methods in equivariant stable homotopy theory to give a partial analysis of $(S^V)^{hC_2}$, which will help us analyze Q and apply Theorem 10.12.

11.1. **Some C_2-homotopy theory for representation spheres.** In this section we write down some classical homotopy theory for the homotopy fixed points $(S^V)^{hC_2}$, where V is a virtual representation of the cyclic group of order 2.

There is a small and geometric model for EC_2; namely, $S^\infty = \cup S^m$ with the antipodal action. In a coincidence forced on us by the fact that C_2 is a very small group, S^∞ is equivariantly homeomorphic to the geometric realization of the standard simplicial model for EC_2 obtained from the bar construction. Thus for any C_2-spectrum X, its homotopy fixed points can be expressed as

$$X^{hC_2} \simeq \mathrm{Tot}(F(C_2^{\bullet+1}, X)) \simeq F(\Sigma_+ S^\infty, X).$$

Indeed, for each m there is an equivalence $\mathrm{Tot}_m(F(C_2^{\bullet+1}, X)) \simeq F_{C_2}(\Sigma^\infty_+ S^m, X)$. We could use either of these descriptions to construct the homotopy fixed point spectral sequence

$$H^s(C_2, \pi_t X) \Longrightarrow \pi_{t-s} X^{hC_2},$$

as the Bousfield-Kan spectral sequence of the respective towers giving X^{hC_2} as a homotopy limit

$$X^{hC_2} \simeq \mathrm{holim}\, \mathrm{Tot}_m(F(C_2^{\bullet+1}, X)) \simeq \mathrm{holim}_m F_{C_2}(\Sigma^\infty_+ S^m, X).$$

If V is a virtual C_2-representation, then S^V is dualizable with Spanier-Whitehead dual S^{-V}; thus we have

$$(S^V)^{hC_2} \simeq \mathrm{holim}_m F(\Sigma^\infty_+ S^m \wedge_{C_2} S^{-V}, S^0).$$

From this we get that the basic objects of study are the spectra

$$\Sigma^\infty_+ S^m \wedge_{C_2} S^{n\sigma}, \quad n, m \in \mathbb{Z},\ m \geq 0$$

and their Spanier-Whitehead duals. As above, σ is the sign representation.

In the following we will confuse the (pointed) projective spaces $\mathbb{R}\mathrm{P}^n$ with their suspension spectra. We will do this throughout, letting context indicate whether we are working with the space or the spectrum.

Let ξ be the tautological line bundle over $\mathbb{R}\mathrm{P}^\infty$ and let T denote the Thom spectrum functor; we arrange our conventions for this functor so that if γ is a bundle of virtual dimension k, then the Thom class is in $H^k(T(\gamma), \mathbb{F}_2)$. By [Hus94, Theorem 1.8 of Chapter 16], we have that for all $n \in \mathbb{Z}$, the Thom spectrum of $n\xi$ is identified as

$$\Sigma^\infty_+ S^\infty \wedge_{C_2} S^{n\sigma} \simeq T(n\xi) \simeq \mathbb{R}\mathrm{P}^\infty_n. \tag{11.3}$$

Here, if $n \geq 0$, then $\mathbb{R}\mathrm{P}^\infty_n = \mathbb{R}\mathrm{P}^\infty / \mathbb{R}\mathrm{P}^{n-1}$ is the (suspension spectrum of) truncated projective space; if $n < 0$, this formula is the definition of $\mathbb{R}\mathrm{P}^\infty_n$. Note that if $n = 0$, $\mathbb{R}\mathrm{P}^\infty_0 \simeq \Sigma^\infty_+ \mathbb{R}\mathrm{P}^\infty$, while for $n = 1$ we have $\mathbb{R}\mathrm{P}^\infty_1 \simeq \Sigma^\infty \mathbb{R}\mathrm{P}^\infty$, which we write as $\mathbb{R}\mathrm{P}^\infty$ according to the convention above.

If ξ_m denotes the restriction of ξ to $\mathbb{R}\mathrm{P}^m$, then

$$\Sigma^\infty_+ S^m \wedge_{C_2} S^{n\sigma} \simeq T(n\xi_m) \simeq \mathbb{R}\mathrm{P}^{m+n}_n, \tag{11.4}$$

where again this formula may be needed to define the truncated projective space. Note again that $H_k(\mathbb{R}\mathrm{P}^{m+n}_n, \mathbb{F}_2) \neq 0$ only for $n \leq k \leq m+n$. Colloquially, we say that this spectrum has bottom cell in dimension n and top cell in dimension $m+n$.

The virtual bundle $\xi_m - 1$ on $\mathbb{R}\mathrm{P}^m$ has finite order; indeed, this order is a number c_m related to the Adams vector field number. The original source for this result is [Ada62], but it can be found conveniently in [Hus94, Chapter 16, Theorem 12.7], with an explicit formula in [Hus94, Chapter 16, Remark 11.1]. This gives *James Periodicity* for truncated projective spaces

$$\Sigma^{c_m} \mathbb{R}\mathrm{P}^{m+n}_n \simeq \mathbb{R}\mathrm{P}^{c_m+m+n}_{c_m+n}. \tag{11.5}$$

In particular, the following classical result will help calculate differentials in the C_2-homotopy fixed point spectral sequence of a suitable representation sphere.

Lemma 11.6. *For all n, there are equivalences of spectra*

$$\Sigma^4\mathbb{R}\mathrm{P}_n^{n+2} \simeq \mathbb{R}\mathrm{P}_{n+4}^{n+6}, \qquad \Sigma^4\mathbb{R}\mathrm{P}_n^{n+3} \simeq \mathbb{R}\mathrm{P}_{n+4}^{n+7}, \qquad \textit{and} \qquad \Sigma^8\mathbb{R}\mathrm{P}_n^{n+7} \simeq \mathbb{R}\mathrm{P}_{n+8}^{n+15}.$$

In addition, we have an equivalence of spectra

$$\mathbb{R}\mathrm{P}^3 \simeq \mathbb{R}\mathrm{P}^2 \vee S^3.$$

Proof. By [Hus94, Chapter 16, Theorem 12.7] the virtual bundles $\xi_2 - 1$ on $\mathbb{R}\mathrm{P}^2$, as well as $\xi_3 - 1$ on $\mathbb{R}\mathrm{P}^3$ have order $c_2 = 4 = c_3$, while $\xi_7 - 1$ on $\mathbb{R}\mathrm{P}^7$ has order $c_m = 8$. So, the first three statements follow from James Periodicity (11.5). For the last statement, note that $\mathbb{R}\mathrm{P}^3$ is the manifold underlying the Lie group $SO(3)$; hence $\mathbb{R}\mathrm{P}^3$ is stably parallelizable and stably the top cell splits off. That $\mathbb{R}\mathrm{P}^3$ is stably parallelizable could also be proved using the fact that the stable tangent bundle of $\mathbb{R}\mathrm{P}^3$ is isomorphic to $4\xi_3$, hence trivial. $\square$

We will need the following formula for the Spanier-Whitehead dual of $\mathbb{R}\mathrm{P}_n^k$

$$D\mathbb{R}\mathrm{P}_n^k = F(\mathbb{R}\mathrm{P}_n^k, S^0) \simeq \Sigma\mathbb{R}\mathrm{P}_{-k-1}^{-n-1}, \tag{11.7}$$

which can found as [Ati61, Theorem 6.1], or can easily be proved given what we've said so far. Indeed, if $n = 0$ this follows from Atiyah Duality and the fact that the stable normal bundle of $\mathbb{R}\mathrm{P}^k$ is $-(k+1)\xi + 1$. For general n and k we use James Periodicity, as in (11.5).

The above material may be assembled to give a formula for the homotopy fixed points. We will only need it for $n = 0$, 2, and 4, but the result is easy to state for all n.

Proposition 11.8. *Let $n \in \mathbb{Z}$. Then the* Tot*-tower of the cosimplicial spectrum $F_{C_2}(C_2^{\bullet+1}, S^{n\sigma})$ is equivalent to the tower of fibrations*

$$\begin{tikzcd}
\cdots \arrow[r] & \Sigma\mathbb{R}\mathrm{P}_{n-m-1}^{n-1} \arrow[r] & \Sigma\mathbb{R}\mathrm{P}_{n-m}^{n-1} \arrow[r] & \cdots \arrow[r] & \Sigma\mathbb{R}\mathrm{P}_{n-2}^{n-1} \arrow[r] & \Sigma\mathbb{R}\mathrm{P}_{n-1}^{n-1} \\
& S^{n-m} \arrow[u] & S^{n-m+1} \arrow[u] & & S^{n-1} \arrow[u] & S^n \arrow[u, "\simeq"]
\end{tikzcd}$$

where the maps from the successive fibers are given by the inclusion of the bottom cell. We have

$$\mathrm{Tot}_m\, F_{C_2}(C_2^{\bullet+1}, S^{n\sigma}) \simeq \Sigma\mathbb{R}\mathrm{P}_{n-m-1}^{n-1}$$

and there is an equivalence

$$(S^{n\sigma})^{hC_2} \simeq \mathrm{holim}_i\, \Sigma\mathbb{R}\mathrm{P}_{-i}^{n-1}\ .$$

Proof. By the discussion at the beginning of this subsection, we have an equivalence

$$\mathrm{Tot}_m(F(C_2^{\bullet+1}, S^{n\sigma})) \simeq F_{C_2}(\Sigma_+^\infty S^m, S^{n\sigma}).$$

We can analyze the right-hand side using (11.4) to obtain

$$F_{C_2}(\Sigma_+^\infty S^m, S^{n\sigma}) \simeq F(\Sigma_+^\infty S^m \wedge_{C_2} S^{-n\sigma}, S^0) \simeq F(\mathbb{R}\mathrm{P}_{-n}^{m-n}, S^0),$$

while (11.7) identifies this with $\Sigma\mathbb{R}\mathrm{P}_{n-m-1}^{n-1}$ as needed. $\square$

Remark 11.9. So far these formulas have only been for multiples of the sign representation, but we have $S^{n\sigma+k} = S^{n\sigma} \wedge S^k$ with the trivial action on S^k. This forces a shift into the Tot-tower; for example, we have

$$(S^{n\sigma+k})^{hC_2} \simeq \mathrm{holim}_i\, \Sigma^{1+k}\mathbb{R}\mathrm{P}^{n-1}_{-i}\ .$$

Now let n be even and we implicitly 2-complete all spectra.

The requirement that n be even implies that the action of C_2 on $H_n(S^{n\sigma}, \mathbb{Z}_2)$ is trivial, so the action on $\pi_* S^{n\sigma} \cong \pi_* S^n$ is also trivial. Choose a generator

$$j_n \in H^0(C_2, \pi_n S^{n\sigma}) \cong \mathbb{Z}_2.$$

We are interested in the fate of this class in the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(C_2, \pi_t S^{n\sigma}) \Longrightarrow \pi_{t-s}(S^{n\sigma})^{hC_2}. \tag{11.10}$$

This is the spectral sequence for the tower of fibrations in Proposition 11.8. Since the class j_n is represented by the generator of

$$\pi_n \Sigma\mathbb{R}\mathrm{P}^{n-1}_{n-1} \cong \pi_n S^n \cong \mathbb{Z}_2,$$

differentials on it come down to whether or not the projection map

$$\Sigma\mathbb{R}\mathrm{P}^{n-1}_{n-m-1} \to S^n$$

to the top cell has a splitting.

