EQUIVARIANT TREES AND PARTITION COMPLEXES

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Abstract. We introduce two definitions of G-equivariant partitions of a finite G-set, both of which yield G-equivariant partition complexes. By considering suitable notions of equivariant trees, we show that G-equivariant partitions and G-trees are G-homotopy equivalent, generalizing existing results for the non-equivariant setting. Along the way, we develop equivariant versions of Quillen's Theorems A and B, which are of independent interest.

1. Introduction

Given a finite set $\mathbf{n} = \{1, \ldots, n\}$, we can consider the set of partitions of $\mathbf{n}$, which has a partial order by coarsening. For example, we have

$$(12)(3)(4) < (12)(34)$$

as partitions of the set $\mathbf{4}$. Thinking of this poset as a category allows us to take its classifying space and obtain a topological space. If we include all partitions, this space is contractible, since the discrete partition consisting of singleton sets is an initial object, and the indiscrete partition consisting of the whole set is a terminal object. Discarding these two partitions results in a poset $\mathcal{P}(\mathbf{n})$; its classifying space $|\mathcal{P}(\mathbf{n})|$ is called a *partition complex*.

This space is of interest in a wide variety of mathematical applications, ranging from combinatorics to algebra to topology. For instance, it has been used to study the Goodwillie derivatives of the identity functor [AM99], [Chi05]; its homology is intimately related to Lie (super)algebras [Wac98], [HW95], [Bar90], [Rob04]; it plays a central role in the study of bar constructions for operads [Chi05], [Fre04]; and it has applications in pure combinatorics [Sta82, §7].

Robinson and Whitehouse [RW96, Rob04] first observed that the data of a partition complex can also be encoded in a suitable category of trees. This comparison was further developed in recent work by Heuts and Moerdijk

Date: February 20, 2023.

2020 *Mathematics Subject Classification.* 55P91, 05A18, 20E08, 05E18.

Key words and phrases. partition complexes, trees, equivariant homotopy theory.

This project began at a workshop funded by NSF RTG grant DMS-1839968. The first-named author was supported by NSF grant DMS-1906281. The third-named author was supported by NSF GRFP grant DGE-1845298. The fourth-named author was partially supported by NSF grant DMS-2104300.

[HM21], with an application to operad theory. Let us briefly summarize these results; more details and formal definitions can be found in Section 2.

Let $\mathcal{T}(\mathbf{n})$ be the category of reduced $\mathbf{n}$-trees of Definition 2.5. There are two ways to show that $\mathcal{P}(\mathbf{n})$ and $\mathcal{T}(\mathbf{n})$ have suitably equivalent geometric realizations, as given by the following zig-zags of topological spaces:

$$|\mathcal{P}(\mathbf{n})| \xleftarrow{\simeq} |\Delta\mathcal{P}(\mathbf{n})| \xrightarrow{\simeq} |\mathcal{T}(\mathbf{n})|$$

$$|\mathcal{P}(\mathbf{n})| \xleftarrow{\cong_{\Sigma_n}} \mathbb{T}(\mathbf{n}) \xrightarrow{\cong_{\Sigma_n}} |\mathcal{T}(\mathbf{n})|.$$

The first zig-zag uses the category of simplices $\Delta\mathcal{P}(\mathbf{n})$ of the nerve of $\mathcal{P}(\mathbf{n})$, and both maps are homotopy equivalences, obtained via an application of Quillen's Theorem A. The argument for why the left-hand map is a homotopy equivalence can be found in [Dug06], while the proof for the right hand map is given by Heuts and Moerdijk [HM21]. As part of our work, we show that these maps can be upgraded to Σ_n-equivariant homotopy equivalences. Notably, this second composite homeomorphism does not arise from a map between categories or simplicial sets.

The second zig-zag instead uses the space $\mathbb{T}(\mathbf{n})$ of measured $\mathbf{n}$-trees given in Definition 2.12, and the maps are Σ_n-equivariant homeomorphisms. The proof for the left-hand map was given by Robinson [Rob04, Theorem 2.7], and we give an argument for the right-hand map in Theorem 6.7.

Our goal in this paper is to show that these results hold in a G-equivariant setting, where G is a finite group. As a first step, we must introduce G-equivariant versions of the structures involved, and we find that there are several possible ways to define both G-equivariant partition complexes and G-trees, depending on "how equivariant" we ask them to be.

Given a finite set A, we can encode a partition of A as a surjective function $A \twoheadrightarrow \mathbf{k}$ for some $\mathbf{k}$. If A is now a finite G-set, this notion of partition still makes sense, as we can consider surjective functions on the underlying sets. Alternatively, we can ask for a non-trivial G-action on the target as well. That is, we can encode a partition of A as a surjective function $A \to B$ where B is some finite G-set, and either ask that the surjective map be equivariant or not. These distinctions are summarized in Table 1, and more details can be found in Subsection 4.1.

	Less equivariant		More equivariant
Partitions	$A \twoheadrightarrow \mathbf{k}$ non-equivariant	$A \twoheadrightarrow B$ non-equivariant	$A \twoheadrightarrow B$ equivariant
Partition complex	$\lvert\mathcal{P}(A)\rvert$	$\lvert\mathcal{P}_G(A)\rvert$	$\lvert\mathcal{P}^G(A)\rvert$

TABLE 1. Equivariant partitions

There are inclusions of poset categories $\mathcal{P}(A) \hookrightarrow \mathcal{P}_G(A) \hookleftarrow \mathcal{P}^G(A)$; however, it turns out that the two extreme cases, $\mathcal{P}(A)$ and $\mathcal{P}^G(A)$, have the most interesting connections to trees. The relevant types of G-trees are summarized in Table 2; see Section 5 for definitions and details.

	Less equivariant	More equivariant
Trees	A-labeled trees with G-action on leaves	A-labeled trees with G-action on entire tree
Category of trees	$\mathcal{T}(A)$	$\mathcal{T}^G(A)$
Space of trees	$\mathbb{T}(A)$	$\mathbb{T}^G(A)$

Table 2. Equivariant trees

Each of these notions of G-trees has the expected interaction with the corresponding notion of G-equivariant partition; we thus obtain two different equivariant analogues of the zig-zags of equivalences above. To prove these results, we develop equivariant versions of Quillen's Theorems A and B (Theorems A.1 and A.6), which we consider of independent interest.

Theorem 1.1 (Corollary 6.6, Theorem 6.7, Theorem 6.10)**.** *There are G-equivariant zig-zags of G-spaces*

$$|\mathcal{P}(A)| \xleftarrow{\simeq_G} |\Delta\mathcal{P}(A)| \xrightarrow{\simeq_G} |\mathcal{T}(A)|$$

$$|\mathcal{P}(A)| \xleftarrow{\cong_G} \mathbb{T}(A) \xrightarrow{\cong_G} |\mathcal{T}(A)|.$$

By taking fixed points, we obtain the analogous zig-zags relating $|\mathcal{P}^G(A)|$, $\mathbb{T}^G(A)$, and $|\mathcal{T}^G(A)|$.

There are many applications of partition complexes and trees in the literature, and we can ask which of these applications have G-equivariant versions. We address two of these questions here. The first is the computation of the homotopy type of a partition complex. In the non-equivariant setting, these homotopy types are given by wedges of spheres; in contrast, the situation for G-partition complexes is much more subtle.

Theorem 1.2 (Proposition 4.5, Proposition 7.5, Proposition 7.11)**.** *Let A be a finite G-set and $\downarrow_H^G A$ be the restriction of A to an H-set for $H \leq G$. Then*

$$|P(A)|^H \simeq |P^H(\downarrow_H^G A)|$$

is non-contractible only if $\downarrow_H^G A \cong_G \coprod_{i=1}^n H/K$ for some $K \leq H$.

This question was also addressed by Arone and Brantner [AB21], and some of our results in Section 7 recover some of theirs, although our proofs are different. The second question is the computation of the homology groups of spaces of trees. In the classical setting, Robinson [Rob04] showed that these groups are related to the Lie algebra operad via a twisted action of the integral sign representation of Σ_n. We obtain an analogous result for our "less equivariant" G-trees by considering instead the integral sign representation of G.

Theorem 1.3 (Theorem 8.2)**.** *For any finite G-set A, there is an isomorphism of G-modules*

$$H^{n-3}(\mathbb{T}(A)) \cong \varepsilon_A^G \otimes \mathrm{Lie}_A,$$

where ε_A^G is the sign representation of G induced by the action on A.

In Proposition 8.3 we also explore the homology of our space of "fully equivariant" G-trees in relation to our understanding of the homotopy type of their corresponding partitions.

Outline of the paper. In Section 2, we summarize the non-equivariant comparison between partition complexes and trees, and in Section 3 we review some of the equivariant homotopy theory that we use. We begin to set up the equivariant picture in Section 4 by defining equivariant partition complexes, and we analogously define equivariant trees in Section 5, and then in Section 6 we establish Theorem 1.1. In Section 7 we discuss the homotopy type of the equivariant partition complexes, and in Section 8 we discuss the equivariant analogues of results relating the homology of spaces of trees to Lie algebras.

Acknowledgements. This project was started at the Collaborative Workshop in Algebraic Topology in August 2022, supported by the Geometry and Topology RTG grant at University of Virginia. We would like to thank the other participants of this workshop for an enjoyable and productive week, and the hosts at the workshop site for their hospitality.

2. A review of the partition complexes and trees

In this section we review partition complexes, categories of trees, and the relationship between them in the non-equivariant setting. We begin with partition complexes. First, let us fix a finite set $\mathbf{n} = \{1, \ldots, n\}$ and consider the poset category $\mathcal{P}(\mathbf{n})$ of non-trivial partitions of $\mathbf{n}$, ordered by coarsening, where we omit the discrete and indiscrete partitions. To turn this category into a topological space, we use the classifying space construction.

Definition 2.1. The *nerve* of a category $\mathcal{C}$, denoted by $N\mathcal{C}$, is the simplicial set whose n-simplices are chains of n composable morphisms

$$c_0 \to c_1 \to \cdots \to c_n,$$

where the degeneracy maps s_j insert the identity $c_j \xrightarrow{\mathrm{id}} c_j$ for $0 \leq j \leq n$ and the face maps d_i compose $c_{i-1} \to c_i \to c_{i+1}$ for $1 \leq i \leq n-1$; the map d_0 forgets the first morphism and d_n forgets the final morphism. The *classifying space* of $\mathcal{C}$ is the geometric realization of the nerve,

$$|\mathcal{C}| := |N\mathcal{C}|.$$

Definition 2.2. The *partition complex* of $\mathbf{n}$ is the classifying space $|\mathcal{P}(\mathbf{n})|$ of $\mathcal{P}(\mathbf{n})$.

Remark 2.3. Other authors, including Heuts and Moerdijk [HM21], use the refinement relation on $\mathcal{P}(\mathbf{n})$ instead. We have chosen to use coarsening since it generalizes more conveniently to the equivariant setting in Section 4. Ultimately, the choice does not matter on the level of classifying spaces.

Definition 2.4. For any category $\mathcal{C}$, the *category of simplices* is the over-category $\Delta\mathcal{C} := \Delta^{(-)} \downarrow N\mathcal{C}$. Explicitly, the objects are the k-simplices of the nerve, i.e., chains of length k of arrows in $\mathcal{C}$, and morphisms are generated by the face and degeneracy maps.

There is a functor $\Delta\mathcal{C} \to \mathcal{C}$ that sends a chain of arrows to its ultimate target, called the *last vertex* functor.

Using the discussion preceding Theorem 2.4 in [Dug06], the last vertex map is homotopy initial (Definition 3.11) and hence by Quillen's Theorem A induces a homotopy equivalence on classifying spaces. It follows that $|\Delta\mathcal{P}(\mathbf{n})|$ is another model for the partition complex.

We now introduce several varieties of trees, studied in [Rob04] and [HM21], that connect to the partition complex.

By a *tree*, we always mean a finite tree whose internal edges are attached to a vertex at both ends, but whose external edges are only attached to a single vertex. One external edge is distinguished as the *root* of the tree, and the other external edges are called *leaves*. The tree is oriented from the leaves down to the root. Additionally, our trees are prohibited from having nullary vertices; see Example 2.7 for a visual.

Notation. For a tree T, we denote by $L(T)$, $V(T)$, and $E^i(T)$ the sets of leaves, vertices, and inner edges of T, respectively.

Definition 2.5. For any $n > 0$, an *n-labeled tree*, or simply $\mathbf{n}$-*tree*, is a tree equipped with a labeling bijection $\mathbf{n} \to L(T)$.

- We say an $\mathbf{n}$-tree is *layered* if there is a constant number of inner edges between any leaf and the root.
- We say an $\mathbf{n}$-tree is *reduced* if there are no unary vertices.
- We say an $\mathbf{n}$-tree is *measured* if it is equipped with the additional data of an assignment $E^i(T) \to (0, 1]$ giving every inner edge a length in $(0, 1]$, such that at least one inner edge has length 1.