We now give a calculation of the E_2-term of (11.10) in a range. Since we are assuming n is even, the E_2-term is a free module of rank one on $j_n \in H^0(C_2, \pi_n S^{n\sigma})$ over the ring $H^*(C_2, \pi_* S^0)$. Let $\eta \in \pi_1 S^0$ and $\nu \in \pi_3 S^0$ be the standard generators and let $h \in H^1(C_2, \mathbb{Z}/2)$ and $g \in H^2(C_2, \mathbb{Z}_2)$ be the generators of the group cohomology rings. Then g is the Bockstein on h, and reduces to $h^2 \in H^2(C_2, \mathbb{Z}/2)$. Finally, let $\alpha = h\eta \in H^1(C_2, \pi_1 S^0)$, and $\beta = h\nu^2 \in H^1(C_2, \pi_3 S^0)$. The following is now a standard calculation. Compare Figure 3.

Proposition 11.11. *There is a map of bigraded rings*

$$\varphi\colon \mathbb{Z}_2[g, \eta, \alpha, \nu, \beta] \longrightarrow H^*(C_2, \pi_* S^0),$$

where the (s,t)-bidegrees of the generators are

$$|g| = (2,0) \quad |\eta| = (0,1) \quad |\alpha| = (1,1) \quad |\nu| = (0,3) \quad |\beta| = (1,6).$$

The ideal

$$I = (2\eta,\ 8\nu,\ 2\nu^2,\ \eta\nu,\ \eta^3 - 4\nu,\ 2\alpha,\ 2g,\ \eta^2 g - \alpha^2,\ \nu\alpha,\ 2\beta)$$

is in the kernel of φ. The map

$$\mathbb{Z}_2[g, \eta, \alpha, \nu, \beta]/I \longrightarrow H^*(C_2, \pi_* S^0)$$

induced by φ is an isomorphism in bidegrees (s,t) with $t \leq 6$.

We now come to our first calculations with (11.10), which will be key to Theorem 11.17 and Theorem 11.21.

The following proposition will be useful because the homotopy fixed point spectral sequences for C_2 acting on S^V for any representation V are modules over the spectral sequence for the C_2 acting trivially on S^0.

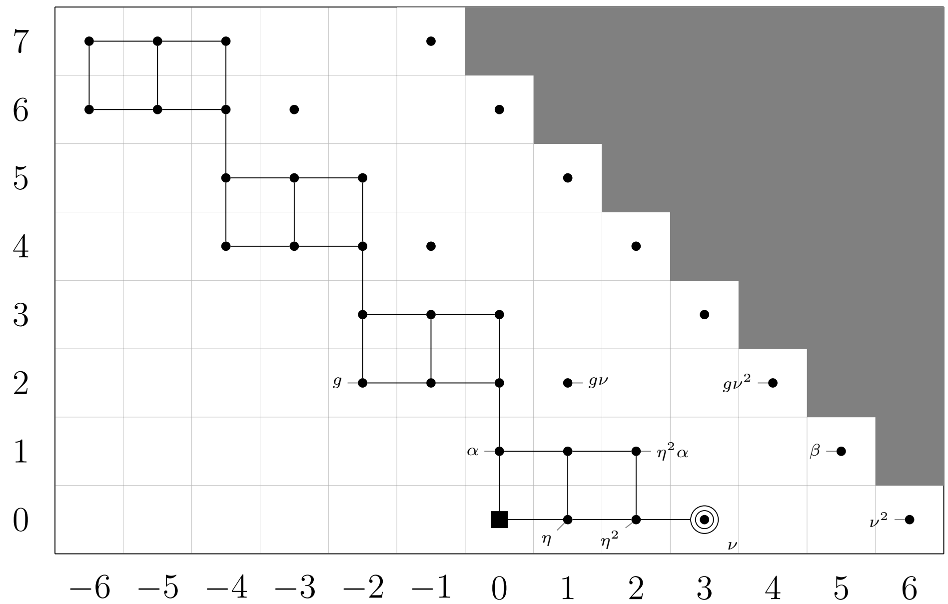

FIGURE 3. The cohomology ring $H^s(C_2, \pi_t S^0)$ in a range. The vertical axis is s and the horizontal axis is $t-s$.

Proposition 11.12. *Let $n \equiv 0 \mod 8$. Then in the homotopy fixed point spectral sequence*

$$E_2^{s,t} = H^s(C_2, \pi_t S^{n\sigma}) \Longrightarrow \pi_{t-s}(S^{n\sigma})^{hC_2}$$

we have $d_r(g^2 j_n) = 0$ for $r \leq 3$.

Proof. Write $n = 8k$. We use the tower of Proposition 11.8. The class $g^2 j_n$ is the residue class of the bottom cell of $\Sigma\mathbb{R}\mathrm{P}^{8k-1}_{8k-5}$ and we are asking if it is in the image of the quotient map

$$\Sigma\mathbb{R}\mathrm{P}^{8k-1}_{8k-8} \longrightarrow \Sigma\mathbb{R}\mathrm{P}^{8k-1}_{8k-5}.$$

From Lemma 11.6 we have that this map is a (de-)suspension of the quotient map

$$\mathbb{R}\mathrm{P}^7_0 \longrightarrow \Sigma\mathbb{R}\mathrm{P}^7_3.$$

If $S^3 \to \mathbb{R}\mathrm{P}^3$ splits off the top the cell then the composition

$$S^3 \longrightarrow \mathbb{R}\mathrm{P}^3 \longrightarrow \mathbb{R}\mathrm{P}^7_0 \longrightarrow \Sigma\mathbb{R}\mathrm{P}^7_3.$$

is non-zero in homology, as needed. □

Proposition 11.13. *Let $n \equiv 2$ modulo 4. Then in the homotopy fixed point spectral sequence*

$$E_2^{s,t} = H^s(C_2, \pi_t S^{n\sigma}) \Longrightarrow \pi_{t-s}(S^{n\sigma})^{hC_2}$$

we have differentials

$$d_2(j_n) = \eta g j_n \quad \text{and} \quad d_2(g j_n) = 0.$$

Proof. Write $n = 4k+2$. We can analyze these differentials as obstructions to lifting along the tower in Proposition 11.8. Note that we have a splitting $\mathbb{RP}^{4k+1}_{4k} \simeq S^{4k} \vee S^{4k+1}$, which gives a lift of j_n as a map $S^{4k+2} \to \Sigma\mathbb{RP}^{4k+1}_{4k}$, and $d_2(j_n)$ is the obstruction to lifting this further to $\Sigma\mathbb{RP}^{4k+1}_{4k-1}$. Now note that

$$0 \neq \mathrm{Sq}^2 : H^{4k-1}(\mathbb{RP}^{4k+1}_{4k-1}, \mathbb{F}_2) \longrightarrow H^{4k+1}(\mathbb{RP}^{4k+1}_{4k-1}, \mathbb{F}_2).$$

This implies that

$$\pi_{4k+2}\Sigma\mathbb{RP}^{4k+1}_{4k-1} \longrightarrow \pi_{4k+2}\Sigma\mathbb{RP}^{4k+1}_{4k+1}$$

is not surjective, since the non-trivial Sq^2 implies that the top cell of $\mathbb{RP}^{4k+1}_{4k-1}$ does not split off. This gives $d_2(j_n) \neq 0$ and, by Proposition 11.11 there is only one non-zero class in $H^2(C_2, \pi_{n+1}S^{n\sigma})$.

For the second statement, let $x : S^{4k} \to \Sigma\mathbb{RP}^{4k+1}_{4k-1}$ be the inclusion of the bottom cell. Then we are asking if x factors through $\Sigma\mathbb{RP}^{4k+1}_{4k-3}$. To see that it does, note that the composition

$$\mathbb{RP}^{4k-1}_{4k-3} \to \mathbb{RP}^{4k+1}_{4k-3} \to \mathbb{RP}^{4k+1}_{4k-1}$$

is non-zero in $H_{4k-1}(-, \mathbb{F}_2)$. Now use that

$$\mathbb{RP}^{4k-1}_{4k-3} \simeq \Sigma^{4k-4}\mathbb{RP}^3 \simeq S^{4k-1} \vee \Sigma^{4k-4}\mathbb{RP}^2$$

to see that x can be factored as the resulting splitting $S^{4k} \to \Sigma\mathbb{RP}^{4k-1}_{4k-3}$ followed by the inclusion $\Sigma\mathbb{RP}^{4k-1}_{4k-3} \to \Sigma\mathbb{RP}^{4k+1}_{4k-3}$. $\square$

By a similar method, we get the following result, crucial to Theorem 11.21.

Proposition 11.14. *Let $n \equiv 4$ modulo 8. Then in the homotopy fixed point spectral sequence*

$$H^s(C_2, \pi_t S^{n\sigma}) \Longrightarrow \pi_{t-s}(S^{n\sigma})^{hC_2}$$

we have $d_r(j_n) = 0$ for $r \leq 3$ and

$$0 \neq d_4(j_n) = \nu g^2 j_n.$$

Moreover, $d_r(g^2 j_n) = 0$ for $r \leq 3$.

Proof. Write $n = 8k+4$. Proving that $d_r(j_n) = 0$ for $r \leq 3$ amounts to lifting the bottom cell $S^{8k+4} \to \Sigma\mathbb{RP}^{8k+3}_{8k+3}$ to $\Sigma\mathbb{RP}^{8k+3}_{8k}$. But by Lemma 11.6, we have

$$\mathbb{RP}^{8k+3}_{8k} \simeq \Sigma^{8k}\mathbb{RP}^3_0 \simeq S^{8k} \vee \Sigma^{8k}\mathbb{RP}^2 \vee S^{8k+3},$$

so there are no obstruction to the needed lift. For the d_4 calculation, notice that

$$0 \neq \mathrm{Sq}^4 : H^{8k-1}(\mathbb{RP}^{8k+3}_{8k-1}, \mathbb{F}_2) \longrightarrow H^{8k+3}(\mathbb{RP}^{8k+3}_{8k-1}, \mathbb{F}_2).$$

It follows

$$\pi_{8k+4}\Sigma\mathbb{RP}^{8k+3}_{8k} \longrightarrow \pi_{8k+4}\Sigma\mathbb{RP}^{8k+3}_{8k+3} \cong \mathbb{Z}_2$$

is surjective but

$$\pi_{8k+4}\Sigma\mathbb{RP}^{8k+4}_{8k-1} \longrightarrow \pi_{8k+4}\Sigma\mathbb{RP}^{8k+3}_{8k+3}$$

is not, implying that $d_4(j_n) \neq 0$. By Proposition 11.11

$$\nu g^2 j_n \in H^4(C_2, \pi_{n+3}S^{n\sigma})$$

is the only non-zero class. Since $d_4(j_n) \neq 0$, the class $g^2\nu j_n$ cannot be the image of d_2 or d_3 and the result follows.

To see that $d_r(g^2 j_n) = 0$ for $r \leq 3$, note from Proposition 11.12 that g^2 is a d_3-cycle in the homotopy fixed point spectral sequence for the trivial C_2-action on S^0. Using the module structure over that spectral sequence, we get the claim. □

11.2. **A non-trivial element in $\kappa_2(G_{48})/\kappa(\mathbb{G}_2^1)$.** After the above interlude on stunted projective spaces and the calculation of differentials in some classical C_2-homotopy fixed points, we are ready to return to the study of our invertible spectrum Q from Definition 11.2.

We set the stage for the proof of Theorem 11.17, to connect it to the material in Section 11.1. We will use the same set-up for Theorem 11.21 as well.

Let $V \in I(C_2)$. In Remark 5.13 we used the map $S^V \to E \wedge S^V$ to construct a map of augmented cosimplicial spectra

$$\begin{tikzcd} S^V \arrow[r] \arrow[d] & F(C_2^{\bullet+1}, S^V)^{C_2} \arrow[d] \\ E \wedge S^V \arrow[r] & F_c(\mathbb{G}_2^{\bullet+1}, E \wedge S^V)^{\mathbb{G}_2} \end{tikzcd} \tag{11.15}$$

Suppose $V = n\sigma - n$. Let $Y_m = \mathrm{Tot}_m\, F_c(\mathbb{G}_2^{\bullet+1}, E \wedge S^V)^{\mathbb{G}_2}$. Then, by Proposition 11.8 and Remark 11.9 we have a commutative diagram of towers

$$\begin{tikzcd} \Sigma^{1-n}\mathbb{R}\mathrm{P}^{n-1}_{n-m-1} \arrow[r] \arrow[d] & Y_m \arrow[d] \\ \Sigma^{1-n}\mathbb{R}\mathrm{P}^{n-1}_{n-m} \arrow[r] & Y_{m-1} \end{tikzcd},$$

which gives the diagram of spectral sequences

$$\begin{tikzcd} E_2^{s,t}(C_2, S^V) = H^s(C_2, \pi_t S^V) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}(S^V)^{hC_2} \arrow[d] \\ E_2^{s,t}(\mathbb{G}_2, E \wedge S^V) = H^s(\mathbb{G}_2, E_t S^V) \arrow[r, Rightarrow] & \pi_{t-s} J(V). \end{tikzcd} \tag{11.16}$$

Now suppose $V \in I(C_2)^2$ and fix a choice of C_2-invariant generator $j_V \in \pi_0 S^V$, which defines a $\mathbb{G}_2$-invariant generator $\iota_V \in E_0 S^V$. Then we have a commutative diagram in group cohomology

$$\begin{tikzcd} H^*(C_2, \pi_* S^0) \arrow[r] \arrow[d, "(j_V)_*"', "\cong"] & H^*(\mathbb{G}_2, E_*) \arrow[d, "(\iota_V)_*"', "\cong"] \\ H^*(C_2, \pi_* S^V) \arrow[r] & H^*(\mathbb{G}_2, E_* S^V). \end{tikzcd}$$

Note that the bottom spectral sequence is isomorphic to the Adams–Novikov spectral sequence for $J(V)$, by Lemma 5.4. We will be interested in the faith of ι_V.

To be specific, recall the map

$$\phi_3^1 : \kappa(G_{48}) \to H^3(\mathbb{G}_2^1, E_2) \cong \mathbb{Z}/2$$

from (10.1) (*c.f.* Definition 3.27). For the element $Q = J(2\sigma - 2)$ from Definition 11.2 of $\kappa_{G_{48}}$, we can understand $\phi_3^1(Q)$ by first analyzing the faith of $\iota_{2\sigma-2}$

in the homotopy fixed point spectral sequence $E_r^{*,*}$ through (11.16). Then we can restrict to the Adams–Novikov spectral sequence for $E^{h\mathbb{G}_2^1} \wedge Q$, which is the $\mathbb{G}_2^1$ homotopy fixed point spectral sequence for $E \wedge S^{2\sigma-2}$ and study the outcome.

In the next subsection, a similar argument will determine that $2Q$, equivalent to $J(4\sigma-4)$ by Proposition 5.10, is a trivial element in the Picard group. In that case, we will be checking that $\iota_{4\sigma-4}$ cannot support differentials.

In Theorem 10.12, we proved that ϕ_3^1 gives an injection

$$\kappa(G_{48})/\kappa(\mathbb{G}_2^1) \longrightarrow \mathbb{Z}/2,$$

and now we are ready to prove surjectivity.

Theorem 11.17. *We have*

$$0 \neq \phi_3^1(Q) \in H^3(\mathbb{G}_2^1, E_2) \cong \mathbb{Z}/2$$

and hence a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \kappa(\mathbb{G}_2^1) \arrow[r] & \kappa(G_{48}) \arrow[r, "\phi_3^1"] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

Proof of Theorem 11.17. Let $V = 2\sigma - 2$. We will show that in the spectral sequence $E_r^{s,t}(\mathbb{G}_2, E \wedge S^V)$, there is a non-trivial differential

$$d_3(\iota_V) = \eta\widetilde{\chi}\iota_V.$$

Because the restriction of $\eta\widetilde{\chi}\iota_V$ is non-zero in $H^*(\mathbb{G}_2^1, E_*S^V)$, this forces the same differential in the homotopy fixed point spectral sequence $E_r^{s,t}(\mathbb{G}_2^1, E \wedge S^V)$. The result then follows from Theorem 10.12.

From here on, we write $j = j_V$, $\iota = \iota_V$, we abbreviate $E_r^{s,t}(S^V) = E_r^{s,t}(C_2, S^V)$ and $E_r^{s,t}(E \wedge S^V) = E_r^{s,t}(\mathbb{G}_2, E \wedge S^V)$, and we refer to Proposition 11.11 for the definitions of elements in $H^*(C_2, \pi_*S^V)$.

We apply the truncation construction of Section 6 to the map of towers (11.15), and let

$$f_* \colon E_{*,\leq 3}^{*,*}(S^V) \to E_{*,\leq 3}^{*,*}(E \wedge S^V)$$

be the resulting map of truncated spectral sequences. We truncate up to 3 since we are interested in $d_3(\iota)$. Then we have a commutative diagram

$$\begin{tikzcd} H^s(C_2, \pi_t S^V) \arrow[r] \arrow[d] & H^s(\mathbb{G}_2, E_t S^V) \arrow[d] \\ E_{2,\leq 3}^{s,t}(S^V) \arrow[r, "f_*"] & E_{2,\leq 3}^{s,t}(E \wedge S^V), \end{tikzcd} \tag{11.18}$$

where the vertical maps are injections for $s \leq 3$ and isomorphisms for $s < 3$. See Lemma 6.6. The truncated spectral sequence $E_{*,\leq 3}^{*,*}(S^V)$ converges to $\pi_*\Sigma^{-1}\mathbb{R}\mathrm{P}_{-2}^1$ by Proposition 11.8 and Remark 11.9.

From Proposition 11.13 we get that gj is a permanent cycle in the truncated spectral sequence

$$E_{2,\leq 3}^{*,*}(S^V) \Longrightarrow \pi_{t-s}\Sigma^{-1}\mathbb{R}\mathrm{P}_{-2}^1$$

detecting a non-zero element of $\pi_{-2}\Sigma^{-1}\mathbb{R}\mathrm{P}^1_{-2}$ which we will denote by $[gj]$. Furthermore, also by Proposition 11.13 we have the differential

$$d_2(j) = \eta g j$$

and so we conclude that $\eta[gj] = 0$ in $\pi_{-1}\Sigma^{-1}\mathbb{R}\mathrm{P}^1_{-2}$.

We now turn to some corresponding elements in the truncated spectral sequence

$$E^{*,*}_{2,\leq 3}(E \wedge S^V) \Longrightarrow \pi_{t-s}\operatorname{Tot}_3(Q).$$

Referring to (11.18) we see that on E_2-terms, $f_*(gj)$ is the class

$$\widetilde{\chi}\iota \in H^2(\mathbb{G}_2, E_0 S^V),$$

which is immediate from the definition of the class $\widetilde{\chi} \in H^2(\mathbb{G}_2, E_0) \cong H^2(\mathbb{G}_2, E_0 S^V)$ as the Bockstein on χ; see Definition 4.4 or [BBG+22, (1.7)]. Now, the class $\widetilde{\chi}\iota \in E^{*,*}_{r,\leq 3}(E \wedge S^V)$ is not hit by a differential for degree reasons; see [BBG+22, Table 3]. Therefore, it detects a non-trivial class in $[\widetilde{\chi}\iota] \in \pi_{-2}\operatorname{Tot}_3(Q)$.

Since $\eta[gj] = 0$ we must have $\eta[\widetilde{\chi}\iota] = 0$. But $\eta[\widetilde{\chi}\iota]$ is detected by

$$\eta\widetilde{\chi}\iota \in H^3(\mathbb{G}_2, E_2 S^V) \cong H^3(\mathbb{G}_2, E_2)$$

again by [BBG+22, §7], or Figure 1. So, $\eta\widetilde{\chi}\iota$ must be killed by a differential and $d_3(\iota) = \eta\widetilde{\chi}\iota$ is the only possibility. □

11.3. **The group** $\kappa(G_{48})$**.** In this section we apply the same type of analysis as in the previous section in order to prove that $Q \in \kappa(G_{48})$ has order 2. This will imply a splitting

$$\kappa(G_{48}) \cong \kappa(\mathbb{G}_2^1) \oplus \mathbb{Z}/2.$$

While the idea of the argument is essentially the same as for Theorem 11.17, it is somewhat harder to accomplish as there are longer differentials to keep track of.

We start with the diagram of augmented cosimplicial spectra from (11.15), with $V = 4\sigma - 4$. Again we abbreviate

$$\begin{aligned} j &= j_{4\sigma-4} \in H^0(C_2, \pi_0 S^{4\sigma-4}) \\ \iota &= \iota_{4\sigma-4} \in H^0(\mathbb{G}_2, E_0 S^{4\sigma-4}). \end{aligned}$$

The goal is to show that ι is a d_7-cycle in the $\mathbb{G}_2$ spectral sequence. Recalling Definition 3.27, this would imply that $2Q$ is an element of $\kappa_{2,8}$, so altogether in $\kappa_8(\mathbb{G}_2^1)$, which is trivial by Theorem 8.16.

The first step is an examination of some useful truncations of the C_2-fixed point spectral sequence of $S^{4\sigma-4}$, whose outcome is Lemma 11.20. Since we are ultimately interested in $d_r(\iota)$ for $r \leq 7$, we will study the 7-truncation. By Proposition 11.14 we know that in the homotopy fixed point spectral sequence

$$H^s(C_2, \pi_t S^{4\sigma-4}) \Longrightarrow \pi_{t-s}(S^{4\sigma-4})^{hC_2}$$

we have $d_4(j) = \nu g^2 j$; we will study a relative truncation in order to analyze the implications of this differential.