An *isomorphism* of (reduced) $\mathbf{n}$-trees is a root-preserving homeomorphism. It is an isomorphism of layered trees if it also preserves the labels, and an isomorphism of measured trees if it preserves edge measurements.

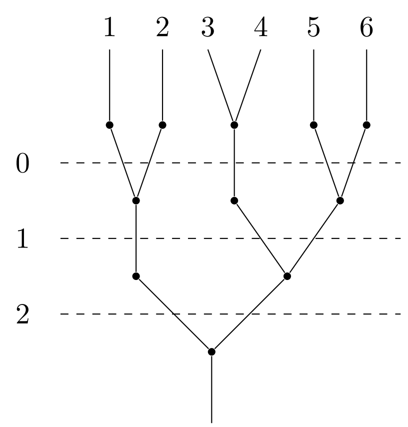
Remark 2.6. What we call reduced $\mathbf{n}$-trees are simply called $\mathbf{n}$-*trees* in [HM21] and [Rob04]. Robinson uses the term *fully grown* $\mathbf{n}$-trees for what we call measured $\mathbf{n}$-trees.

Let us look at these different kinds of trees in more depth. First, we observe that the category of simplices $\Delta\mathcal{P}(\mathbf{n})$ is isomorphic to the category of (isomorphism classes of) layered $\mathbf{n}$-trees, with face maps contracting an entire layer and degeneracy maps inserting a layer of unary edges.

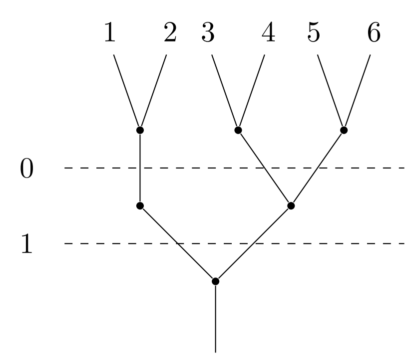
Example 2.7. Let $\mathbf{n} = \mathbf{6}$ and consider the 2-simplex

$$(1)(2)(34)(5)(6) \leq (12)(34)(56) \leq (12)(3456)$$

in $N\mathcal{P}(\mathbf{6})$. This chain of partitions corresponds to the layered tree with 3 internal layers below:



Here, layer 0 corresponds to (1)(2)(34)(5)(6), layer 1 to (12)(34)(56), and layer 2 to (12)(3456). The face map d_0 contracts the 0-th layer, i.e. all the edges that intersect with the dashed line labeled by 0, resulting in the tree

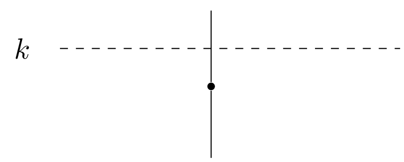


that corresponds to the chain $(12)(34)(56) \leq (12)(3456)$. We leave it to the reader to compute the other face maps as well as the degeneracy maps.

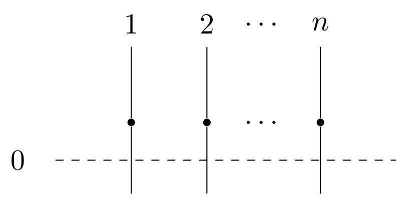
We say a layered tree is *non-degenerate* if its associated simplex is. Visually, this condition means that there is no layer whose vertices are all unary.

Additionally, a layered tree is *elementary* if every layer contains exactly one non-unary vertex. A vertex is in a layer if it is the source of an edge in the layer. Both examples above are non-degenerate, but neither is elementary.

Remark 2.8. The exclusion of the trivial partitions in $\mathcal{P}(\mathbf{n})$ imposes restrictions on what a layered tree can look like before the first layer and after the final layer. Specifically, excluding the coarsest partition means we do not allow the layer closest to the root in any k-simplex to be degenerate:

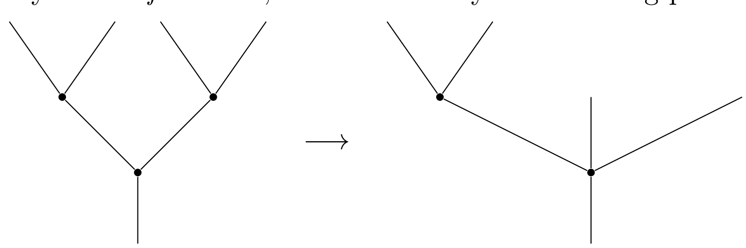


and excluding the finest partition means we do not allow the 0-th layer to be degenerate:

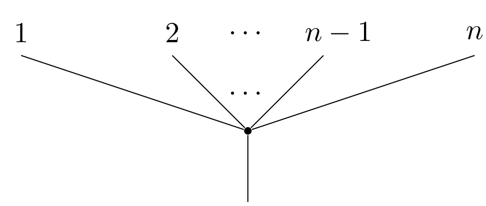


Remark 2.9. Note that that a k-simplex in $\Delta\mathcal{P}(\mathbf{n})$ determines a layered $\mathbf{n}$-tree only up to label-preserving isomorphism. For example, in the second layered tree from Example 2.7, we may swap the labelings of 3 and 4 and, independently, 5 and 6.

Definition 2.10. Let $\mathcal{T}(\mathbf{n})$ denote the poset whose objects are isomorphism classes of reduced $\mathbf{n}$-trees, where there is a unique morphism $T \to T'$ if T' can be obtained from T by contracting a collection of inner edges. In this case we say T' is a *face* of T, as illustrated by the following picture:



This category has an terminal object, the *corolla* $C_{\mathbf{n}}$,



but for the rest of this paper we omit this object from $\mathcal{T}(\mathbf{n})$.

Remark 2.11. The category $\mathcal{T}(\mathbf{n})$ is the *opposite* of the category denoted by $\mathcal{T}_+(n)$ in [HM21], due to our choice of ordering $\mathcal{P}(\mathbf{n})$ via coarsening; see Remark 2.3. As a consequence, these arrows are the opposite of the maps of trees in the dendroidal category Ω.

In [HM21], Heuts and Moerdijk show that the functor

$$\Delta\mathcal{P}(\mathbf{n}) \longrightarrow \mathcal{T}(\mathbf{n})$$

that collapses unary vertices and forgets layers is homotopy final (Definition 3.11), and so again induces a homotopy equivalence $|\mathcal{P}(\mathbf{n})| \simeq |\mathcal{T}(\mathbf{n})|$ on classifying spaces.

Finally, we discuss the space of measured $\mathbf{n}$-trees, following [RW96] and [Rob04].

Definition 2.12. We denote the space of (isomorphism classes of) measured $\mathbf{n}$-trees by $\mathbb{T}(\mathbf{n})$. It is defined as the simplicial complex whose vertices are the measured $\mathbf{n}$-trees with exactly one inner edge. A k-simplex of $\mathbb{T}(\mathbf{n})$ corresponds to a shape of fully grown tree with $k+1$ inner edges, whose vertices are obtained by collapsing all but one inner edge which is then assigned weight 1.

Points in such a simplex consist of measured $\mathbf{n}$-trees of that shape. In other words, points in a simplex are obtained by assigning lengths to all the inner edges in that simplex shape, and in turn, these lengths determine the barycentric coordinates of the point.

Remark 2.13. Alternatively and equivalently, [RW96] defines the topology on $\mathbb{T}(\mathbf{n})$ as given by the cubical complex structure where two trees are in the same open cube if there is a label-preserving homeomorphism between them which also preserves edges of length 1. The internal edge lengths determine the coordinates within each cube. We do not make use of that description here.

Robinson produces an explicit homeomorphism $\mathbb{T}(\mathbf{n}) \to |\mathcal{P}(\mathbf{n})|$ in [Rob04, Theorem 2.7]. Moreover, a similar argument shows there is a homeomorphism $\mathbb{T}(\mathbf{n}) \to |\mathcal{T}(\mathbf{n})|$ as well; see Theorem 6.7. All together, we have a zig-zag of homeomorphisms

$$|\mathcal{P}(\mathbf{n})| \longleftarrow \mathbb{T}(\mathbf{n}) \longrightarrow |\mathcal{T}(\mathbf{n})|;$$

however, it does not appear to arise from any functors between these categories. In the same paper, Robinson also shows that $\mathbb{T}(\mathbf{n})$ is a simplicial complex with the homotopy type of a wedge of spheres and studies the connection between measured $\mathbf{n}$-trees and Lie representations.

3. Background on equivariant homotopy theory

In equivariant homotopy theory, we consider familiar objects like sets, spaces, or categories, except now we give these objects the extra structure of a group action. Throughout this paper, we assume the group G is finite.

This idea of equipping an object in a category $\mathcal{C}$ with a G-action is nicely encapsulated by a functor from the one-object groupoid whose morphism group is G. We use BG to refer to both the one-object category and the classifying space of this category, clarifying if it is not clear from context which we mean.

Definition 3.1. A *$\mathcal{C}$-object with G-action* is an object of $\mathrm{Fun}(BG,\mathcal{C})$, the category of functors from BG to $\mathcal{C}$. *Equivariant morphisms*, or simply *G-morphisms*, are natural transformations of these functors, and we typically denote the resulting category by $G\mathcal{C}$.

We primarily consider the following examples.

- For $\mathcal{C} = \mathcal{S}et$ the category of sets, a *G-set* is a set A together with a G-action map $G \times A \to A$ so that $e \cdot a = a$ and $(gg') \cdot a = g \cdot (g' \cdot a)$ for all $a \in A$ and $g, g' \in G$. A *G-map* of G-sets is a set map $f\colon A \to A'$ so that $g \cdot f(a) = f(g \cdot a)$ for all $a \in A$ and $g \in G$. The category of G-sets is denoted by $G\mathcal{S}et$. We can also restrict to $\mathcal{C} = \mathcal{F}in$, the category of finite sets, to get a category of finite G-sets, denoted by $G\mathcal{F}in$.
- For $\mathcal{C} = \mathcal{T}op$ the category of compactly generated weak Hausdorff spaces, a *G-space* is a topological space X along with a continuous map $G \times X \to X$, where G is given the discrete topology. A *G-map* of G-spaces is a continuous map which is equivariant on the underlying sets. The category of G-spaces is denoted by $G\mathcal{T}op$.
- For $\mathcal{C} = \mathcal{C}at$ the category of small categories, a *category with G-action* is a category $\mathcal{D}$ with action functors $(g\cdot)\colon \mathcal{D} \to \mathcal{D}$ for each $g \in G$ so that $(e\cdot) = \mathrm{id}_{\mathcal{D}}$ and $(g\cdot) \circ (g'\cdot) = (gg'\cdot)$. A *$G$-functor* is a functor $F\colon \mathcal{D} \to \mathcal{D}'$ so that $g \cdot F(d) = F(g \cdot d)$ and $g \cdot F(f) = F(g \cdot f)$ for all objects d of $\mathcal{D}$, all morphisms f of $\mathcal{D}$, and all $g \in G$. This data assembles into a category, denoted by $G\mathcal{C}at$.

Remark 3.2. What we call categories with G-action are sometimes called *strict* G-categories, and G-functors between them are called *strict* G-functors. In some situations, it can be helpful to consider *pseudo* G-categories where the G-actions are only associative and unital up to natural isomorphism. In all of the examples in this paper the actions are strictly associative and unital so we do not need to make this distinction.

3.1. **Preliminaries on equivariant topological spaces.** We briefly review some basic ideas in the context of G-spaces, specifically, although the results we cite here have analogues in the setting of G-sets and G-categories. Our exposition primarily follows [May96], and another well-known reference is [LMS80].

Many non-equivariant constructions on spaces work equally well equivariantly. For example, if X and Y are two G-spaces, then G acts diagonally on their product $X \times Y$ and by conjugation on the set $\mathrm{Hom}_{\mathcal{T}op}(X, Y)$, i.e., for any $f \colon X \to Y$, we get $g \cdot f \colon X \to Y$ by

$$X \xrightarrow{g^{-1}} X \xrightarrow{f} Y \xrightarrow{g} Y.$$

Additionally, in the equivariant setting, we have access to new structures that can be associated to subgroups $H \leq G$.

Definition 3.3. Let X be a G-space and $H \leq G$.

- The *H-fixed points* of X are given by the space
$$X^H := \{x \in X \mid h \cdot x = x \text{ for all } h \in H\}.$$
- The *H-orbits* of X, denoted by X/H, is the quotient space of X by the equivalence relation generated by $x \sim h \cdot x$ for all $h \in H$.
- For $x \in X$, the *isotropy subgroup*, or *stabilizer*, of x is
$$G_x := \{g \in G \mid g \cdot x = x\} \leq G.$$

Note that $x \in X^H$ precisely when $H \leq G_x$. Both X^H and X/H have the structure of $W_G H$-spaces, where $W_G H = N_G H/H$ is the *Weyl group* of H in G. Here, $N_G H$ denotes the normalizer of H in G.