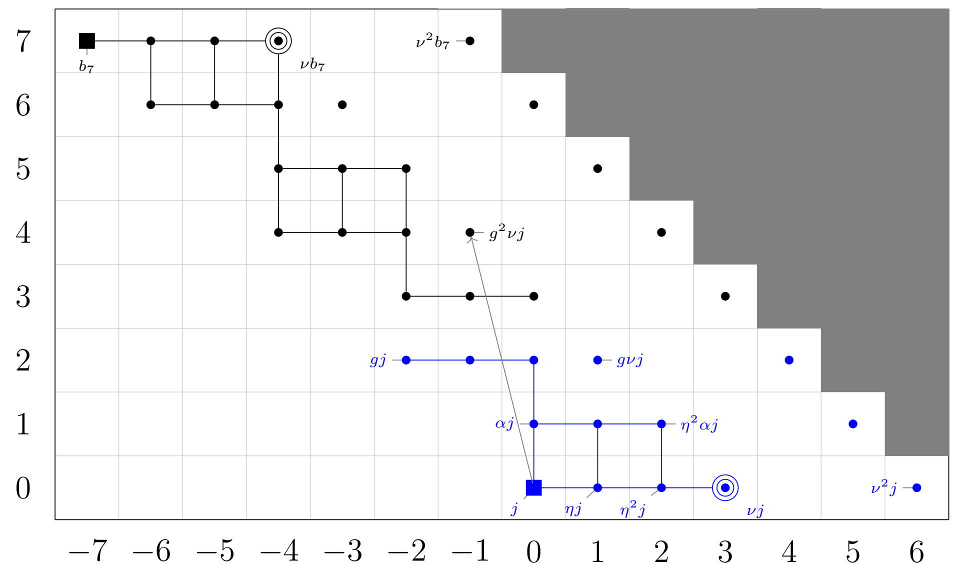

FIGURE 4. The spectral sequence for the cofiber sequence of towers $\mathbf{X}_3^7 \longrightarrow \mathbf{X}_{\le 7} \longrightarrow \mathbf{X}_{\le 2}$. Classes in blue are in the spectral sequence for $\mathbf{X}_{\le 2}$, classes in black are in that of $\mathbf{X}_3^7$. The classes combined give the E_2-page of the spectral sequence of $\mathbf{X}_{\le 7}$. A $\blacksquare$ is a copy of $\mathbb{Z}_2$, a $\bullet$ is a $\mathbb{Z}/2$. The circled $\bullet$ represent $\mathbb{Z}/8$. We have only drawn one differential in the spectral sequence of $\mathbf{X}_{\le 7}$.

The Tot tower $\mathbf{X}$ for $(S^{4\sigma-4})^{hC_2}$ is a (-4)-desuspension of the tower given in Proposition 11.8. The 7-truncation $\mathbf{X}_{\le 7}$ of this tower is given by the tower

$$\begin{tikzcd}
\Sigma^{-3}\mathbb{R}\mathrm{P}^3_{-4} \arrow[r] & \Sigma^{-3}\mathbb{R}\mathrm{P}^3_{-3} \arrow[r] & \Sigma^{-3}\mathbb{R}\mathrm{P}^3_{-2} \arrow[r] & \Sigma^{-3}\mathbb{R}\mathrm{P}^3_{-1} \arrow[r] & \cdots \arrow[r] & \Sigma^{-3}\mathbb{R}\mathrm{P}^3_{3} \\
S^{-7} \arrow[u] & S^{-6} \arrow[u] & S^{-5} \arrow[u] & S^{-4} \arrow[u] & & S^{0} \arrow[u, "\simeq"']
\end{tikzcd}$$

and the relative truncation $\mathbf{X}_3^7$ is given by
(11.19)

$$\begin{tikzcd}
\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4} \arrow[r] & \Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-3} \arrow[r] & \Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-2} \arrow[r] & \Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-1} \arrow[r] & \Sigma^{-3}\mathbb{R}\mathrm{P}^0_{0} \\
S^{-7} \arrow[u] & S^{-6} \arrow[u] & S^{-5} \arrow[u] & S^{-4} \arrow[u] & S^{-3}. \arrow[u, "\simeq"']
\end{tikzcd}$$

We have a fiber sequence of very short towers

$$\mathbf{X}_3^7 \longrightarrow \mathbf{X}_{\le 7} \longrightarrow \mathbf{X}_{\le 2}$$

which, after taking inverse limits (which amounts to taking the top space in these finite towers) gives the standard cofiber sequence

$$\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4} \to \Sigma^{-3}\mathbb{R}\mathrm{P}^3_{-4} \to \Sigma^{-3}\mathbb{R}\mathrm{P}^3_{1} \to \Sigma^{-2}\mathbb{R}\mathrm{P}^0_{-4}.$$

For example, $\mathrm{Tot}(\mathbf{X}_3^7) \simeq \Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4}$. Note $\Sigma^{-3}\mathbb{R}\mathrm{P}^3_1 = \Sigma^{-3}\mathbb{R}\mathrm{P}^3 \simeq S^0 \vee \Sigma^{-3}\mathbb{R}\mathrm{P}^2$.

From Lemma 6.6, for $N = 2$ we get an identification

$$E_2^{s,t}(\mathbf{X}_{\leq 2}) = \begin{cases} H^s(C_2, \pi_t S^0), & s \leq 2; \\ 0, & s > 2, \end{cases}$$

while for $N = 7$, we get

$$E_2^{s,t}(\mathbf{X}_{\leq 7}) = \begin{cases} H^s(C_2, \pi_t S^0), & s \leq 6; \\ \pi_t S^0, & s = 7; \\ 0, & s > 7. \end{cases}$$

In addition, by Lemma 6.8, we have

$$E_2^{s,t}(\mathbf{X}_3^7) = \begin{cases} H^s(C_2, \pi_t S^0), & 3 \leq s \leq 6; \\ \pi_t S^0, & s = 7; \\ 0, & \text{otherwise}. \end{cases}$$

Figure 4 gives a concise summary of the E_2-terms of the respective spectral sequences: the part in blue depicts the spectral sequence for $\mathbf{X}_{\leq 2}$, the part in black depicts the one for $\mathbf{X}_3^7$, and everything together gives the spectral sequence for $\mathbf{X}_{\leq 7}$. Differentials are not depicted in this figure, other than the d_4-differential on j, amounting to the image of j under the boundary map.

Using the nomenclature of Proposition 11.11 we now have the following lemma, illustrated in Figure 4.

Lemma 11.20. *In the spectral sequence*

$$E_2^{s,t}(\mathbf{X}_{\leq 2}) \Longrightarrow \pi_{t-s}\Sigma^{-3}\mathbb{R}\mathrm{P}^3$$

the class j is a permanent cycle detecting a homotopy class $[j_2] \in \pi_0\Sigma^{-3}\mathbb{R}\mathrm{P}^3$, which is non-zero in homology.

The class g^2j is a non-trivial permanent cycle in the spectral sequence

$$E_2^{s,t}(\mathbf{X}_3^7) \Longrightarrow \pi_{t-s}\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4}.$$

The image $\delta_*[j_2]$ of $[j_2]$ in $\pi_{-1}\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4}$ is detected by the class $[\nu g^2 j]$ in the spectral sequence

$$E_2^{s,t}(\mathbf{X}_3^7) \Longrightarrow \pi_{t-s}\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4}.$$

More precisely, $\delta_*[j_2] = \nu([g^2j] + a[\nu b_7])$, where $a \in \mathbb{Z}/2$, and $b_7 \in E_2^{7,0}(\mathbf{X}_3^7)$.

Proof. We apply Lemma 6.9 to the tower $\mathbf{X}$ with $K = 3$, $M = 7$, $r = 4$, $x = j$, and $y = \nu g^2 j$, as we know from Proposition 11.14 that $d_4(j) = \nu g^2 j$. This gives us that $\delta_*[j_2]$ is indeed detected by $[\nu g^2 j]$. Since there is nothing in filtrations 5 and 6 contributing to $\pi_{-1}\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4}$, we conclude that $\delta_*[j_2] = [\nu g^2 j] + [z_7]$, for some $z_7 \in E_2^{7,6}(\mathbf{X}_3^7)$.

First, we note that the class $[\nu g^2 j] \in \pi_{-1}\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4}$ is the ν-multiple of $[g^2j]$, which follows from the fact that $[g^2j]$ is a non-trivial permanent cycle in the spectral sequence for $\mathbf{X}_3^7$, again by combination of Proposition 11.14 and Lemma 6.9.

Second, $z_7 = a\nu^2 b_7$, where b_7 represents the bottom cell of $\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4}$, and $a \in \mathbb{Z}/2$. Note that νb_7 cannot be hit by a differential: while there are potentially non-trivial d_2 and d_3 with targets in bidegree $(s,t) = (7,3)$, they both have sources in groups of order at most 2, thus cannot conspire to kill the generator νb_7 of $E_2^{7,3}(\mathbf{X}_3^7) \cong \mathbb{Z}/8$. Consequently, $[z_7] = a\nu[\nu b_7]$, and all together, $\delta_*[j_2]$ is a multiple of ν. $\square$

Theorem 11.21. *The spectrum $2Q = Q \wedge Q$ is trivial in κ_2.*

Proof. Write

$$A^\bullet = \{(E^{\bullet+1} \wedge S^{4\sigma-4})^{h\mathbb{G}_2}\}$$

for the cosimplicial spectrum giving the homotopy fixed point spectral sequence. Write $\mathbf{Y}_\bullet$ for the associated tower. As discussed in the beginning of this subsection, it suffices to show that $d_r(\iota) = 0$ for $r \le 7$ in the associated spectral sequence, i.e. the $\mathbb{G}_2$ homotopy fixed point spectral sequence. For that, it suffices to show that ι is a permanent cycle in the truncated spectral sequence for $\mathbf{Y}_{\le 7}$, and that is what we will prove.

Consider the truncated towers

$$\mathbf{Y}_3^7 \to \mathbf{Y}_{\le 7} \to \mathbf{Y}_{\le 2}.$$

Then (11.15) gives us a diagram

$$\begin{tikzcd} \Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4} \arrow[r] \arrow[d, "f"] & \Sigma^{-3}\mathbb{R}\mathrm{P}^3_{-4} \arrow[r] \arrow[d, "f"] & \Sigma^{-3}\mathbb{R}\mathrm{P}^3 \arrow[d, "f"] \\ Y_3^7 \arrow[r] & Y_7 \arrow[r] & Y_2 \end{tikzcd} \tag{11.22}$$

where $Y_3^7 = \operatorname{holim} \mathbf{Y}_3^7$ and so on. We have a diagram of spectral sequences

$$\begin{tikzcd} E_2^{s,t}(\mathbf{X}_{\le 2}) \arrow[r, Rightarrow] \arrow[d, "f_*"] & \pi_{t-s}\Sigma^{-3}\mathbb{R}\mathrm{P}^3 \arrow[d, "f_*"] \\ E_2^{s,t}(\mathbf{Y}_{\le 2}) \arrow[r, Rightarrow] & \pi_{t-s} Y_2. \end{tikzcd}$$

By Lemma 11.20, we have that $j \in H^0(C_2, S^{4\sigma-4})$ is a permanent cycle in the top spectral sequence, giving a non-trivial class $[j_2] \in \pi_0\Sigma^{-3}\mathbb{R}\mathrm{P}^3$, which is in fact a choice of splitting of the top cell. This implies that $f_*(j) = \iota$ is also a permanent cycle in the bottom spectral sequence, giving a homotopy class of $[\iota_2] \in \pi_0 Y_2$.