Remark 3.4. If the action of G on X is transitive, that is, if X is the orbit of x for any $x \in X$, then $X \cong_G G/G_x$ by the Orbit-Stabilizer Theorem.

Remark 3.5. Note that the G-fixed points of the conjugation action on $\mathrm{Hom}_{\mathcal{T}op}(X, Y)$ are precisely the G-maps $\mathrm{Hom}_{G\mathcal{T}op}(X, Y)$.

The functors $G\mathcal{T}op \to \mathcal{T}op$ that take a G-space X to its H-fixed points and H-orbits are the right and left adjoints, respectively, of the functor $\mathcal{T}op \to G\mathcal{T}op$ that gives a space the trivial G-action. That is, for a space A with trivial G-action, we have

$$\mathrm{Hom}_{G\mathcal{T}op}(A, X) \cong \mathrm{Hom}_{\mathcal{T}op}(A, X^G) \quad \text{and} \quad \mathrm{Hom}_{G\mathcal{T}op}(X, A) \cong \mathrm{Hom}_{\mathcal{T}op}(X/G, A).$$

Given $H \leq G$, we can also consider the *restriction* functor $\downarrow_H^G \colon G\mathcal{T}op \to H\mathcal{T}op$ that only remembers the H-action. This functor admits a left adjoint

$$\uparrow_H^G \colon H\mathcal{T}op \to G\mathcal{T}op$$

called *induction*. Given an H-space Y, the induction of Y is the balanced product

$$\uparrow_H^G (Y) = G \times_H Y = G \times Y/\sim,$$

where $\sim$ is the relation generated by $(g, h \cdot y) \sim (gh, y)$ for $g \in G$, $y \in Y$, and $h \in H$. If X is a G-space, rather than just an H-space, then

$$G \times_H X \cong_G G/H \times X$$

as G-spaces.

Definition 3.6. A *homotopy between G-maps $X \to Y$* is a homotopy $H\colon X \times I \to Y$ that is also a G-map, where I is given the trivial G-action. A G-map $f\colon X \to Y$ is a *(weak) G-equivalence* if it is a (weak) equivalence upon passage to H-fixed points $f^H\colon X^H \to Y^H$ for each $H \leq G$.

Taking $H = e$, we see that such an f needs to be a homotopy equivalence of the underlying spaces. In light of the definition above, much of equivariant homotopy theory amounts to non-equivariant homotopy theory of fixed-point spaces.

3.2. **Preliminaries on equivariant classifying spaces.** In this subsection, we establish some basic facts about classifying spaces of categories with G-action.

Definition 3.7. Let $\mathcal{C}$ be a category with G-action. The nerve of $\mathcal{C}$ is the same simplicial set from Definition 2.1, but now is equipped with a G-action given objectwise, with

$$g \cdot (c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n) = gc_0 \xrightarrow{gf_1} gc_1 \xrightarrow{gf_2} \dots \xrightarrow{gf_n} gc_n.$$

This action makes $N\mathcal{C}$ a G-object in $s\mathcal{S}et$, or equivalently a simplicial G-set, which in turn makes $|\mathcal{C}|$ a G-space.

This construction is functorial, in that a G-functor induces a G-map of classifying spaces. Note that a G-functor also restricts to a functor on H-fixed points $\mathcal{C}^H \to \mathcal{D}^H$, so we also get maps on fixed points of nerves and classifying spaces. On nerves, we have that $(N\mathcal{C})^H = N(\mathcal{C}^H)$, and the following proposition implies that taking fixed points also commutes with taking classifying spaces.

Proposition 3.8. *For any $H \leq G$ and simplicial G-space X, taking H-fixed points commutes with geometric realization, i.e., there is a homeomorphism $|X^H| \cong |X|^H$.*

Proof. There is a natural inclusion $|X^H| \to |X|^H$. This map is also surjective because any equivalence class $[x, t] \in |X|^H$ can be represented by a H-fixed point of X, since one of the faces of x must be H-fixed. □

A *G-natural transformation* $\eta\colon F \Rightarrow F'$ can be defined as a G-functor $\eta\colon \mathcal{C} \times \{0 \to 1\} \to \mathcal{D}$ so that $\eta(-, 0) = F$ and $\eta(-, 1) = F'$. Here $\{0 \to 1\}$ is the poset category with trivial G-action. A routine check shows that this data is equivalent to the usual data of a natural transformation so that $\eta_{g\cdot c}(-) = \eta_c(g \cdot -)$ for each component η_c.

Proposition 3.9. *A G-natural transformation $\eta\colon F \Rightarrow F'$ induces a G-homotopy $|F| \simeq |F'|$.*

Proof. By functoriality and the fact that $|\mathcal{C} \times \{0 \to 1\}| \cong |\mathcal{C}| \times |\{0 \to 1\}|$ [Seg68, §2], the map $|\eta|\colon |\mathcal{C}| \times I \to |\mathcal{D}|$ is a homotopy between $|\eta(-, 0)| = |F|$ and $|\eta(-, 1)| = |F'|$. It is a G-homotopy by functoriality of taking classifying spaces and the fact that G acts trivially on $|\{0 \to 1\}| = I$. □

For the following corollary, recall that a *lax inverse* of a functor $F\colon \mathcal{C} \to \mathcal{D}$ is a functor $F'\colon \mathcal{D} \to \mathcal{C}$ such that both FF' and $F'F$ have natural transformations to or from the relevant identity functors. There are no conditions imposed on the interactions of these natural transformations, so this notion is strictly weaker than that of an adjunction, although adjoint pairs and equivalences of categories are examples of lax inverses.

Corollary 3.10. *The realization of a G-functor that admits an equivariant lax inverse is a G-homotopy equivalence.*

Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *homotopy initial* (respectively, *homotopy final*) if the overcategories $F \downarrow d$ (respectively, undercategories $d \downarrow F$) are contractible for every object d of $\mathcal{D}$. Quillen's Theorem A [Qui73, §1] shows that such a functor induces a homotopy equivalence on classifying spaces. We can generalize this notion to G-functors.

Definition 3.11. A G-functor $F\colon \mathcal{C} \to \mathcal{D}$ between G-categories is *G-homotopy initial* (respectively, *G-homotopy final*) if the overcategories $F \downarrow d$ (respectively undercategories $d \downarrow F$) are G_d-contractible for every object d of $\mathcal{D}$.

Remark 3.12. In [DM16], Dotto and Moi instead use the terminology *left G-cofinal* rather than G-initial, and *right G-cofinal* rather than G-final.

In Appendix A, we prove that the realization of a G-homotopy initial or final functor is a G-equivalence on classifying spaces, in the form of an equivariant version of Quillen's Theorem A.

When $\mathcal{C}$ is a category with G-action, its category of simplices inherits a G-action and the last vertex functor φ is a G-functor. In Corollary A.3, we show that this functor is G-homotopy initial (Definition 3.11), which has the following consequence for the partition complex.

Corollary 3.13. *The last vertex functor $\Delta\mathcal{P}(\mathbf{n}) \to \mathcal{P}(\mathbf{n})$ is Σ_n-homotopy initial.*

4. G-partition complexes

We now introduce equivariant versions of partition complexes; that is, we develop an analogue of $\mathcal{P}(\mathbf{n})$, where the finite set $\mathbf{n}$ is replaced with a G-set A such that $|A| = n$.

To figure out what we mean by a partition of a G-set A, we first note that the data of a partition of $\mathbf{n}$ can be encoded as the equivalence class of a surjective function $\mathbf{n} \twoheadrightarrow \mathbf{k}$, modulo the action by Σ_k on $\mathbf{k}$. As an example, the partition

$$(12)(345)(6)$$

can be expressed as the function $\mathbf{6} \twoheadrightarrow \mathbf{3}$ given by

$$1,2 \mapsto 1,\ \ 3,4,5 \mapsto 2,\ \ 6 \mapsto 3.$$

The role of the equivalence relation is to identify this mapping with, for example, the map

$$1, 2 \mapsto 2,\ \ 3, 4, 5 \mapsto 1,\ \ 6 \mapsto 3,$$

that determines the same partition.

From this perspective, there are several natural ways to extend this notion to account for a G-action:

- through non-equivariant functions $A \twoheadrightarrow \mathbf{k}$ where $\mathbf{k}$ has the trivial G-action;
- through G-maps $A \twoheadrightarrow B$ where A and B are G-sets; or
- through non-equivariant functions $A \twoheadrightarrow B$ where A and B are G-sets.

We focus on the first two notions; see Remark 5.16 for a discussion on why we choose to ignore the third.

4.1. G**-partitions.** We now explore the first notion of G-partitions described above.

Definition 4.1. For any G-set A, let $\mathcal{P}(A)$ denote the G-poset of non-trivial partitions of $\downarrow_e^G A$, the underlying set of A, ordered by coarsening.

Equivalently, we can describe $\mathcal{P}(A)$ as the category whose objects are equivalence classes of non-equivariant surjections $A \twoheadrightarrow \mathbf{k}$ modulo the action by Σ_k, and arrows $(A \twoheadrightarrow \mathbf{k}) \to (A \twoheadrightarrow \mathbf{j})$ are factorizations

$$\begin{tikzcd} & A \arrow[dl, two heads] \arrow[dr, two heads] & \\ \mathbf{k} \arrow[rr, two heads] & & \mathbf{j}. \end{tikzcd}$$

As in the non-equivariant case, the trivial partitions $A \to |A|$ and $A \to \mathbf{1}$ are excluded.

Note that this data is well-defined and indeed forms a poset, since by surjectivity of $A \twoheadrightarrow \mathbf{k}$, any two such maps $\mathbf{k} \twoheadrightarrow \mathbf{j}$ must agree, and this factorization determines a unique factorization between any two elements of the equivalence classes of the legs. Moreover, $\mathcal{P}(A)$ is a G-poset, since an element $g \in G$ acts on $\mathcal{P}(A)$ by precomposition with its inverse; that is, g sends $A \twoheadrightarrow \mathbf{k}$ to $A \xrightarrow{g^{-1}} A \twoheadrightarrow \mathbf{k}$.

The following result gives us information about how to relate $\mathcal{P}(A)$ and $\mathcal{P}(\mathbf{n})$, as well as their categories of simplices. The proof is omitted, as it merely consists of a detailed unpacking of the definitions involved.

Lemma 4.2. *Let A be a G-set with $|A| = n$, and let $\alpha\colon G \to \Sigma_n$ denote the group homomorphism encoding the G-action. Then*

$$\mathcal{P}(A) = \alpha^* \mathcal{P}(\mathbf{n})$$

and

$$\Delta\mathcal{P}(A) = \alpha^* \Delta\mathcal{P}(\mathbf{n}).$$

Just as for $P(\mathbf{n})$, the last vertex functor for $P(A)$ is G-homotopy initial; see Corollary A.3.

Corollary 4.3. *For any G-set A, the last vertex functor $\Delta\mathcal{P}(A) \to \mathcal{P}(A)$ is G-homotopy initial; in particular, it is a homotopy equivalence.*

The second notion of equivariant partitions is as follows.

Definition 4.4. Let $\mathcal{P}^G(A)$ denote the poset of non-trivial equivariant partitions of A, ordered by coarsening.

In other words, $\mathcal{P}^G(A)$ is the category whose objects are equivalence classes of G-surjections between G-sets $A \twoheadrightarrow B$ modulo the action by $\mathrm{Aut}_G(B)$, and whose arrows are factorizations $A \twoheadrightarrow B' \twoheadrightarrow B$ for which all maps are equivariant. The trivial partitions given by G-isomorphisms $A \xrightarrow{\cong} B$ and by the constant map $A \to \mathbf{1}$ are excluded.

Since the objects of $\mathcal{P}^G(A)$ consist of G-maps, the natural G-action on $\mathcal{P}^G(A)$ is the trivial one.

4.2. **Interactions through fixed points.** Studying the fixed points of these G-posets yields interesting interactions between our different notions of G-partitions.

Proposition 4.5. *For any $H \leq G$ there is an equivalence of categories*

$$\mathcal{P}(A)^H \simeq \mathcal{P}^H(\downarrow_H^G A).$$

Proof. To simplify notation, we leave the $\downarrow_H^G$ implicit and simply treat A as an H-set. We begin by defining an auxiliary category $\mathcal{P}^H_{\mathrm{ord}}(A)$ which has as objects the equivalence classes of H-surjections $f\colon A \twoheadrightarrow B$, where B is an H-set equipped with a total ordering. The morphisms in this category are the same as those of $\mathcal{P}^H(A)$; in particular, we do not require morphisms to respect the ordering. With these morphisms, one can see that the functor $\mathcal{P}^H_{\mathrm{ord}}(A) \to \mathcal{P}^H(A)$ that forgets the orderings is an equivalence of categories. It remains to check that $\mathcal{P}^H_{\mathrm{ord}}(A)$ is categorically equivalent to $\mathcal{P}(A)^H$.

Given $f\colon A \twoheadrightarrow B$ in $\mathcal{P}^H_{\mathrm{ord}}(A)$, the total ordering on B determines a unique bijection $B \xrightarrow{\cong} \mathbf{k_B}$ where $k_B = |B|$. Define a functor $F\colon \mathcal{P}^H_{\mathrm{ord}}(A) \to \mathcal{P}(A)^H$ which sends the class of a map $f\colon A \twoheadrightarrow B$ to the class of

$$A \xrightarrow{f} B \xrightarrow{\cong} \mathbf{k_B}.$$

Note that $F(f)$ is H-fixed because for any $h \in H$, the fact that $hfh^{-1} = f$ implies that $F(f)$ and $F(f) \circ h^{-1}$ are the same up to an automorphism of $\mathbf{k_B}$, namely the one determined by h. Similar reasoning shows that F is well defined, since varying the representative $f\colon A \twoheadrightarrow B$ of an equivalence class by an H-automorphism of B only changes the value of $F(f)$ by an automorphism of $\mathbf{k_B}$.

If $s\colon B \twoheadrightarrow B'$ defines a morphism in $\mathcal{P}^H_{\mathrm{ord}}(A)$, we define $F(s)$ to be the unique map that fills the following square:

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "\cong"'] & B' \arrow[d, "\cong"] \\ \mathbf{k_B} \arrow[r, "F(s)"'] & \mathbf{k'_B}. \end{tikzcd}$$

We want to show that F is an equivalence of categories. Note that since $\mathcal{P}^H_{\mathrm{ord}}(A)$ is equivalent to a poset, its hom-sets all have size 0 or 1, and so the functor F is automatically faithful.

First we show that F is surjective on objects. Let $f\colon A \twoheadrightarrow \mathbf{k}$ represent an object in $\mathcal{P}(A)^H$, which means that for each $h \in H$ there exists a (necessarily unique) bijection $\sigma_h\colon \mathbf{k} \to \mathbf{k}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, two heads, "f"'] & A \arrow[d, two heads, "f"] \\ \mathbf{k} \arrow[r, "\sigma_h"'] & \mathbf{k}. \end{tikzcd}$$

Thus $\mathbf{k}$ is endowed with the H-action given by $hi = \sigma_h(i)$ for all $i \in \mathbf{k}$ and $h \in H$. Note that the uniqueness of σ_h ensures that $\sigma_e = \mathrm{id}$ and $\sigma_{h_1}\sigma_{h_2} = \sigma_{h_1h_2}$ and we do indeed get an H-action. This action is defined so that $f\colon A \twoheadrightarrow \mathbf{k}$ is an H-map which determines an object in $\mathcal{P}^H_{\mathrm{ord}}(A)$ whose image under F is equal to $f\colon A \twoheadrightarrow \mathbf{k}$.

It remains to show that F is full. Given a morphism $\varphi\colon \mathbf{k} \twoheadrightarrow \mathbf{j}$ between objects $f\colon A \twoheadrightarrow \mathbf{k}$ and $f'\colon A \twoheadrightarrow \mathbf{j}$ in $\mathcal{P}(A)^H$, consider the following diagram:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, two heads, "f"] \arrow[dd, two heads, bend right=60, "f'"'] & A \arrow[d, two heads, "f"'] \arrow[dd, two heads, bend left=60, "f'"] \\ \mathbf{k} \arrow[r, "\sigma_h^{\mathbf{k}}"] \arrow[d, two heads, "\varphi"] & \mathbf{k} \arrow[d, two heads, "\varphi"'] \\ \mathbf{j} \arrow[r, "\sigma_h^{\mathbf{j}}"] & \mathbf{j}. \end{tikzcd}$$

The lower square commutes when precomposed with the surjection f, which implies that the square itself commutes and thus φ is an H-map when $\mathbf{k}$ and $\mathbf{j}$ are given H-actions as above. This data determines a map φ' between the corresponding objects $f\colon A \twoheadrightarrow \mathbf{k}$ and $f'\colon A \twoheadrightarrow \mathbf{j}$ in $\mathcal{P}^H_{\mathrm{ord}}(A)$ with $F(\varphi') = \varphi$, and hence F is full. $\square$

The next result now follows directly.

Corollary 4.6. *For any $H \le G$ there is an equivalence of categories*

$$\Delta\mathcal{P}(A)^H \simeq \Delta\mathcal{P}^H(\downarrow^G_H A).$$

5. G-trees

Having defined several notions of equivariant partitions, we now present the corresponding notions of trees in this equivariant context. We refer the reader back to Definition 2.5 for the analogous non-equivariant definitions.

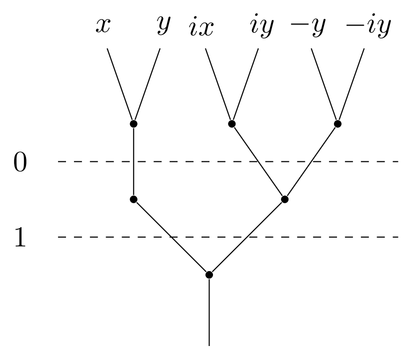
Definition 5.1. For any finite G-set A, an *A-labeled tree*, or simply *A-tree*, is a tree equipped with a non-equivariant labeling bijection from A to the leaves of T. We say an A-tree is *layered, reduced*, or *measured* if the underlying $|A|$-tree is.

An *isomorphism* of (reduced) A-trees is a root-preserving homeomorphism. It is an isomorphism of layered A-trees if it also preserves the labels, and an isomorphism of measured A-trees if it preserves edge measurements.

First, we observe that, as in the non-equivariant case, the category of simplices $\Delta\mathcal{P}(A)$ may be described as the category of (isomorphism classes of) *layered A-trees*.

Example 5.2. Let $G = \Sigma_6$ and $A = \mathbf{6} = \{1, 2, 3, 4, 5, 6\}$. Then both trees from Example 2.7 are examples of layered $\mathbf{6}$-trees.

Example 5.3. Let $G = C_4 = \{1, i, -1, -i\}$ and $A = \{x = -x, ix = -ix, y, -y, iy, -iy\} = C_4 \amalg C_4/C_2$. Then



is the layered A-tree corresponding to the chain of partitions

$$(x, y)(ix, iy)(-y, -iy) < (x, y)(ix, iy, -y, -iy).$$

Equivalently, reading down the layers of this tree, we see that this chain corresponds to the string $A \twoheadrightarrow \mathbf{3} \to \mathbf{2}$, where A, $\mathbf{3}$, and $\mathbf{2}$ correspond to the leaves, the inner edges in layer 0, and the inner edges in layer 1, respectively.

Note that the labeling of the leaves need not correspond in any way to the symmetry of the tree.

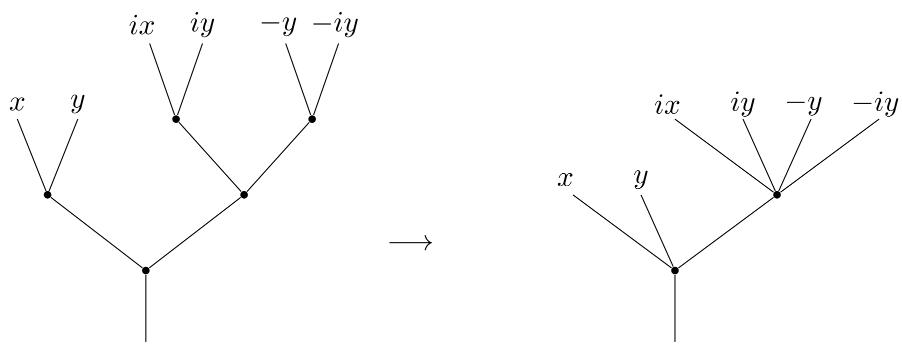
As before, layered A-trees are defined up to label-preserving isomorphism, so, for example, we may swap the labels ix and iy, and independently $-y$ and $-iy$ in the above example.

Next, we consider the category of reduced A-trees.

Definition 5.4. We denote by $\mathcal{T}(A)$ the category whose objects are isomorphism classes of reduced A-trees T, and where there is a unique morphism $T \to T'$ if T' can be obtained from T by contracting a collection of inner edges, and call T' a *face* of T. As we did non-equivariantly, we omit the terminal object given by the corolla tree with no internal edges.

The poset $\mathcal{T}(A)$ naturally has an action by G, where g acts on objects by sending $(T, f\colon A \to L(T))$ to (T, fg^{-1}).

Example 5.5. Let G and A be as in Example 5.3. Then there is a map



in $\mathcal{T}(A)$.

Finally, we consider the G-space of measured A-trees.

Definition 5.6. We denote the G-space of (isomorphism classes of) measured A-trees by $\mathbb{T}(A)$. It is defined as the simplicial complex whose vertices are the measured A-trees with exactly one inner edge. An n-simplex of $\mathbb{T}(A)$ corresponds to a fully grown tree with $n+1$ inner edges whose vertices are obtained by collapsing all but one inner edge which is then assigned weight 1. Points in such a simplex consist of measured A-trees of that shape. In other words, points in a simplex are obtained by assigning lengths to all the inner edges in that simplex shape, and in turn, these lengths determine the barycentric coordinates of the point.

The group G acts on a point of $\mathbb{T}(A)$ by acting on the underlying A-labeled tree.

Analogously to Lemma 4.2, we can establish the following relationship between these new notions of equivariant trees and the classical notions reviewed in Section 2.

Lemma 5.7. *Let A be a G-set with $|A| = n$, and let $\alpha\colon G \to \Sigma_n$ denote the group homomorphism encoding the G-action. Then there is an isomorphism of G-categories*

$$\mathcal{T}(A) \cong_G \alpha^* \mathcal{T}(\mathbf{n})$$

and a G-homeomorphism between spaces

$$\mathbb{T}(A) \cong_G \alpha^* \mathbb{T}(\mathbf{n}).$$

In order to visualize the equivariant partitions introduced in Definition 4.4 properly, we need a corresponding more equivariant notion of A-tree.

Definition 5.8. A *G-tree* is a tree equipped with a G-action through root-preserving automorphisms which endows the sets of leaves, (inner) edges, and vertices with a G-action. An *A-labeled G-tree* is a G-tree equipped with an equivariant labeling bijection between A and the G-set of leaves.

- We say an A-labeled G-tree is *layered* or *reduced* if the underlying $|A|$-tree is.
- We say an A-labeled G-tree is *G-elementary* if each layer has a unique G-orbit of vertices that are non-unary.
- We say an A-labeled G-tree is *G-measured* if the length assignment $E^i(T) \to (0,1]$ is G-equivariant.

An *isomorphism* of (reduced) G-trees is a G-homeomorphism that preserves the root. It is an isomorphism of layered A-labeled G-trees if it also preserves labels, and an isomorphism of G-measured A-labeled G-trees if it preserves edge measurements.

Remark 5.9. Note that this notion of G-tree is distinct from the notion with the same name in the work of the second-named author and Pereira; see [BP22, §2.2]. There, the above trees would be examples of "trees with G-action", while the term G-tree would refer to "orbits" of trees, say $G \cdot_H T$ for some tree T with H-action.

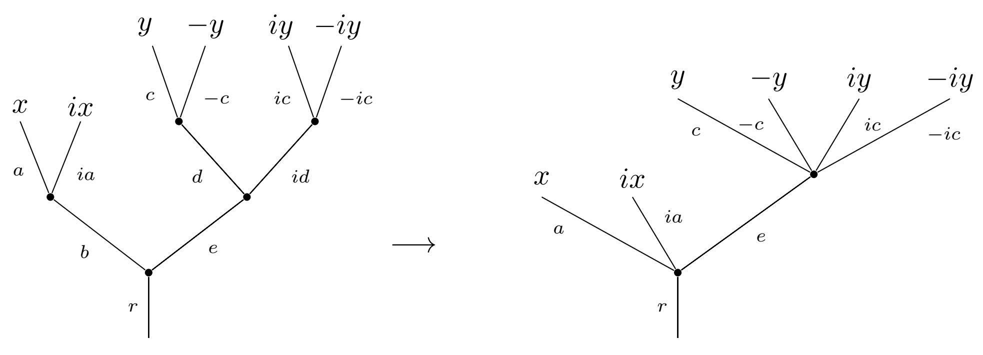
As before, we can associate categories and spaces to the three types of additional structure on G-trees.

Definition 5.10. First, the category of simplices $\Delta\mathcal{P}^G(A)$ may be described as the category of (isomorphism classes of) *layered A-labeled G-trees*, where faces and degeneracies again collapse or add layers.