We also have a diagram of spectral sequences

$$\begin{tikzcd} E_2^{s,t}(\mathbf{X}_3^7) \arrow[r, Rightarrow] \arrow[d, "f_*"] & \pi_{t-s}\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4} \arrow[d, "f_*"] \\ E_2^{s,t}(\mathbf{Y}_3^7) \arrow[r, Rightarrow] & \pi_{t-s} Y_3^7, \end{tikzcd}$$

and we next study the image of $\delta_*[j_2] \in \pi_{-1}\Sigma^{-3}\mathbb{R}\mathrm{P}^0_{-4}$ under f_*. By Lemma 11.20, $\delta_*[j_2] = \nu([g^2 j] + a[\nu b_7])$.

First, note that since $f_*([\nu b_7])$ has filtration 7, it must actually be zero as $E_2^{7,3}(\mathbf{Y}_3^7) = 0$. This implies that

$$f_*(\delta_*[j_2]) = \nu f_*([g^2 j]).$$

From Theorem 6.1.6 and Proposition 5.3.1 of [BGH22] we have that

$$0 = \widetilde{\chi}^2 \in H^4(\mathbb{G}_2, \mathbb{W}) \cong H^4(\mathbb{G}_2, E_0)$$

and hence

$$0 = \widetilde{\chi}^2 \iota = f_*(g^2 j) \in H^4(\mathbb{G}_2, E_0 S^{4\sigma-4}) = E_2^{4,0}(\mathbf{Y}_3^7).$$

This implies that $f_*[g^2 j]$ must be detected in filtration greater than 4 in the spectral sequence for Y_3^7. Thus $f_*[g^2 j]$ could be detected in $E_2^{s,s-4}(\mathbf{Y}_3^7)$, with $4 < s \leq 7$. By Lemma 6.8, and since $\pi_1 E = 0 = \pi_3 E$, the only such group which is non-zero is the case $s = 6$, given by

$$E_2^{6,2}(\mathbf{Y}_3^7) = H^6(\mathbb{G}_2, E_2).$$

Thus $f_*[g^2 j]$ must have filtration 6. From [BBG+22, Table 2], we know that ν-multiplication

$$H^6(\mathbb{G}_2, E_2) \xrightarrow{\nu} H^7(\mathbb{G}_2, E_6)$$

is trivial (since every class in $H^6(\mathbb{G}_2, E_2)$ is a multiple of η), thus $\nu f_*[g^2 j] = f_*(\nu[g^2 j])$ must be in higher filtration, i.e. in filtration at least 8. However, there is nothing in filtration 8 in the homotopy of Y_3^7, so $f_*(\nu[g^2 j]) = 0$.

Therefore, the image of $[\iota_2] := f_*[j_2]$ in ΣY_3^7 is

$$\delta_*[\iota_2] = \delta_* f_*[j_2] = f_*(\nu[g^2 j]) = 0.$$

Now we apply the second part of Lemma 6.9 to conclude that ι is a permanent cycle in the spectral sequence for Y_7 as needed. □

Now the following calculation of $\kappa(G_{48})$ is immediate.

Corollary 11.23. *The short exact sequence*

$$0 \longrightarrow \kappa(\mathbb{G}_2^1) \longrightarrow \kappa(G_{48}) \xrightarrow{\phi_3^1} \mathbb{Z}/2 \longrightarrow 0$$

splits and there is an isomorphism

$$\kappa(G_{48}) \cong \kappa(\mathbb{G}_2^1) \oplus \mathbb{Z}/2 \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^3.$$

We end this section with an analysis of the descent filtration of $\kappa(G_{48})$.

Theorem 11.24. *In the filtration*

$$0 \subseteq \kappa_7(G_{48}) \subseteq \kappa_5(G_{48}) \subseteq \kappa_3(G_{48}) = \kappa(G_{48}) \cong \mathbb{Z}/8 \times (\mathbb{Z}/2)^3$$

we have isomorphisms

$$\begin{aligned} \kappa_5(G_{48}) &= \kappa_5(\mathbb{G}_2^1) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2 \\ \kappa_7(G_{48}) &= \kappa_7(\mathbb{G}_2^1) \cong \mathbb{Z}/2 \\ \kappa_s(G_{48}) &= \kappa_s(\mathbb{G}_2^1) = 0, \qquad s > 7. \end{aligned}$$

Furthermore,

$$\kappa(G_{48})/\kappa_5(G_{48}) \xrightarrow[\cong]{\phi_3} \mathbb{Z}/2\{\eta\widetilde{\chi}\} \times \mathbb{Z}/2\{\zeta\langle\widetilde{\chi}, 2, \eta\rangle\}$$

$$\kappa_5(G_{48})/\kappa_7(G_{48}) \xrightarrow[\cong]{\phi_5} \mathbb{Z}/4\{\zeta e\nu\} \times \mathbb{Z}/2\{\zeta\widetilde{\chi}\eta^2\}$$

$$\kappa_7(G_{48}) \xrightarrow[\cong]{\phi_7} \mathbb{Z}/2\{\zeta e\eta^3\}.$$

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \kappa_5(\mathbb{G}_2^1) \arrow[r] \arrow[d, "\cap"] & \kappa(\mathbb{G}_2^1) \arrow[r, "\phi_3"] \arrow[d, "\cap"] & H^3(\mathbb{G}_2, E_2) \arrow[d, "="] \\
0 \arrow[r] & \kappa_5(G_{48}) \arrow[r] & \kappa(G_{48}) \arrow[r, "\phi_3"] & H^3(\mathbb{G}_2, E_2)
\end{tikzcd}$$

where both ϕ_3 mean their restrictions to $\kappa(G_{48})$ and $\kappa(\mathbb{G}_2^1)$. Both horizontal sequences are exact by Lemma 3.28. By [BBG+22, §7] (see also Figure 1),

$$H^3(\mathbb{G}_2, E_2) \cong \mathbb{Z}/2\{\zeta\langle\widetilde{\chi}, 2, \eta\rangle\} \oplus \mathbb{Z}/2\{\eta\widetilde{\chi}\}.$$

From Theorem 8.16, we have that $\phi_3(\kappa(\mathbb{G}_2^1)) = \mathbb{Z}/2\{\zeta\langle\widetilde{\chi}, 2, \eta\rangle\}$ and Theorem 11.17 shows that $\phi_3(Q) = \eta\widetilde{\chi}$. By chasing in the above diagram, we see that

$$\kappa_5(G_{48}) = \kappa_5(\mathbb{G}_2^1).$$

We then conclude that $\kappa_s(G_{48}) = \kappa_s(\mathbb{G}_2^1)$ for $s \geq 5$, and the calculation of ϕ_5 and ϕ_7 follows from Theorem 8.16. □

12. Picard elements detected by $E^{hG_{48}}$, and the calculation of κ_2

Now that we have determined the subgroup $\kappa(G_{48}) \subseteq \kappa_2$, it remains to compute the quotient and study the extension in order to understand κ_2 completely.

In this section, we will show that $\kappa_2/\kappa(G_{48})$ is $\mathbb{Z}/8$ in two steps: first, we show that the upper bound on this quotient is $\mathbb{Z}/8$ in Proposition 12.14, and then we exhibit a generator of order 8 in Theorem 12.21. Then, in Proposition 12.27 we show that the extension problem is trivial and finally, we conclude the paper by analyzing the descent filtration on κ_2 in Theorem 12.28.

The elements of κ_2 detected by $E^{hG_{48}}$ are those $X \in \kappa_2$ which do not have an $E^{hG_{48}}$-orientation in the sense of Definition 3.3. We will be looking at the obstructions for having such an orientation. By Proposition 3.5, those obstructions come in the form of differentials on a $\mathbb{G}_2$-invariant generator

$$\iota_X \in H^0(\mathbb{G}_2, E_0X) \subseteq H^0(G_{48}, E_0X)$$

in the homotopy fixed point spectral sequence

$$E_2^{s,t}(G_{48}, X) \cong H^s(G_{48}, E_tX) \Rightarrow \pi_{t-s}(E^{hG_{48}} \wedge X). \tag{12.1}$$

We have an isomorphism of this E_2-page with $E_2^{s,t}(G_{48}, S^0)$ determined by the choice of ι_X. To understand the possible fates of ι_X, we'll need a detailed analysis of the homotopy fixed point spectral sequence for $E^{hG_{48}}$, over which (12.1) is a module. We recall some details about that spectral sequence in the first subsection.

12.1. **The homotopy fixed point spectral sequence for $E^{hG_{48}}$.** The complete calculation of homotopy fixed point spectral sequence $E_*^{*,*}(G_{48}, S^0)$ can be found in citeDuanKongLiLuWang. This is closely related to any calculation of the homotopy groups of various version of topological modular forms spectrum, since the $K(2)$-localization of tmf at $p = 2$ is $E^{hG_{48}}$. See, for example, [DFHH14, Bau08, Sto14, BR21]. A basic point is that this is a spectral sequence of rings; in particular the differentials satisfy the Leibniz rule.

The cohomology ring $H^*(G_{48}, E_*)$ is described in Theorem 2.15 and the accompanying Figure 5 of [BG18] and we content ourselves with a summary here. There are elements

$$c_4 \in H^0(G_{48}, E_8) \qquad c_6 \in H^0(G_{48}, E_{12})$$
$$\Delta \in H^0(G_{48}, E_{24}) \qquad j \in H^0(G_{48}, E_0)$$

so that

$$E_2^{0,t} = H^0(G_{48}, E_t) = \mathbb{Z}_2[[j]][c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 - (12)^3\Delta, c_4^3 - j\Delta)$$

and

$$H^*(G_{48}, E_*) \cong H^0(G_{48}, E_*)[\eta, \nu, \mu, \epsilon, \kappa, \bar{\kappa}]/R, \tag{12.2}$$

where R is a rather elaborate list of relations. The classes

$$\eta \in H^1(G_{48}, E_2) \qquad \nu \in H^1(G_{48}, E_4)$$
$$\epsilon \in H^2(G_{48}, E_{10}) \qquad \kappa \in H^2(G_{48}, E_{16})$$
$$\bar{\kappa} \in H^4(G_{48}, E_{24})$$

are named for the elements of π_*S^0 that they detect. The class $\mu \in H^1(G_{48}, E_6)$ has the property that $d_3(\mu) = \eta^4$. Crucial among these generators is $\bar{\kappa} \in H^4(G_{48}, E_{24})$, and one important property it has is that multiplication by $\bar{\kappa}$ on $H^*(G_{48}, E_*)$ is onto in cohomological degree greater than 4 and has no annihilators of positive cohomological dimension.

If r is even, then $d_r = 0$. The odd differentials form a complicated but well-understood pattern. Crucially, Δ^8 is a permanent cycle and the entire spectral sequence is periodic of period 192 in degree t.

Here and elsewhere we will use the standard convention of writing elements by their names at E_2 even though they may no longer be products at E_r.

The d_3 differentials are determined by

$$d_3(\mu) = \eta^4, \qquad d_3(c_6) = c_4\eta^3 \tag{12.3}$$

and the fact that all other generators are d_3-cycles.

The d_5 and d_7 differentials are determined by the formulas

$$\begin{aligned} &d_5(\Delta) = \overline{\kappa}\nu, \\ &d_7(4\Delta) = \overline{\kappa}\eta^3, \qquad &d_7(2\Delta^2) = \Delta\overline{\kappa}\eta^3, \\ &d_7(4\Delta^3) = \Delta^2\overline{\kappa}\eta^3, \qquad &d_7(\Delta^4) = \Delta^3\overline{\kappa}\eta^3, \end{aligned} \tag{12.4}$$

the multiplicative structure, and the fact that they vanish on all the remaining generators.

Next are the d_9 differentials, regarding which we need only the following

$$d_9(\Delta^2\epsilon) = \bar{\kappa}^2\kappa\eta, \qquad d_9(\Delta^6\epsilon) = \Delta^4\bar{\kappa}^2\kappa\eta. \tag{12.5}$$

In fact, there are considerably more differentials, but those are not needed for our arguments below. We will focus here on the calculation in a small range, which we present in Figure 5, and some key properties summarized in the following result.

Proposition 12.6. *In the homotopy fixed point spectral sequence*

$$E_2^{s,t}(G_{48}, S^0) = H^s(G_{48}, E_t) \Longrightarrow \pi_{t-s} E^{hG_{48}} \tag{12.7}$$

we have the following:

(1) *The non-zero differentials that originate on the zero line are d_3, d_5 and d_7.*
(2) *The spectral sequence collapses at the $r = 24$ page and*

$$E_\infty^{s,*}(G_{48}, S^0) = 0 \quad \text{if} \quad s \geq 23.$$

(3) *For $r > 7$, we have $E_r^{r,r-1}(G_{48}, S^0) = 0$.*
(4) *For $s > 0$, $E_\infty^{s,s}(G_{48}, S^0) = 0$.*
(5) *The spectrum $E^{hG_{48}}$ is 192-periodic, with a class $\Delta^8 \in \pi_{192} E^{hG_{48}}$ a periodicity generator. Thus the composition*

$$\Sigma^{192} E^{hG_{48}} \xrightarrow{\Delta^8 \wedge \iota} E^{hG_{48}} \wedge E^{hG_{48}} \xrightarrow{m} E^{hG_{48}},$$

where m is the ring spectrum multiplication, is an equivalence.

12.2. **Establishing an upper bound for $\kappa_2/\kappa(G_{48})$.** In this subsection we will produce an injective homomorphism $\kappa_2/\kappa(G_{48}) \to \mathbb{Z}/8$, see Proposition 12.14.

So we fix $X \in \kappa_2$, let $\iota_X \in E_0 X$ be a $\mathbb{G}_2$-invariant generator for X, and we examine what can happen to ι_X in the spectral sequence (12.1). The first result handles d_3.

Lemma 12.8. *In the homotopy fixed point spectral sequence*

$$H^s(G_{48}, E_t X) \Longrightarrow \pi_{t-s}(E^{hG_{48}} \wedge X)$$

we have $d_3(\Delta^k \iota_X) = 0$ for all k.

Proof. Since Δ is a d_3 cycle (see (12.3)), it suffices to prove that

$$d_3(\iota_X) \in H^3(G_{48}, E_4 X) \cong \mathbb{F}_2[[j]]\eta^3 \frac{c_4 c_6}{\Delta} \iota_X$$

is zero. A priori, we have

$$d_3(\iota_X) = f(j)\eta^3 \frac{c_4 c_6}{\Delta} \iota_X$$

for some power series $f(j) \in \mathbb{F}_2[[j]]$. We apply d_3 again, and use the fact that $\frac{c_6^2}{\Delta} \equiv j \mod 2$ to obtain that $f(j)$ satisfies

$$f(j) + j f(j)^2 = 0.$$

This implies that $f(j) = 0$. □

Remark 12.9. The reader is encouraged to compare this computation of $d_3(\iota_X)$ with the computation of d_3 in the Picard spectral sequence in [MS16, Theorem 8.2.2].

Remark 12.10. Now that we know $d_3(\iota_X) = 0$, (12.2) and the differential patterns of (12.3), (12.4) and (12.5) (depicted in Figure 5) give that

$$d_5(\iota_X) = a\Delta^{-1}\bar{\kappa}\nu\iota_X, \qquad \text{for some } a \in \mathbb{Z}/4$$

and if $a = 0$,

$$d_7(\iota_X) = b\Delta^{-1}\bar{\kappa}\eta^3\iota_X, \qquad \text{for some } b \in \mathbb{Z}/2.$$

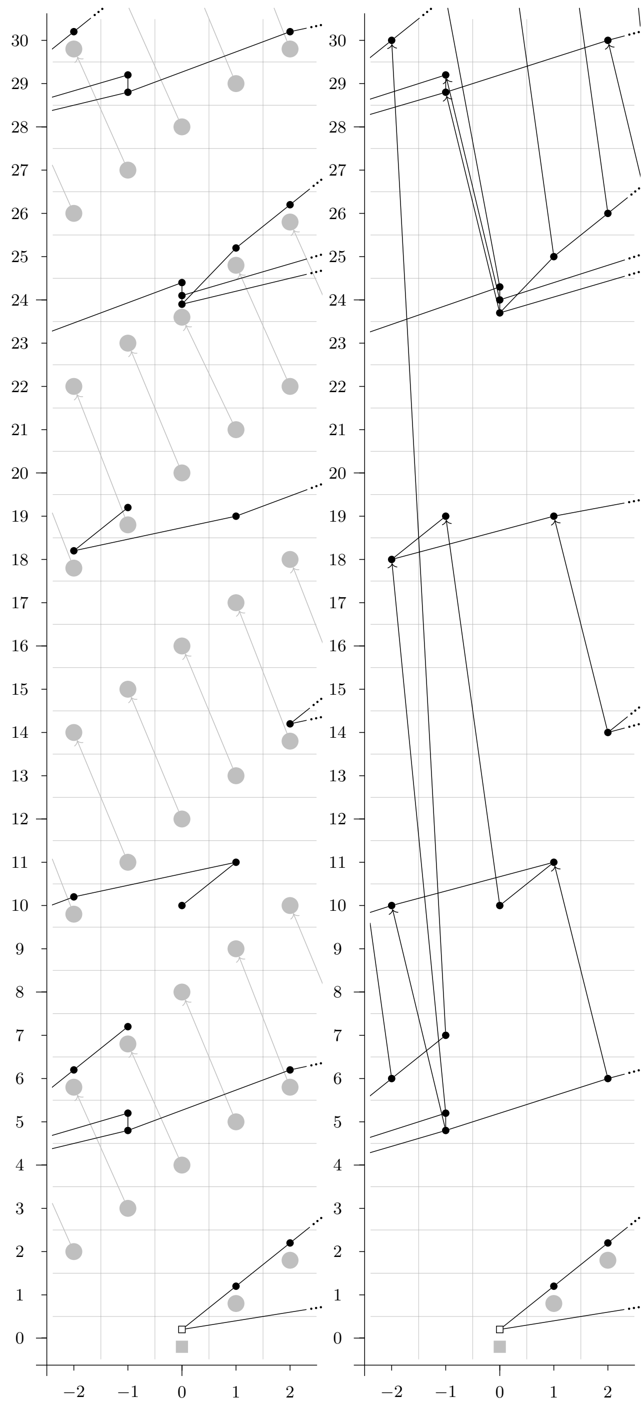

FIGURE 5. The homotopy fixed point spectral sequence (12.7) for $\pi_* E^{hG_{48}}$ in a small range. The x-axis is $t-s$ and the y-axis is s. The circles represent $j\mathbb{F}_2[[j]]$ and bullets represent $\mathbb{F}_2$. n bullets connected by a vertical line represent $\mathbb{Z}/2^n$. Lines of slope 1 and 1/3 represent multiplication by η and ν, respectively.

We use this observation to prove the next result, which finds that some Δ-multiple of ι_X must be a d_7-cycle.

Lemma 12.11. *Let $X \in \kappa_2$ and let $\iota_X \in E_0X$ be a $\mathbb{G}_2$-invariant generator. Then there is an integer k so that*

$$d_5(\Delta^k \iota_X) = 0 = d_7(\Delta^k \iota_X).$$

Proof. In the homotopy fixed point spectral sequence for $E^{hG_{48}}$ we have $d_5(\Delta) = \bar{\kappa}\nu$, see (12.4). Then

$$d_5(\Delta^k) = \begin{cases} k\Delta^{k-1}\bar{\kappa}\nu, & k \not\equiv 0 \text{ modulo } 4; \\ 0, & k \equiv 0 \text{ modulo } 4. \end{cases}$$

and

$$d_7(\Delta^k) = \Delta^{k-1}\bar{\kappa}\eta^3, \qquad k \equiv 4 \text{ modulo } 8.$$

We use this and the differential pattern of Remark 12.10. Suppose $d_5(\iota_X) = a\Delta^{-1}\bar{\kappa}\nu\iota_X$ with $0 \le a \le 3$; then $d_5(\Delta^{-a}\iota_X) = 0$, and

$$d_7(\Delta^{-a}\iota_X) = c\Delta^{-a-1}\bar{\kappa}\eta^3\iota_X, \qquad c \in \mathbb{Z}/2.$$

If $c = 0$ we are done. If not, $d_5(\Delta^{-a-4}\iota_X) = 0$ as well and

$$d_7(\Delta^{-a-4}\iota_X) = 0.$$

□

Let ι_X be a $\mathbb{G}_2$-invariant generator for X. By Lemma 12.11, we can find an integer k such that $\Delta^k\iota_X$ is a d_7-cycle in the homotopy fixed point spectral sequence $E_r^{s,t}(G_{48}, X)$ (12.1). It turns out that this implies $\Delta^k\iota_X$ is a permanent cycle, which has the following consequence.

Proposition 12.12. *Let $X \in \kappa_2$, and let k be an integer such that $\Delta^k\iota_X$ is a d_7-cycle. Then $\Delta^k\iota_X$ a permanent cycle extending to an equivalence of $E^{hG_{48}}$-modules $\Sigma^{24k}E^{hG_{48}} \simeq E^{hG_{48}} \wedge X$.*

Proof. To see that $\Delta^k\iota_X$ is a permanent cycle, we use an adaptation of the usual proof that Δ^8 is permanent cycle in the spectral sequence for $E^{hG_{48}}$. If $\Delta^k\iota_X$ is a d_r-cycle, then we have an isomorphism of $E_r^{*,*}(G_{48}, S^0)$-modules

$$E_r^{*,*}(G_{48}, X) \cong E_r^{*,*}(G_{48}, S^{24k}) \cong E_r^{*,*-24k}(G_{48}, S^0).$$

We know this is the case for $r = 7$. The image of $\Delta^k\iota_X$ under d_r then lives in the group

$$E_r^{r,r+24k-1}(G_{48}, X) \cong E_r^{r,r-1}(G_{48}, S^0),$$

but from Proposition 12.6(3), we know when $r > 7$, the groups $E_r^{r,r-1}(G_{48}, S^0)$ vanish. Thus we conclude that we cannot have any further non-trivial differentials on $\Delta^k\iota_X$. □

As a first corollary of this result, we get a bound on the descent filtration on κ_2.