Second, let $\mathcal{T}^G(A)$ denote the category of isomorphism classes of A-labeled G-trees, again excluding the A-corolla. There is a unique morphism $T \to T'$ if T' can be obtained from T by contracting a collection of inner edges, and in this case we call T' a (G-equivariant) *face* of T.

Third, let $\mathbb{T}^G(A)$ denote the space of (isomorphism classes of) measured A-labeled G-trees. In this case, the vertices are measured G-trees with exactly one orbit of inner edges. The description of a generic simplex in $\mathbb{T}^G(A)$, as well as that of a generic point, mimics the one in Definition 5.6.

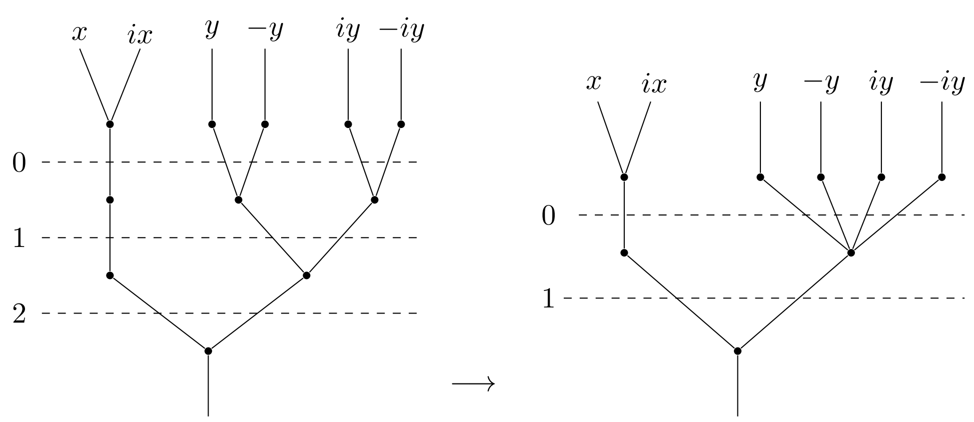
Example 5.11. Let $G = \{1, i, -1, -i\}$, and $A = \{x, ix, y, -y, iy, -iy\}$ as in Example 5.3. None of the A-trees from Example 5.3 or Example 5.5 may be endowed with a G-action such that the A-labeling is G-equivariant. However, consider the following relabeling of the trees from Example 5.5:



We have additionally named the edges of the tree to indicate the G-action. There is an arrow between these two trees in $\mathcal{T}^G(A)$.

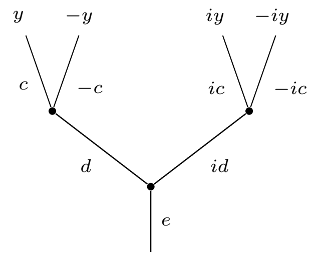
However, if we had only collapsed the edge labeled by d on the left, the resulting tree would not have a compatible G-action, and thus would not be a G-tree. We thus observe that we must collapse an entire orbit of inner edges.

Example 5.12. With a slight modification, the map from Example 5.11 is also a map of elementary layered A-labeled G-trees. With G and A as before, consider the following trees:



For readability, we have dropped the names of the edges indicating the action by G; however, the action is just as it was previously. Additionally, this map is between layered trees, as the arrow simply collapses the layer 1 on the left. Finally, these trees are both G-elementary; in particular, even though there are two non-unary vertices in layer 1 in the tree on the left, this tree is still G-elementary since those two vertices are in the same G-orbit.

Example 5.13. With G and A as in Example 5.11, the tree



is a vertex in $\mathbb{T}^G(A)$, as $|E^i(T)/G| = 1$. However, the underlying A-tree is not a vertex in $\mathbb{T}(A)$.

Remark 5.14. Note that, just as with $\mathcal{P}^G(A)$, the natural G-actions on $\Delta\mathcal{P}^G(A)$, $\mathcal{T}^G(A)$, and $\mathbb{T}^G(A)$ are the trivial ones.

These different varieties of trees are strongly related, as indicated by the following result.

Lemma 5.15. *For any $H \leq G$ there is an isomorphism of categories*

$$\mathcal{T}(A)^H \cong \mathcal{T}^H(\downarrow_H^G A)$$

and a homeomorphism of spaces

$$\mathbb{T}(A)^H \cong \mathbb{T}^H(\downarrow_H^G A).$$

Proof. We describe the homeomorphism of spaces; the isomorphism of categories is very similar with the slight wrinkle that we must consider an auxiliary category to define our functors as in the proof of Proposition 4.5.

Given an A-labeled H-tree T, forgetting the H-action on T, but remembering the G-action on A, determines a measured A-tree we denote by $\varphi(T)$. Since the isomorphism class of T as an A-labeled H-tree is smaller than the isomorphism class of T as a measured A-tree, this asignment determines a well-defined continuous map

$$\varphi\colon \mathbb{T}^H(\downarrow_H^G A) \to \mathbb{T}(A).$$

Given an A-labeled H-tree T, observe that the H-action is determined entirely by the action of H on the leaves, and thus by the structure of T as simply an A-tree. Said another way, T being a H-tree is a property, not additional structure, which implies that φ is injective. Since both spaces are finite simplicial complexes, they are compact Hausdorff and so injectivity implies that φ is a homeomorphism onto its image.

It remains to prove that $\operatorname{im}(\varphi)$ is equal to $\mathbb{T}(A)^H$. Note that for any A-labeled H-tree T, the H-action on T fixes the isomorphism class of T as an A-tree. Thus the image of φ is contained in the H-fixed points of $\mathbb{T}(A)$. Conversely, if a measured A-tree $(T, f\colon A \to L(T))$ is H-fixed, then for each $h \in H$, $h \cdot T = (T, fh^{-1})$ is in the same equivalence class as T, so there exists a tree automorphism σ_h such that $fh^{-1} = \sigma_h f$. These σ_h then define an H-action on T such that f is an H-map. We denote the resulting A-labeled H tree by $\psi(T)$. We have $\varphi(\psi(T)) = T$, so we have shown that $\mathrm{im}(\varphi) = \mathbb{T}(A)^H$. □

One conclusion we draw from this result is that, given an A-labeled tree, there exists at most one G-action on the tree making the labeling G-equivariant.

Remark 5.16. The third option proposed for equivariant partitions at the beginning of Section 4 was non-equivariant surjections $A \twoheadrightarrow B$ between G-sets. Using this notion in practice leads to several complications, often due to the fact that the G-actions and fixed points do not correspond to natural constructions.

First, in order to build a new G-poset structure $\mathcal{P}_G(A)$ with these objects, the arrows must be triangles such that the map $B \to B'$ is G-equivariant, which implies that we must take as objects $A \twoheadrightarrow B$ modulo the G-automorphisms of B. If such an equivalence class $[A \twoheadrightarrow B]$ is H-fixed, the representing map need not be H-equivariant; instead, following the proof of Proposition 4.5, it implies that the G-action on B extends to a $G \times H$-action, and the map is H-equivariant with respect to the "diagonal" H-action on B.

Finally, the trees that correspond to this structure are seemingly problematic, as G-trees equipped with a non-equivariant A-labeling of the leaves, modulo G-automorphisms of the G-tree. Describing the elements of an such equivalence class is a non-trivial exercise. Once again, the H-fixed points correspond to $G \times H$-trees such that the A-labeling is H-equivariant with respect to the diagonal action.

6. Comparison of G-partition complexes and G-trees

In this section, we use the equivariant version of Quillen's Theorem A to establish G-homotopy equivalences between the classifying spaces of the equivariant partition complex and several notions of equivariant trees.

To that end, let $\varphi\colon \Delta\mathcal{P}(\mathbf{n}) \to \mathcal{T}(\mathbf{n})$ denote the functor from [HM21] that collapses unary vertices and forgets layerings. Given a G-set A, Lemmas 4.2 and 5.7 imply that this functor induces a G-functor

$$\varphi\colon \Delta\mathcal{P}(A) \to \mathcal{T}(A). \tag{6.1}$$

Theorem 6.2. *The G-functor φ from Equation (6.1) is G-homotopy final.*

Proof. We follow the proof in [HM21], but take into account the potential orbital nature of T.

Fix a tree T in $\mathcal{T}(A)$ and $H \le G_T = \mathrm{Stab}_G(T)$. We must show that $(T \downarrow \varphi)^H$ is contractible. We first note that T is an H-tree by Lemma 5.15, and following Corollary 4.6, we define an *H-layering of T* to be a layered A-labeled H-tree S, thought of as an object of $\Delta\mathcal{P}(A)^H = \Delta\mathcal{P}^H(\downarrow^G_H A)$, such that $\varphi(S) = T$. Second, let $\Lambda^H(T) \subseteq N\mathcal{P}^H(\downarrow^G_H A)$ denote the sub-simplicial set spanned by the H-equivariant faces of H-layerings of T. Equivalently,

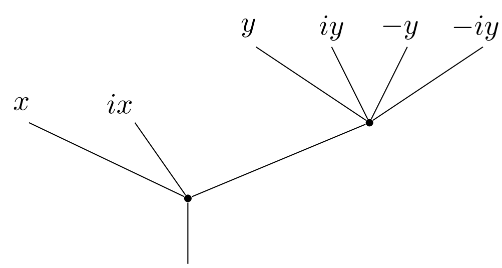
$\Lambda^H(T)$ is generated by the elementary H-layerings of T, all of which live in simplicial degree $|V(T)/H|-2$. Note that a simplex $S' \in \Lambda^H(T)$ is the face of a unique non-degenerate H-layering S of T, as the face of a layering of T is the layering of a unique face of T, and thus S' induces a canonical H-map $T = \varphi(S) \to \varphi(S')$ in $\mathcal{T}^H(A)$.

We can then see that

$$(T \downarrow \varphi)^H = T \downarrow \varphi^H \cong \Delta \downarrow \Lambda^H(T).$$

Let $V^L(T)$ denote the H-set of maximal vertices of T, i.e. vertices whose inputs are all leaves. For any $Hv \in V^L(T)/H$, let $\Lambda^H_{Hv}(T) \subseteq \Lambda^H(T)$ denote the sub-simplicial set generated by the elementary H-layerings for which the vertex orbit Hv is in the top layer. Then $\Lambda^H(T) = \bigcup_{V^L(T)/H} \Lambda^H_{Hv}(T)$. But $\Lambda^H_{Hv}(T)$ is the cone on $\Lambda^H(\partial_{Hv}T)$, where $\partial_{Hv}T$ is the tree obtained from T by removing all the vertices in Hv and their incoming edges. Additionally, given distinct orbits $Hv_1, \ldots, Hv_n$, we have that their intersection $\bigcap_{i=1,\ldots,n} \Lambda^H_{Hv_i}(T)$ is the cone on $\Lambda(\partial_{Hv_1} \ldots \partial_{Hv_n} T)$. Thus $\Lambda^H(T)$ is contractible, and hence so is $\Delta \downarrow \Lambda^H(T) \cong (T \downarrow \varphi)^H$. □

Example 6.3. Let $G = \{1, i, -1, -i\}$ and $A = \{x, ix, y, -y, iy, -iy\}$ as in Example 5.11. Consider the tree T below:



.

Both trees from Example 5.12 are in $\Lambda^G(T)$: the source is an actual G-layering of T, while the target is a face.

Remark 6.4. For an H-tree T with orbital representation T/H, $\Lambda^H(T)$ is not equal to $\Lambda(T/H)$, as unary vertices in T/H can correspond to (an orbit of) non-unary vertices in T. Thus, we cannot reduce the proof of Theorem 6.2 to the non-equivariant case, even though the argument of the proof seems to follow as if we could.

Remark 6.5. Considering $\mathbf{n}$ with the natural Σ_n-action, this result implies that $\varphi\colon \Delta\mathcal{P}(\mathbf{n}) \to \mathcal{T}(\mathbf{n})$ is Σ_n-homotopy final.

Combining Theorems 6.2 and A.2 and Corollary A.3 yields the following comparison.

Corollary 6.6. *There is a natural zig-zag of G-functors*

$$\mathcal{P}(A) \xleftarrow{\simeq} \Delta\mathcal{P}(A) \xrightarrow{\simeq} \mathcal{T}(A)$$

that induce G-homotopy equivalences on classifying spaces.

As in the non-equivariant case, we also have an independent chain of a G-homeomorphisms between related spaces.

Theorem 6.7. *There are G-homeomorphisms*

$$|\mathcal{P}(A)| \cong_G \mathbb{T}(A) \cong_G |\mathcal{T}(A)|.$$

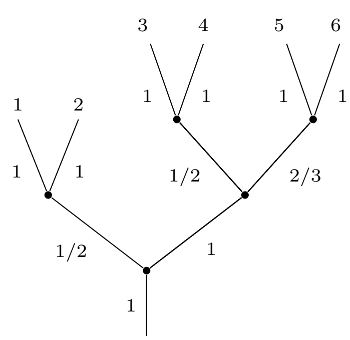
Proof. The first G-homeomorphism follows from [Rob04, Theorem 2.7] and Lemmas 4.2 and 5.7, since the restriction of a Σ_n-homeomorphism is a G-homeomorphism. For the second, we define a Σ_n-homeomorphism $F\colon \mathbb{T}(\mathbf{n}) \to |\mathcal{T}(\mathbf{n})|$ below of a similar flavor, from which the result follows.

Given a measured $\mathbf{n}$-tree T, we get a family of $\mathbf{n}$-trees $S(t)$, for $0 \le t \le 1$, by collapsing all inner edges with lengths less than t and forgetting the remaining lengths. This family in fact produces a chain of $\mathbf{n}$-trees, and the barycentric coordinate of $F(T)$ with respect to S is given by the amount of time $S(t) = S$.

Conversely, given a (strict) chain of $\mathbf{n}$-trees and barycentric coordinates $(S_0 < S_1 < \cdots < S_n, (\ell_0, \ldots, \ell_n))$, define the measured $\mathbf{n}$-tree T to have underlying $\mathbf{n}$-tree S_0, with the weights of $E^i(S_n)$ equal to 1, and for $0 \le k \le n-1$, the weights of $E^i(S_k) \setminus E^i(S_{k+1})$ equal to $1 - \sum_{i=k+1}^{n} \ell_i$. Here, we are using the fact that if T' is a face of T then there is a canonical inclusion $E^i(T') \subseteq E^i(T)$.

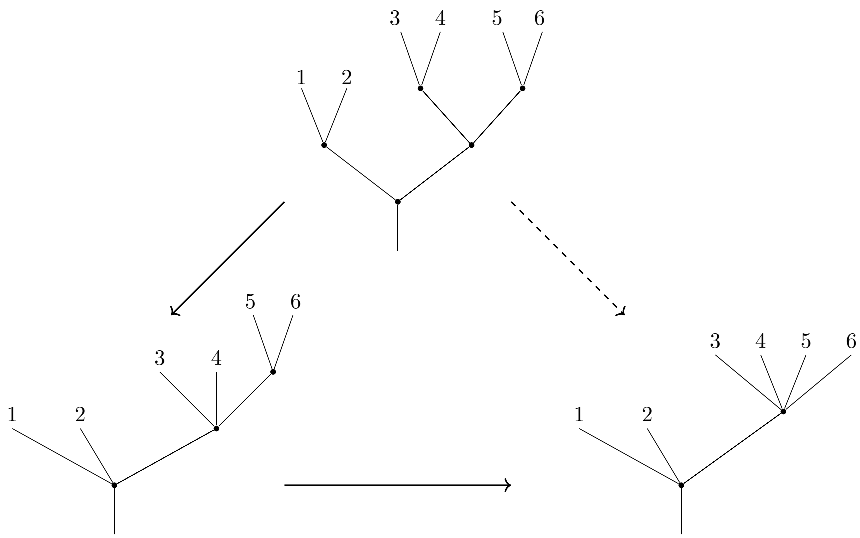
It is straightforward to check that these maps are continuous, Σ_n-equivariant, and inverse to one another. □

Example 6.8. Consider the following element of $\mathbb{T}(\mathbf{6})$:



.

The map in the proof above sends this element to the 2-simplex of $|\mathcal{T}(A)|$



,

with barycentric coordinates $(1/2, 1/6, 1/3)$.

Remark 6.9. The composite map $|\mathcal{T}(\mathbf{n})| \to |\mathcal{P}(\mathbf{n})|$ is not simplicial, as it does not even send vertices to vertices. For example, the height 3 binary tree with 4 leaves gets sent to the chain of partitions $(1)(234) < (1)(2)(34)$ with barycentric coordinates $(1/2, 1/2)$.

Taking fixed points yields similar results to the above comparing G-equivariant partitions and G-trees.

Theorem 6.10. *For any G-set A:*

(a) The functor

$$\varphi\colon \Delta\mathcal{P}^G(A) \longrightarrow \mathcal{T}^G(A)$$

is homotopy final, and so in particular induces a homotopy equivalence on classifying spaces.

(b) There is a natural zig-zag of functors

$$\mathcal{P}^G(A) \xleftarrow{\simeq} \Delta\mathcal{P}^G(A) \xrightarrow{\simeq} \mathcal{T}^G(A)$$

that induce homotopy equivalences on classifying spaces.

(c) There are homeomorphisms

$$|\mathcal{P}^G(A)| \cong \mathbb{T}^G(A) \cong |\mathcal{T}^G(A)|.$$

Proof. Using Corollary 4.6 and Lemma 5.15 and the fact that the fixed points of a G-homotopy initial (respectively, final functor) is homotopy initial (respectively, final), part (a) follows from Theorem 6.2, part (b) from (a) and Corollary A.3, and part (c) from Proposition 3.8 and Theorem 6.7. $\square$

7. The G-homotopy type of $\mathcal{P}(A)$

In this section we use tools developed above to study the homotopy type of the partition complexes $|\mathcal{P}(A)|$ and $|\mathcal{P}^G(A)|$. These two spaces are related by Propositions 3.8 and 4.5, which identify $|\mathcal{P}(A)|^H \simeq |\mathcal{P}^H(\downarrow_H^G A)|$ for all $H \leq G$. Since the G-homotopy type of $|\mathcal{P}(A)|$ depends on the ordinary homotopy type of its fixed points, one should view computations of $|\mathcal{P}^H(A)|$ as stepping stones to understanding the G-homotopy type of $|\mathcal{P}(A)|$.

When $G = \Sigma_n$, computations of the G-homotopy type of $\mathcal{P}(\mathbf{n})$ have been carried out by Arone and Brantner [AB21]. Our results are similar, but our proofs are different and make use of our explicit descriptions of the fixed point categories of $\mathcal{P}(A)$.

A study of the category $\mathcal{P}^G(A)$ reveals that its homotopy type depends heavily on the G-set A; more precisely, on whether A is H-isovariant for some subgroup $H \leq G$.

Definition 7.1. A finite G-set A is *H-isovariant* if there is a G-isomorphism $A \cong \amalg_{i=1}^n G/H$ for some n.

With this in mind, we divide our approach in two cases.

7.1. Case 1: A is not H-isovariant.

We first prove that if A is not H-isovariant for any $H \leq G$, then $\mathcal{P}^G(A)$ is contractible.

Note that in this case A must have at least two orbits as otherwise, by Remark 3.4, we would have $A \cong G/G_a$ for any $a \in A$. Thus A would be G_a-isovariant, which would imply, in particular, that the category $\mathcal{P}_2^G(A)$ appearing in the following lemma is non-empty.

Lemma 7.2. *Suppose that A is not H-isovariant for any $H \leq G$. Let $\mathcal{P}_2^G(A) \subseteq \mathcal{P}^G(A)$ denote the full subcategory on objects $A \twoheadrightarrow B$ where B has at least two G-orbits. Then $\mathcal{P}_2^G(A)$ is contractible.*

Proof. Let $\mathcal{C} \subseteq \mathcal{P}_2^G(A)$ be the full subcategory on objects $f\colon A \twoheadrightarrow B$ where B has trivial G action. Since A is not G-isovariant and has at least two orbits, the partition $A \twoheadrightarrow A/G$ is neither discrete nor indiscrete and thus is an object in $\mathcal{C}$. This object is initial in $\mathcal{C}$ and so $\mathcal{C}$ is contractible.

Let $I\colon \mathcal{C} \to \mathcal{P}_2^G(A)$ denote the inclusion; we want to show that this functor is a homotopy equivalence. By Quillen's Theorem A, it suffices to prove that for any object $f\colon A \twoheadrightarrow B$ in $\mathcal{P}_2^G(A)$, the category $f \downarrow I$ has an initial object. By the definition of $\mathcal{P}_2^G(A)$, the G-set B must have at least two orbits so $|B/G| > 1$. Let $\pi\colon B \to B/G$ denote the quotient map. The pair $(\pi f\colon A \twoheadrightarrow B/G, B \twoheadrightarrow B/G)$ is an object in $f \downarrow I$ and is initial since any equivariant map from A to a set with trivial G-action which factors through B must also factor through B/G. □

Since almost all G-sets have more than one orbit, $\mathcal{P}_2^G(A) \subseteq \mathcal{P}^G(A)$ is a rather large subcategory. We will see presently that the inclusion of this subcategory induces a homotopy equivalence whenever A is not H-isovariant for any $H \leq G$. The argument follows Quillen's Theorem A:

if $I\colon \mathcal{P}_2^G(A) \to \mathcal{P}^G(A)$ is the inclusion, we show that the overcategory $I \downarrow (f\colon A \twoheadrightarrow B)$ is contractible for any $f\colon A \twoheadrightarrow B$ in $\mathcal{P}^G(A)$. When $A = \amalg_{i=1}^n G/H$ is H-isovariant, our arguments show that the overcategory $I \downarrow f$ is either contractible or categorically equivalent to the partition poset $\mathcal{P}(\mathbf{n})$.

Before proceeding, we need some notation.

Definition 7.3. Let $H \leq G$ be a proper subgroup and let A be a finite G-set. We say that A is H*-induced* if there is an G-map $A \twoheadrightarrow G/H$.

Remark 7.4. Let A' be an H-set, and let $*$ denote the H-set with one point and trivial action. Applying the induction functor $\uparrow_H^G\colon H\mathcal{F}in \to G\mathcal{F}in$ to the map $A' \to *$ yields a G-map $\uparrow_H^G (A') \to \uparrow_H^G (*) \cong G/H$. This construction actually gives an equivalence of categories $H\mathcal{F}in \simeq G\mathcal{F}in \downarrow (G/H)$, which justifies our terminology for H-induced sets. In particular, A is H-induced if and only if there is a finite H-set A' with A isomorphic to $\uparrow_H^G (A')$.

The following result is equivalent, by Proposition 4.5 above, to Lemma 6.3 in [AB21] in the case where $G = \Sigma_n$.

Proposition 7.5. *If A is not H-isovariant for any $H \leq G$ then $\mathcal{P}^G(A)$ is contractible.*

Proof. Let $I\colon \mathcal{P}_2^G(A) \to \mathcal{P}^G(A)$ denote the inclusion of the full subcategory on objects $A \twoheadrightarrow B$ where B has at least two orbits. We want to show, under our hypotheses, that I induces a homotopy equivalence so the result follows from Lemma 7.2. We prove that for any $f\colon A \twoheadrightarrow B$ in $\mathcal{P}^G(A)$, the undercategory $I \downarrow f$ is contractible, and hence our claim follows from (the dual of) Quillen's Theorem A.

If $f\colon A \twoheadrightarrow B$ is an object in $P_2^G(A)$ then $I \downarrow f$ is contractible since the identity on f is a terminal object. Suppose then that $f\colon A \twoheadrightarrow B$ is not in $\mathcal{P}_2^G(A)$. Then B has a single orbit, and by Remark 3.4 we may assume without loss of generality that $B = G/H$ for some proper subgroup $H \leq G$. In particular, A is H-induced and so there is a finite H-set A' so that $A \cong \uparrow_H^G (A')$.

We claim there is an equivalence of categories $I \downarrow f \simeq \mathcal{P}_2^H(A')$. If so, then the fact that A is not H-isovariant implies A' is not K-isovariant for any $K \leq H$, and so $\mathcal{P}_2^H(A')$ is contractible by Lemma 7.2.

An object in the category $I \downarrow f$ consists of a commutative triangle

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[dr, two heads, "f"'] & B' \arrow[d, two heads] \\ & G/H \end{tikzcd}$$

in $G\mathcal{F}in$ such that B' has more than one orbit. The claim follows from the observation that $I \downarrow f$ is equivalent to the subcategory of $(f\colon A \twoheadrightarrow G/H) \downarrow (\mathcal{S}et^G \downarrow (G/H))$ consisting of surjections from A onto objects with at least two orbits. Since the equivalence $H\mathcal{F}in \simeq G\mathcal{F}in \downarrow (G/H)$ preserves both surjections and objects with at least two orbits, we see that $I \downarrow f$ is equivalent to the subcategory of $A' \downarrow H\mathcal{F}in$ with only surjections onto H-sets with more than one orbit, which is the definition of $\mathcal{P}_2^H(A')$. □

Remark 7.6. In the notation of the above proof, it is always true that $I \downarrow f \simeq \mathcal{P}_2^H(A')$. When A is H-isovariant, A' is a trivial H-set and we have an equivalence of categories $\mathcal{P}_2^H(A') \simeq \mathcal{P}(|A'|)$, which is generally not contractible.

7.2. **Case 2: A is H-isovariant.** We now turn our attention to studying $\mathcal{P}^G(A)$ when A is H-isovariant.

The simplest case is when $A = G/H$ is a transitive G-set, and we can identify $\mathcal{P}^G(A)$ with an equivalent category. In order to do this, let us first establish the following convention.