Corollary 12.13. *The subgroup $\kappa_{2,8} \subset \kappa_2$ is trivial.*

Proof. Suppose $X \in \kappa_{2,8}$, so that by definition, $d_r(\iota_X) = 0$ for $r \le 7$ in the homotopy fixed point spectral sequence $E_r^{s,t}(\mathbb{G}_2, X)$. Then we get that the image of ι_X in the homotopy fixed point spectral sequence $E_r^{s,t}(G_{48}, X)$ is also a d_7-cycle, but then Proposition 12.12 shows that ι_X is a permanent cycle in the latter spectral sequence. Hence, $X \in \kappa_8(G_{48}) = \kappa_{2,8} \cap \kappa(G_{48})$. But from Theorem 11.24, we know that the group $\kappa_8(G_{48})$ is trivial, implying that X is trivial. $\square$

Note that since $E^{hG_{48}}$ is $24 \times 8 = 192$-periodic, the integer k in Proposition 12.12 is only well-defined modulo 8.

Proposition 12.14. *Define a homomorphism $\kappa_2 \to \mathbb{Z}/8$ by sending X to k, where $E^{hG_{48}} \wedge X \simeq \Sigma^{24k} E^{hG_{48}}$. This gives an injection*

$$\kappa_2/\kappa(G_{48}) \longrightarrow \mathbb{Z}/8.$$

Proof. Suppose $X \in \kappa_2$ goes to 0; that is, we have an equivalence of $E^{hG_{48}}$-module spectra $E^{hG_{48}} \simeq E^{hG_{48}} \wedge X$. Then Lemma 3.8, with the input of Proposition 12.6(4) implies that X is in $\kappa(G_{48})$. $\square$

Remark 12.15. Let $\mathrm{Pic}^0(E^{hG_{48}})$ denote the Picard group of invertible $E^{hG_{48}}$-modules N so that E_*N is in even degrees. This is equivalent to the Picard group $\mathrm{Pic}^0_{G_{48}}(E)$ of invertible $K(2)$-local E-modules M in G_{48}-spectra so that π_*M is in even degrees. Then $\pi_0 M$ is an invertible E_0-module equipped with a compatible action of G_{48}. Thus we have a map

$$\mathrm{Pic}^0(E^{hG_{48}}) \to H^1(G_{48}, E_0^\times).$$

Using methods similar to [Kar10, Proposition 5.2] at the prime 3, or the uncompleted version in [MS16, Appendix B], one can show that the target is isomorphic to $\mathbb{Z}/12$ generated by the image of $\Sigma^2 E^{hG_{48}}$. In particular, the map is onto and we have a short exact sequence

$$0 \to K \to \mathrm{Pic}^0(E^{hG_{48}}) \to H^1(G_{48}, E_0^\times) \to 0.$$

Using the techniques of [MS16] and [HMS17], and using the same input we used above, one can also show that K is $\mathbb{Z}/8$ generated by $\Sigma^{24} E^{hG_{48}}$; in particular, $\mathrm{Pic}^0(E^{hG_{48}}) \cong \mathbb{Z}/96$ and hence $\mathrm{Pic}(E^{hG_{48}}) \cong \mathbb{Z}/192$ generated by $\Sigma E^{hG_{48}}$. Theorem 12.21 below shows that the induced map $\kappa_2/\kappa(G_{48}) \to K$ is an isomorphism.

Note that $E^{hG_{48}} \simeq L_{K(2)} TMF$ and the analogous result for the Picard group of TMF-modules is [MS16, Theorem 8.2.2].

12.3. **A generator for $\kappa_2/\kappa(G_{48})$.** In the previous section we established that the quotient $\kappa_2/\kappa(G_{48})$ is at most $\mathbb{Z}/8$. The goal of this section is to show that it is exactly $\mathbb{Z}/8$ by showing that there exists $P \in \kappa_2$ such that $P \wedge E^{hG_{48}} \simeq \Sigma^{-24} E^{hG_{48}}$.

The invertible spectrum P that we'll use for this purpose comes from Gross-Hopkins duality [HG94c, HG94a]. This seminal work implies that at each height n, the spectrum I_n obtained as the Brown-Comenetz dual of the n-th monochromatic layer of the sphere, is an invertible object in the $K(n)$-local category; a more hands-on presentation of this fact can be found in [Str00]. For each n, there is a decomposition of I_n as a smash product

$$I_n \simeq S^{n^2-n} \wedge S\langle \det \rangle \wedge P_n, \tag{12.16}$$

where $S\langle\det\rangle$ is the determinant sphere, as constructed in [BBGS22], for example, and P_n is an element of κ_n. The invertibility of I_n then implies that for any $K(n)$-local spectrum Z, we have

$$I_n Z = F(Z, I_n) \simeq F(Z, L_{K(n)}S^0) \wedge I_n \simeq D_n Z \wedge \Sigma^{n^2-n} S^0\langle\det\rangle \wedge P_n, \tag{12.17}$$

where the smash product is understood to be $K(n)$-local.

The P we consider at height 2 is exactly this P_2. We can find a minimum for its order in κ_2 by studying the interplay between the Spanier-Whitehead dual $D_2 E^{hG_{48}} = F(E^{hG_{48}}, L_{K(n)}S^0)$ and the Gross-Hopkins dual $I_2 E^{hG_{48}} = F(E^{hG_{48}}, I_2)$ of $E^{hG_{48}}$. The Spanier-Whitehead as well as Gross-Hopkins duals of $E^{hG_{48}}$ are well known.

Theorem 12.18 ([Bob20])**.** *There is an equivalence of $E^{hG_{48}}$-modules*

$$D_2 E^{hG_{48}} \simeq \Sigma^{44} E^{hG_{48}}.$$

See also [BGHS22, Theorem 13.25] for a different proof of Theorem 12.18.

The next result identifies the Gross-Hopkins dual of $E^{hG_{48}}$. It has a long history but has not been recorded in the literature in the precise form that we use, and here we present a proof analogous to the prime 3 version given in [Beh06, Proposition 2.4.1]. The key 2-primary calculations are coming from [Pha21].

Theorem 12.19. *There is an equivalence of $E^{hG_{48}}$-modules*

$$I_2 E^{hG_{48}} \simeq \Sigma^{22} E^{hG_{48}}.$$

Proof. Let $A(1)$ be a finite 2-local spectrum with the property that $H^*(A(1), \mathbb{F}_2)$ is free on a generator in degree zero over the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2. We have some choice here, and we choose a version of $A(1)$ which is self-dual; in [Pha21] these are called $A_1[10]$ and $A_1[01]$. Either choice will do, and it suffices to show the claimed equivalence holds after smashing with $A(1)$.

We have that $DA(1) = F(A(1), S^0) = \Sigma^{-6}A(1)$. Since $A(1)$ is a type 2 complex $L_1(E^{hG_{48}} \wedge A(1)) \simeq *$ and

$$I_2(E^{hG_{48}} \wedge A(1)) = I_{\mathbb{Z}/2^\infty}(E^{hG_{48}} \wedge A(1)).$$

Here, $I_{\mathbb{Z}/2^\infty}X$ denotes the Brown-Comenetz dual of X, whose homotopy groups are $(\pi_*X)^\vee = \mathrm{Hom}(\pi_*X, \mathbb{Z}/2^\infty)$. Using Proposition 12.12, (12.17), and the fact that $A(1)$ is a finite complex, we have

$$\begin{aligned} I_{\mathbb{Z}/2^\infty}(E^{hG_{48}} \wedge A(1)) &\simeq S^2 \wedge S^0\langle\det\rangle \wedge P \wedge \Sigma^{44}E^{hG_{48}} \wedge \Sigma^{-6}A(1) \\ &\simeq \Sigma^{40}E^{hG_{48}} \wedge P \wedge A(1) \\ &\simeq \Sigma^{40+24k}E^{hG_{48}} \wedge A(1) \end{aligned}$$

with k to be determined. This gives an equation

$$\begin{aligned} [\pi_{-n}(E^{hG_{48}} \wedge A(1))]^\vee &\cong \pi_{n-40-24k}(E^{hG_{48}} \wedge A(1)) \\ &\cong \pi_{n+152-24k}(E^{hG_{48}} \wedge A(1)). \end{aligned}$$

The last isomorphism uses the periodicity of $E^{hG_{48}}$. Investigating [Pha21, Figures 22-25] for low values of n we see that this is only possible if $k \equiv -1$ modulo 8. Specifically, we see that if $1 \leq n \leq 15$, then

$$\pi_{-n}(E^{hG_{48}} \wedge A(1)) = \pi_{192-n}(E^{hG_{48}} \wedge A(1)) = 0.$$

In the same range, $\pi_{n+152-24k}(E^{hG_{48}} \wedge A(1)) = 0$ only if $k = -1$. $\square$

Remark 12.20. An alternative proof would be to cite Greenlees [Gre16, Example 4.4], for the fact that tmf is Gorenstein self-dual, which is also done in much more detail in the newer [BR21, BGR22]. Then one can apply [GS18, Proposition 4.1] to obtain that the non-connective, non-periodic Tmf is Anderson self-dual, and then $K(2)$-localize to obtain that $E^{hG_{48}} = L_{K(2)}Tmf$ is Gross-Hopkins self-dual as claimed.

The most conceptual proof would use Serre duality on a suitable cover of the compactified moduli stack of elliptic curves. The analogue at the prime 3 was done in [Sto12], while the algebraic calculations needed at $p = 2$ are set up in [Sto14].

Combining these two results give the main result of this section.

Theorem 12.21. *Let $P = P_2 \in \kappa_2$ be determined by the equation (12.16) at $n = 2$, i.e.*

$$I_2 \simeq S^2 \wedge S^0\langle \det \rangle \wedge P.$$

Then we have an equivalence of $E^{hG_{48}}$-modules

$$\Sigma^{-24} E^{hG_{48}} \simeq E^{hG_{48}} \wedge P$$

and, thus, an isomorphism

$$\kappa_2/\kappa(G_{48}) \cong \mathbb{Z}/8.$$

Proof. We set $Z = E^{hG_{48}}$ in (12.17), and then combine with Theorem 12.18 and Theorem 12.19 to get

$$\Sigma^{22} E^{hG_{48}} \simeq \Sigma^{44} E^{hG_{48}} \wedge \Sigma^2 S^0\langle \det \rangle \wedge P.$$

Then we note that $S^0\langle \det \rangle \wedge E^{hG_{48}} \simeq E^{hG_{48}}$ since G_{48} is in the kernel of the determinant map; see [BBGS22, Corollary 3.11]. This then reduces to the indicated equivalence.

The final isomorphism follows from Proposition 12.14, since 8 is the smallest integer m such that $E^{hG_{48}} \wedge P^{\wedge m} \simeq E^{hG_{48}}$. $\square$

Remark 12.22. In the homotopy fixed point spectral sequence $E_r^{*,*}(G_{48}, P)$, we have that ι_P is a d_3-cycle and $d_5(\iota_P)$ is a generator of $E_5^{5,4}(G_{48}, P) \cong \mathbb{Z}/4\{k\nu\}$. Compare Remark 12.10.

Remark 12.23. A similar comparison of the effect of P_2 in E^{hC_2} is done in [HLS21, Theorem 6.6], but that result shows that C_2 can only see that the order of P_2 is divisible by 2.

Remark 12.24. It is natural to ask whether we can construct an element of κ_2 which maps to a generator of $\kappa_2/\kappa(G_{48}) \cong \mathbb{Z}/8$ using the J construction of Section 5. We have not been able to do quite this much, but here is what we *can* do towards that goal.