Remark 7.7. If H and K are subgroups of G, the collection of G-maps from G/H to G/K is in bijection with the set of $g \in G$ such that $gHg^{-1} \subseteq K$. When $H \leq K$, the G-map $G/H \to G/K$ corresponding to $eHe^{-1} \subseteq K$ is given by $gH \mapsto gK$, and we call this map the *canonical quotient*.

Proposition 7.8. *There is an equivalence of categories between the poset $\mathcal{P}^G(G/H)$ and the poset $S(G,H)$ of subgroups K of G such that $H \lneq K \lneq G$.*

Proof. Define a functor $I\colon S(G,H) \to \mathcal{P}^G(G/H)$ that sends a subgroup K to the class of the canonical quotient $G/H \twoheadrightarrow G/K$. On morphisms, I sends an inclusion of subgroups $K \leq K'$ to the canonical quotient $G/K \twoheadrightarrow G/K'$.

As the domain category is a poset, I is necessarily faithful. To see that I is full, note that a morphism in $\mathcal{P}^G(G/H)$ between canonical quotients $(G/H \twoheadrightarrow G/K) \to (G/H \twoheadrightarrow G/K')$ corresponds to a map $G/K \to G/K'$ sending eK to eK'. Such a map exists, and is a canonical quotient, if and only if $K \leq K'$.

Finally, we show I is essentially surjective on objects. If $f\colon G/H \twoheadrightarrow B$ is surjective, then B must be a transitive G-set. By the Orbit-Stabilizer Theorem, $B \cong G/K$ where K is the stabilizer of $f(eH)$. Since the stabilizer of eH is H, we have $H \leq K$ and f is equivalent to the canonical quotient $G/H \twoheadrightarrow G/K$. □

Remark 7.9. The space $|\mathcal{P}^G(G/H)|$ is generally non-contractible. For example, when $G = \Sigma_3$, the space $\mathcal{P}^{\Sigma_3}(\Sigma_3/e)$ is equivalent to four points. Interestingly, understanding the general homotopy type of the realization of the posets $S(G,H)$ is an open problem. When G is solvable, Kratzer and Thévenaz have shown that $|S(G,e)|$ is homotopy equivalent to a wedge of equidimensional spheres [KT85]. However, such a result does not hold for general G; Kramarev and Lokutsievskiy have shown that when $G = PSL(2,\mathbb{F}_7)$, the space $|S(G,e)|$ is homotopy equivalent to a wedge of 48 copies of S^1 and 48 copies of S^2 [KL08].

We are left to understand the homotopy type of $\mathcal{P}^G(A)$ when A is H-isovariant with more than one orbit. In Proposition 7.11 below, we show that the homotopy type of $\mathcal{P}^G(A)$ for such A is entirely determined by the subgroup $H \leq G$ and the number of orbits. When $H = G$, we recover the non-equivariant partition complex, so for the remainder of the section we assume $H < G$ is a proper subgroup.

First, we fix some notation. For any object α in $\mathcal{P}^G(A)$, let $\alpha^\perp$ denote the collection of objects in $\mathcal{P}^G(A)$ that are orthogonal to α. Thinking of $\mathcal{P}^G(A)$ as a poset, an element β is in $\alpha^\perp$ if there is no element ω which is either a lower or upper bound for β and α.

Lemma 7.10. *Let $A = \amalg_{i=1}^n G/H$, with $n > 1$, and let $\alpha\colon A \twoheadrightarrow \amalg_{i=1}^n G/G$ be the union of n collapse maps. Then $\alpha^\perp \subseteq \mathcal{P}^G(A)$ consists of all objects $\beta\colon A \twoheadrightarrow B$ where $B \cong G/H$.*

Proof. Suppose that $\beta\colon A \twoheadrightarrow B$ represents an object in $\alpha^\perp$. Then B must have only one orbit; otherwise, the map $A \twoheadrightarrow B \twoheadrightarrow B/G$ is an upper bound for α and β. Thus $B \cong G/K$ for some subgroup $K \leq G$. Since there is a G-map $A \to G/K$, it must be the case that H is subconjugate to K.

If K is not conjugate to H, the map $A \twoheadrightarrow \amalg_{i=1}^n G/K$ is a lower bound for β and α. It follows that everything in $\alpha^\perp$ is of the form in the statement. That all such objects are in $\alpha^\perp$ follows from similar arguments. $\square$

Proposition 7.11. *There is a homotopy equivalence*

$$|\mathcal{P}^G(\amalg_{i=1}^n G/H)| \simeq \bigvee_{|W_G(H)|^{n-1}} |\mathcal{P}^G(G/H)|^\diamond \wedge |\mathcal{P}(\mathbf{n})|^\diamond$$

where $W_G(H) = N_G(H)/H$ is the Weyl group and $(-)^\diamond$ denotes the unreduced suspension.

Proof. Let $\alpha \in \mathcal{P}^G(A)$ be as in Lemma 7.10. By [AB21, 3.5] (see also [BW83, 4.2]), there is a homotopy equivalence

$$|\mathcal{P}^G(A)| \simeq \bigvee_{\beta \in \alpha^\perp} |(\beta \downarrow \mathcal{P}^G(A))_\times|^\diamond \wedge |(\mathcal{P}^G(A) \downarrow \beta)_\times|^\diamond,$$

where the $\times$ denotes that we are considering the subcategory of the slice category which does not contain the initial or final objects. We remove these objects so that the categories are not contractible.

By Lemma 7.10, we have that an arbitrary $\beta \in \alpha^\perp$ is of the form $A \twoheadrightarrow G/H$, and it is straightforward to check that $(\beta \downarrow \mathcal{P}^G(A))_\times \simeq \mathcal{P}^G(G/H)$ and $(\mathcal{P}^G(A) \downarrow \beta)_\times \simeq \mathcal{P}(n)$. It remains to check how many isomorphism classes of objects are in $\alpha^\perp$. Note that every element $\beta \in \alpha^\perp$ is represented by an object in $\mathrm{Hom}_G(A, G/H)$. Since A is a disjoint union of n copies of G/H, we have $\mathrm{Hom}_G(A, G/H) \cong \mathrm{Aut}_G(G/H)^n \cong W_G(H)^n$. Finally, we need to take the quotient by the subgroup of automorphisms of the target, which is the diagonal copy of $W_G(H)$. $\square$

Remark 7.12. The splitting of Proposition 7.11 is similar to a result of [AB21]. Let A be an H-isovariant G-set and let $n = |A|$ and $|G/H| = d$. The action of G on G/H induces an inclusion $G \subseteq \Sigma_d$. The G-action on A induces an inclusion $G \subseteq \Sigma_n$ which, up to relabeling, factors as

$$G \subseteq \Sigma_d \subseteq \Sigma_d^{\frac{n}{d}} \subseteq \Sigma_n,$$

where the second inclusion is the diagonal embedding. Using this embedding, [AB21, Theorem 6.2] identifies

$$|\mathcal{P}(A)|^G \cong \uparrow^{W_{\Sigma_n}(G)}_{W_{\Sigma_d}(G)\times\Sigma_{\frac{n}{d}}} |\mathcal{P}^G(G/H)|^\diamond \wedge |\mathcal{P}(\mathbf{n})|^\diamond,$$

where $\uparrow$ is the induction functor on based spaces. We can compare this result directly with Proposition 7.11, as induction on based spaces is given by taking wedges. Counting the number of wedge summands in both presentations, we obtain a combinatorial identity

$$\frac{|W_{\Sigma_n}(G)|}{|W_{\Sigma_d}(G)| \cdot (\frac{n}{d})!} = |W_G(H)|^{n-1}$$

that must hold whenever G acts H-isovariantly on a set with n elements.

In many cases of interest, Proposition 7.11 suffices to show that $|\mathcal{P}^G(A)|$ has the homotopy type of a wedge of equidimensional spheres.

Corollary 7.13. *If G is a solvable group, $H \leq G$ is normal, and A is H-isovariant then $|\mathcal{P}^G(A)|$ is homotopy equivalent to a wedge of equidimensional spheres.*

Proof. As noted in Remark 7.9, when G is solvable the homotopy type of $|\mathcal{P}^G(G/e)|$ is a wedge of equidimensional spheres. Since H is normal, $Q = G/H$ is a solvable group and we have equivalences of categories

$$\mathcal{P}^G(G/H) \cong S(G,H) \cong S(Q,e) \cong \mathcal{P}^Q(Q/e),$$

and thus $|\mathcal{P}^G(G/H)|$ has the homotopy type of a wedge of equidimensional spheres. It is well-known that the homotopy type of $|\mathcal{P}(\mathbf{n})|$ is also a wedge of equidimensional spheres; see, for example [Rob04]. The claim now follows immediately from Proposition 7.11 and the facts that the smash product distributes over wedges and the smash product of two spheres is a sphere. □

8. Connections to Cohomology and Lie algebras

Non-equivariantly, the cohomology of the space of trees is related to certain integral representations of the symmetric group Σ_n coming from Lie algebra theory. In this section we recall this result, following Robinson [Rob04], and explain how our work relates to it. All cohomology groups in this section are integral.

Before proceeding, some remarks are in order regarding the way our work fits into the general context of equivariant cohomology theories. For a G-space X, there are three standard ways that the action of G induces additional structure on homology. The most straightforward, and the one we focus on, is that for all $g \in G$, the maps $g\colon X \to X$ induce a G-action on $H^*(X)$ giving it the structure of a graded G-module. Two other common ways of treating equivariant cohomology theories are known as Borel cohomology and Bredon cohomology [May96], but we do not consider these notions here. The interested reader can find computations of the Bredon homology of partition complexes in the case where $G = \Sigma_n$ in work of Arone, Dwyer, and Lesh [ADL16], [ADL21].

We now recall the work of Robinson on computations of the Σ_n-module structure on the cohomology of the ordinary partition complex $\mathcal{P}(\mathbf{n})$. For a fixed n, write $\mathcal{L}_n$ for the free Lie algebra on a set of n generators $\{x_1, \ldots, x_n\}$. The *n-linear part* of $\mathcal{L}_n$ is the subgroup $\mathrm{Lie}_n \leq \mathcal{L}_n$ generated by Lie monomials containing every generator x_i exactly once. The standard left action of the symmetric group Σ_n on the set $\{x_1, \ldots, x_n\}$ extends to an action on Lie_n that we call the *integral Lie representation* of Σ_n. The collection of $\mathbb{Z}[\Sigma_n]$-modules $\{\mathrm{Lie}_n\}$ forms a symmetric operad in abelian groups whose algebras are Lie algebras.

Let ε^{Σ_n} denote the integral sign representation of Σ_n. The following theorem is proved in [Rob04, Theorem 4.1].

Theorem 8.1. *There is an isomorphism of Σ_n-modules*

$$H^{n-3}(\mathbb{T}(\mathbf{n})) \cong \varepsilon^{\Sigma_n} \otimes \mathrm{Lie}_n \,.$$

We would like to prove an analogous result when $\mathbf{n}$ is replaced by a G-set A for some finite group G. The first step is to find suitable replacements for the Σ_n-representations Lie_n and ε^{Σ_n}. Given a G-set A, let $\alpha\colon G \to \Sigma_n$ be the homomorphism that realizes the action of G on A. Implicitly, this homomorphism depends on a choice of total ordering for A, but we do not use this additional information.

Let $\mathcal{L}_A$ denote the free Lie algebra on the set A. Since $\mathcal{L}_A$ is generated as a Lie algebra by a set in bijection with A, it inherits a natural G-action. We define the A-linear part of $\mathcal{L}_A$ to be the G-subgroup $\mathrm{Lie}_A \leq \mathcal{L}_A$ generated by Lie monomials containing every generator of $\mathcal{L}_A$ exactly once. This G-submodule plays the role of Lie_n in the equivariant setting.

To replace the sign representation, we define the A-sign representation ε^G_A of G. Let G act on the free abelian group V generated by A. A choice of ordering for A corresponds to a choice of ordered basis for V, and thus gives matrix representations for the action of each $g \in G$. Define $\varepsilon^G_A(g) = \det(g) = \pm 1$ for all $g \in G$, and consider this action as a 1-dimensional G-representation. Note that while this definition requires a choice of ordering for A, the G-representation ε^G_A is independent of this choice, since any two choices of ordering yield actions on V that are conjugate.

It is not hard to show there are isomorphisms of G-modules

$$\mathrm{Lie}_A \cong \alpha^*(\mathrm{Lie}_n) \quad \text{and} \quad \varepsilon^G_A \cong \alpha^*(\varepsilon^{\Sigma_n}),$$

where α^* is the functor that restricts a Σ_n-module to a G-module along the homomorphism $\alpha\colon G \to \Sigma_n$. The next proposition uses these isomorphisms to give an equivariant analogue of Theorem 8.1.

Theorem 8.2. *There is an isomorphism of G-modules*

$$H^{n-3}(\mathbb{T}(A)) \cong \varepsilon_A^G \otimes \mathrm{Lie}_A\,.$$

Proof. Unwinding the definition, we see there are isomorphisms of G-modules

$$\alpha^*(\varepsilon_A^{\Sigma_n} \otimes \mathrm{Lie}_n) \cong \alpha^*(\varepsilon^{\Sigma_n}) \otimes \alpha^*(\mathrm{Lie}_n) \cong \varepsilon_A^G \otimes \mathrm{Lie}_A\,.$$

Since $\varepsilon^{\Sigma_n} \otimes \mathrm{Lie}_n \cong H^{n-3}(\mathbb{T}(\mathbf{n}))$, the result now follows from Lemma 5.7, together with the fact that for any Σ_n-space Y there is an isomorphism of G-modules $H^*(\alpha^*Y) \cong \alpha^*H^*(Y)$. This last isomorphism follows from an isomorphism at the level of singular cochains. □

We conclude this section with some comments on the cohomology of the space of equivariant trees $\mathbb{T}^G(A)$. Here we rely on the homeomorphism $\mathbb{T}^G(A) \cong |\mathcal{P}^G(A)|$ from Theorem 6.10. Using Proposition 7.5 we see that the homology of this space is often trivial.

Proposition 8.3. *Suppose A is a G-set that is not H-isovariant for any $H \leq G$. Then the reduced homology $\widetilde{H}^*(\mathbb{T}^G(A)) = 0$.*

When A is H-isovariant, its homology is non-trivial and, by Proposition 7.11, it is determined by the homotopy types of $|\mathcal{P}^G(G/H)|$ and $|\mathcal{P}(|\mathcal{A}|)|$. As noted in Remark 7.9, the homotopy type of $|\mathcal{P}^G(G/H)|$ is not known in general, and so we are unable to compute the homology of these spaces completely. In nice cases, we can use Corollary 7.13 to compute the cohomology when the G-set A is free.

Corollary 8.4. *If G is a solvable group, $H \leq G$ is normal, and A is H-isovariant, then the reduced cohomology of $\mathbb{T}^G(A)$ is a finitely generated free abelian group concentrated in a single dimension.*

Appendix A. Equivariant versions of Theorems A and B

Quillen's Theorems A and B [Qui73, §1] play central roles in Quillen's work on higher algebraic K-theory, but are also widely applied outside of that context. In this appendix, we prove the analogous theorems for G-functors between categories with G-action.

A.1. **Equivariant Theorem A.** Quillen's Theorem A is a useful tool for determining whether a functor between categories induces a homotopy equivalence on their classifying spaces. In particular, Theorem A is used by Heuts and Moerdijk in their comparison of partition complexes and trees [HM21]. For our equivariant version of this comparison, we need a suitable equivariant analogue, which we prove here. In the special case of posets, an equivariant version of Theorem A was proved by Thévanaz and Webb [TW91].

The idea behind Quillen's Theorem A is that we can determine whether a functor $F\colon \mathcal{C} \to \mathcal{D}$ is a homotopy equivalence by looking at the classifying spaces of the undercategory $d \downarrow F$ for all objects d of $\mathcal{D}$. Recall that the objects of $d{\downarrow}F$ are pairs $(c, g\colon d \to Fc)$, where c is an object of $\mathcal{C}$ and g is a morphism in $\mathcal{D}$, and that a morphism between (c, g) and (c', g') in $d{\downarrow}F$ is a map $f\colon c \to c'$ such that the following triangle commutes in $\mathcal{D}$:

$$\begin{tikzcd} & d \arrow[dl, "g"'] \arrow[dr, "g'"] & \\ Fc \arrow[rr, "Ff"'] & & Fc'. \end{tikzcd}$$

The original statement of Quillen's Theorem A is as follows.

Theorem A.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If $d \downarrow F$ is contractible for every object d of $\mathcal{D}$, then F induces a homotopy equivalence $|F|\colon |\mathcal{C}| \to |\mathcal{D}|$.*

As noted by Quillen [Qui73], the dual statement where we assume F is homotopy initial, rather than homotopy final, also holds by an analogous proof.

We want an equivariant analogue of this theorem. Observe that if $F\colon \mathcal{C} \to \mathcal{D}$ is a G-functor between categories with a G-action, then the fiber $d \downarrow F$ has an action of the isotropy subgroup $G_d = \{g \in G \mid g \cdot d = d\}$. For any $H \leq G_d$ we can compute the fixed point category $(d \downarrow F)^H$. Note that we have equalities

$$(d \downarrow F)^H = d \downarrow F^H,$$

since both categories consist of the pairs of objects $(c, \psi\colon d \to Fc)$ in $d \downarrow F$ with $hc = c$ and $h\psi = \psi$ for all $h \in H$.

We now want to ask that each fiber $d{\downarrow}F$ is G_d*-contractible*, meaning that the homotopy equivalence $|d{\downarrow}F| \to *$ restricts to homotopy equivalences of fixed points $|d{\downarrow}F|^H \to *$ for all $H \leq G_d$. In other words, we want F to be G-homotopy final in the sense of Definition 3.11. Setting $H = e$, we see the fibers all need to be contractible, as in the non-equivariant version.

We can thus state the equivariant version of Quillen's Theorem A as follows.

Theorem A.2 (Equivariant Theorem A)**.** *If $F\colon \mathcal{C} \to \mathcal{D}$ is G-homotopy initial or G-homotopy final, then $|F|\colon |\mathcal{C}| \to |\mathcal{D}|$ is a G-homotopy equivalence.*

Proof. We focus our attention on the case where F is G-homotopy final, as the dual result follows similarly by replacing the use of (non-equivariant) Theorem A with its dual theorem.

To conclude $|F|$ is a G-homotopy equivalence, we need to show that $|F|^H\colon |\mathcal{C}|^H \to |\mathcal{D}|^H$ is a homotopy equivalence for all $H \leq G$. Since taking fixed points commutes with classifying spaces by Proposition 3.8, we can equivalently show that $|F^H|\colon |\mathcal{C}^H| \to |\mathcal{D}^H|$ is a homotopy equivalence, which we can do by applying (non-equivariant) Theorem A. Note that if

$d \in \operatorname{ob} \mathcal{D}^H$, then we must have $H \le G_d$, and

$$|d{\downarrow}F|^H = |(d{\downarrow}F)^H| = |d{\downarrow}(F^H)|.$$

Then by assumption, $d \downarrow F^H = (d \downarrow F)^H$ is contractible, so we may apply Theorem A to conclude $|F^H|\colon |\mathcal{C}^H| \to |\mathcal{D}^H|$ is a homotopy equivalence, which completes the proof. ☐

Finally, we include the following consequence of Theorem A.2 that we use in this paper.

Corollary A.3. *For any category $\mathcal{C}$ with G-action, the functor*

$$F\colon \Delta\mathcal{C} \to \mathcal{C},$$

sending a chain to its final element, is G-homotopy initial and hence induces a G-equivalence on classifying spaces.

Proof. From the definitions, one can check that $F \downarrow d = \Delta(\mathcal{C} \downarrow d)$ for any object d of $\mathcal{C}$, and so $(F \downarrow d)^H = F^H \downarrow d = \Delta(\mathcal{C}^H \downarrow d)$ for all $H \le G_d$. As noted in, for example, the discussion in [Dug06] before Theorem 2.4, the (non-equivariant) last vertex map induces a weak equivalence on nerves. Hence $(F \downarrow d)^H$ and $\mathcal{C}^H \downarrow d$ have equivalent nerves, and thus $(F \downarrow d)^H$ is contractible, as desired. ☐

A.2. **Equivariant Theorem B.** We may similarly prove an equivariant version of Quillen's Theorem B, which gives a sufficient condition to model the homotopy fiber of $|F|\colon |\mathcal{C}| \to |D|$ as a classifying space. The original statement of Quillen's Theorem B is as follows.

Theorem A.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and suppose that for every morphism $d \to d'$ in $\mathcal{D}$, the induced map $|d' \downarrow F| \to |d \downarrow F|$ is a homotopy equivalence. Then the pullback square*

$$\begin{tikzcd} {|d{\downarrow}F|} \arrow[r] \arrow[d] & {|\mathcal{C}|} \arrow[d, "F"] \\ {|d{\downarrow}\mathrm{id}_{\mathcal{D}}|} \arrow[r] & {|\mathcal{D}|} \end{tikzcd}$$

is a homotopy pullback.

Since the identity on d is an initial object, $|d{\downarrow}\mathrm{id}_{\mathcal{D}}|$ is contractible, so the inclusion $|d \downarrow F| \to \operatorname{hfib}(|F|)$ is a homotopy equivalence. As with Theorem A, there is also a dual version of Theorem B.

To prove an equivariant version of Theorem B, we need a lemma about the homotopy fiber of a G-functor.

Lemma A.5. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a G-functor and let $d \in \operatorname{ob}(\mathcal{D})$. Then the homotopy fiber $\operatorname{hfib}_d(|F|)$ is a G_d-space with*

$$\operatorname{hfib}_d(|F|)^H \simeq \operatorname{hfib}_d(|F^H|)$$

for every $H \le G_d$.

Proof. We can model the homotopy fiber $\mathrm{hfib}_d(|F|)$ as

$$\{d\} \times^h_{|\mathcal{D}|} |\mathcal{C}| = \{(d, \gamma, c) \mid c \in |\mathcal{C}|, \gamma \in |D|^I, \gamma(0) = d, \gamma(1) = |F|(c)\}.$$

This space has a G_d-action $g \cdot (d, \gamma, c) = (d, g \cdot \gamma, g \cdot c)$ where $(g \cdot \gamma)(t) = g \cdot \gamma(t)$ for all $t \in I$. A point $(d, \gamma, c) \in \mathrm{hfib}_d(|F|)$ is thus H-fixed if and only if $c \in |\mathcal{C}|^H = |\mathcal{C}^H|$ and γ is a path in $|\mathcal{D}|^H = |\mathcal{D}^H|$, which is to say $(d, \gamma, c) \in \mathrm{hfib}_d(|F^H|)$. Thus $\mathrm{hfib}_d(|F|)^H$ models $\mathrm{hfib}_d(|F^H|)$ for $H \leq G_d$. □

Theorem A.6 (Equivariant Theorem B)**.** *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a G-functor and every morphism $d \to d'$ in $\mathcal{D}$ induces a $G_d \cap G_{d'}$-equivalence $|d' {\downarrow} F| \to |d {\downarrow} F|$. Then for each object d of $\mathcal{D}$, the inclusion $|d {\downarrow} F| \to \mathrm{hfib}_d(|F|)$ is a G_d-equivalence.*

Proof. For an object d of $\mathcal{D}$, we want to show that $|d {\downarrow} F|^H \to \mathrm{hfib}_d(|F|)^H$ is a homotopy equivalence for each $H \leq G_d$ by applying non-equivariant Theorem B to $F^H\colon \mathcal{C}^H \to \mathcal{D}^H$. Note that for any object d in $\mathcal{D}$ and $H \leq G_d$, we have that d is an object in $\mathcal{D}^H$ and

$$|d {\downarrow} F|^H = |(d {\downarrow} F)^H| = |d {\downarrow} (F^H)|.$$

In order to apply Theorem B, we need to know that every morphism $d_1 \to d_2$ in $\mathcal{D}^H$ induces an equivalence $|d_2 {\downarrow} F^H| \to |d_1 {\downarrow} F^H|$. This equivalence holds by assumption, since if d_1 and d_2 are objects of $\mathcal{D}^H$, then $H \leq G_{d_1} \cap G_{d_2}$. Hence Theorem B allows us to conclude that for any object d of $\mathcal{D}^H$, the pullback

$$\begin{tikzcd} {|d {\downarrow} F^H|} \arrow[r] \arrow[d] & {|\mathcal{C}^H|} \arrow[d, "F^H"] \\ {|d {\downarrow} F^H|} \arrow[r] & {|\mathcal{D}^H|} \end{tikzcd}$$

is a homotopy pullback, which is to say the inclusion $|d {\downarrow} F^H| \to \mathrm{hfib}_d(|F^H|)$ is a homotopy equivalence. By Lemma A.5, it follows that $|d \downarrow F|^H \to \mathrm{hfib}_d(|F|)^H$ is an equivalence. This argument applies to any $H \leq G_d$, and therefore $|d {\downarrow} F| \to \mathrm{hfib}_d(|F|)$ is a G_d-equivalence. □

Remark A.7. There is a dual version of Theorem B for $F \downarrow d$, where we instead assume each morphism $d \to d'$ in $\mathcal{D}$ induces a $G_d \cap G_{d'}$-equivalence $|F {\downarrow} d| \to |F {\downarrow} d'|$.

Remark A.8. As is true non-equivariantly, the equivariant version Theorem B could be used to give an alternative proof of the equivariant version of Theorem A.

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