Recall from (7.2) that $\mathbb{S}_2$ has an open normal subgroup K with the property that the composition

$$G_{24} \longrightarrow \mathbb{S}_2 \longrightarrow \mathbb{S}_2/K$$

is an isomorphism, which produces a splitting $\mathbb{S}_2 \cong K \rtimes G_{24}$. Unfortunately, K is not Galois-invariant, so we cannot write G_{24} or G_{48} as the quotient of $\mathbb{G}_2$. Still, we can use the J-construction for the quotient $\mathbb{S}_2/K \cong G_{24}$ to produce an invertible $E^{h\mathbb{S}_2}$-module, which if we knew is Galois invariant, would be a generator of the quotient $\kappa_2/\kappa(G_{48})$.

Let $\mathbb{H}$ be real quaternion algebra; this a four dimensional irreducible representation of Q_8

$$\mathbb{H} \cong \mathbb{R}\{1, i, j, k\}.$$

The action of C_3 on Q_8 which permutes i, j, k extends $\mathbb{H}$ to a G_{24}-representation. We can let $\mathbb{S}_2$ act on $\mathbb{H}$ via the quotient $q\colon \mathbb{S}_2 \to \mathbb{S}_2/K \cong G_{24}$. Let $V = \mathbb{H} - 4$ and let

$$J(V) = (E \wedge S^V)^{h\mathbb{S}_2} \simeq (E^{hK} \wedge S^V)^{hG_{24}},$$

compare (5.1); however, this is not quite the same construction as (5.1), since $\mathbb{S}_2/K$ is not a quotient of $\mathbb{G}_2$. Nonetheless, $J(V)$ is a $K(2)$-local invertible $E^{h\mathbb{S}_2}$-module, and

$$E^{hK} \wedge J(V) \simeq E^{hK}.$$

Furthermore, by Proposition 5.12, we have an equivalence

$$E^{hG_{24}} \wedge_{E^{h\mathbb{S}_2}} (E \wedge S^V)^{h\mathbb{S}_2} \simeq (E \wedge S^V)^{hG_{24}}$$

In Propositions 13.22 and 13.23 [BGHS22], all elements of the form $(E \wedge S^V)^{hG_{24}}$ in the Picard group $\mathrm{Pic}(E^{hG_{24}})$ of invertible $E^{hG_{24}}$-modules are computed. This is done by showing that the J homomorphism factors as[1]

$$\begin{tikzcd}
RO(G_{24}) \arrow[r, "J"] \arrow[d, "\psi"] & \mathrm{Pic}(E^{hG_{24}}). \\
\mathbb{Z} \oplus \mathbb{Z}/8 \arrow[r, two heads] & (\mathbb{Z} \oplus \mathbb{Z}/8)/(24,1) \cong \mathbb{Z}/192 \arrow[u, "\subseteq"']
\end{tikzcd}$$

Here, the map

$$\psi\colon RO(G_{24}) \to \mathbb{Z} \oplus \mathbb{Z}/8$$

is defined by $\psi(W) = (\dim W, \lambda(W))$ for a certain characteristic class λ which has been normalized so that $\lambda(\mathbb{H}) = 1$. Therefore, $\psi(V) = (0, 1)$. From this, it follows that

$$(E \wedge S^V)^{hG_{24}} \simeq \Sigma^{-24} E^{hG_{24}}.$$

If we could find an $X \in \kappa_2$ such that $E^{h\mathbb{S}_2} \wedge X \simeq J(V)$ then X would be a generator for $\kappa_2/\kappa(G_{48})$. This can be phrased as the question of whether $J(V) \in \mathrm{Pic}(E^{h\mathbb{S}_2})$ is Galois invariant.

[1]There is a slight gap in [BGHS22, Proposition 3.23 (2)]. We only know that the right vertical map is an inclusion, as there is no proof in the literature that $\mathrm{Pic}(E^{hG_{24}}) \cong \mathbb{Z}/192$.

12.4. **The group κ_2.** In this final section we show that the extension

$$0 \longrightarrow \kappa(G_{48}) \longrightarrow \kappa_2 \longrightarrow \mathbb{Z}/8 \longrightarrow 0 \tag{12.25}$$

from Theorem 12.21 is split, and we relate these groups to the descent filtration. We begin with the following intermediate results, which examine the descent filtration and decompose the subgroup $\kappa_{2,5} \subseteq \kappa_2$. Recall that $\kappa_5(G_{48}) = \kappa(G_{48}) \cap \kappa_{2,5}$.

Proposition 12.26. *There is a short exact sequence*

$$0 \to \kappa_5(G_{48}) \to \kappa_{2,5} \to \mathbb{Z}/8 \to 0,$$

where $\kappa_{2,5} \to \mathbb{Z}/8$ is the composite $\kappa_{2,5} \hookrightarrow \kappa_2 \twoheadrightarrow \kappa_2/\kappa(G_{48}) \cong \mathbb{Z}/8$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
 & \kappa_5(G_{48}) \arrow[r] \arrow[d, hook] & \kappa(G_{48}) \arrow[r, "\phi_3"] \arrow[d, hook] & H^3(\mathbb{G}_2, E_2) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \kappa_{2,5} \arrow[r] & \kappa_2 \arrow[r, "\phi_3"] & H^3(\mathbb{G}_2, E_2), &
\end{tikzcd}$$

The lower sequence is exact by Lemma 3.28. That the top ϕ_3 is onto follows from Theorem 11.24, and the rest of the top sequence is exact because $\kappa_5(G_{48}) = \kappa(G_{48}) \cap \kappa_{2,5}$. In particular, the bottom ϕ_3 must be onto as well, and then the Snake Lemma gives the needed isomorphism

$$\kappa_{2,5}/\kappa_5(G_{48}) \cong \kappa_2/\kappa(G_{48}) \cong \mathbb{Z}/8.$$ □

In order to give a splitting of (12.25), we will first split its restriction from Proposition 12.26.

Proposition 12.27. *The short exact sequence*

$$0 \to \kappa_5(G_{48}) \to \kappa_{2,5} \to \mathbb{Z}/8 \to 0$$

splits.

Proof. From Theorem 11.24, we have that $\kappa_5(G_{48}) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$. The result will follow if we prove that $\kappa_{2,5}$ has exponent 8.

Lemma 3.28 provides us with an exact sequence

$$0 \to \kappa_{2,7} \to \kappa_{2,5} \to E_5^{5,4}(\mathbb{G}_2, S^0).$$

The right most term $E_5^{5,4}(\mathbb{G}_2, S^0)$ has exponent 4; see Figure 1. Therefore, it suffices to prove that $\kappa_{2,7}$ has exponent 2. For this, we note that the map

$$\phi_7 \colon \kappa_{2,7} \longrightarrow E_7^{7,6}$$

is injective as a consequence of Corollary 12.13. But $E_7^{7,6}(\mathbb{G}_2, S^0)$ has exponent 2, from Figure 1. So, it follows that $\kappa_{2,7}$ has exponent 2 and we are done. □

The final calculation of κ_2 is now an immediate corollary.

Theorem 12.28. *The short exact sequence*

$$0 \longrightarrow \kappa(G_{48}) \longrightarrow \kappa_2 \longrightarrow \mathbb{Z}/8 \longrightarrow 0.$$

splits and gives an isomorphism

$$\kappa_2 \cong \kappa(G_{48}) \oplus \mathbb{Z}/8 \cong (\mathbb{Z}/8)^2 \oplus (\mathbb{Z}/2)^3.$$

Proof. That the short exact sequence splits follows from Proposition 12.26 and Proposition 12.27. For the calculation of $\kappa(G_{48})$, see Corollary 11.23. □

We end by recording the descent filtration on κ_2.

Theorem 12.29. *In the filtration*

$$0 \subseteq \kappa_{2,7} \subseteq \kappa_{2,5} \subseteq \kappa_{2,3} = \kappa_2 \cong (\mathbb{Z}/8)^2 \times (\mathbb{Z}/2)^3$$

we have isomorphisms

$$\begin{aligned} \kappa_{2,5} &\cong (\mathbb{Z}/8)^2 \times \mathbb{Z}/2 \\ \kappa_{2,7} &\cong (\mathbb{Z}/2)^2 \\ \kappa_{2,s} &= 0, \qquad s > 7. \end{aligned}$$

Furthermore,

$$\kappa_2/\kappa_{2,5} \xrightarrow[\cong]{\phi_3} \mathbb{Z}/2\{\widetilde{\chi}\eta\} \times \mathbb{Z}/2\{\zeta\langle\widetilde{\chi}, 2, \eta\rangle\}$$

$$\kappa_{2,5}/\kappa_{2,7} \xrightarrow[\cong]{\phi_5} \mathbb{Z}/4\{k\nu\} \times \mathbb{Z}/4\{\zeta e\nu\} \times \mathbb{Z}/2\{\zeta\widetilde{\chi}\eta^2\}$$

$$\kappa_{2,7} \xrightarrow[\cong]{\phi_7} \mathbb{Z}/2\{k\eta^3\} \times \mathbb{Z}/2\{\zeta e\eta^3\}.$$

Proof. The group $\kappa_{2,5}$ is of the given form by Proposition 12.27 and Theorem 11.24. The proof of Proposition 12.26 also gave the isomorphism of $\kappa_2/\kappa_{2,5}$ as claimed.

Now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \kappa_5(G_{48}) \arrow[r] \arrow[d, "\phi_5"] & \kappa_{2,5} \arrow[r] \arrow[d, "\phi_5"] & \mathbb{Z}/8 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/4\{\zeta e\nu\} \times \mathbb{Z}/2\{\zeta\widetilde{\chi}\eta^2\} \arrow[r] & \mathrm{im}(\phi_5) \arrow[r] \arrow[d, hook] & E_5^{5,4}(G_{48}, S^0) \cong \mathbb{Z}/4\{k\nu\} \arrow[r] & 0, \\
& & E_5^{5,4}(\mathbb{G}_2, S^0) & &
\end{tikzcd}$$

where the left vertical map is surjective by Theorem 11.24, and the right vertical map is surjective by Remark 12.22. This gives the image of the middle ϕ_5, which is the quotient $\kappa_{2,5}/\kappa_{2,7}$ as claimed.

The Snake Lemma for the kernels in the above diagram now gives a short exact sequence

$$0 \to \mathbb{Z}/2 \cong \kappa_7(G_{48}) \to \kappa_{2,7} \to \mathbb{Z}/2 \to 0,$$

which has to split since $\phi_7 : \kappa_{2,7} \to E_7^{7,6}(\mathbb{G}_2, S^0)$ is injective and $E_7^{7,6}(\mathbb{G}_2, S^0)$ has exponent 2. In fact, $E_5^{7,6}(\mathbb{G}_2, S^0) \cong \mathbb{Z}/2\{k\eta^3\} \times \mathbb{Z}/2\{\zeta e\eta^3\}$, so we conclude that $E_5^{7,6}(\mathbb{G}_2, S^0) \cong E_7^{7,6}(\mathbb{G}_2, S^0)$ and that ϕ_7 must be onto.

Finally, we have that $\kappa_{2,s}$ is trivial for $s > 7$ by Corollary 12.13. □

Remark 12.30. Note that this proof shows that ϕ_5 does not surject onto the group $E_5^{5,4}(\mathbb{G}_2, S^0)$, which was computed in [BBG+22], see Figure 1. Namely, the class $\eta^2 e$ is not in the image of ϕ_5.

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