

# BOTT PERIODICITY AND THE INDEX OF ELLIPTIC OPERATORS

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## Introduction

IN an expository article (1) I have indicated the deep connection between the Bott periodicity theorem (on the homotopy of the unitary groups) and the index of elliptic operators. It is the purpose of this paper to elaborate on this connection and in particular to show how elliptic operators can be used to give a rather direct proof of the periodicity theorem. As hinted at in (1) the merit of such a proof is that it immediately extends to all the various generalizations of the periodicity theorem. Thus we obtain the 'Thom isomorphism' theorem together with its equivariant and real forms.

The equivariant case is particularly noteworthy because for this no proof of the Thom isomorphism theorem is known (even when the base space is a point) which does not use elliptic operators. In fact a main purpose of this paper is to present the proof for the equivariant case. This proof supersedes an earlier (unpublished) proof (7) which, though relying on elliptic operators, was more indirect than our present one.

Besides the fundamental use of elliptic operators there is another novel feature of our treatment. This is that we exploit the multiplicative structure of  $K$ -theory to produce a short-cut in the formal proof of the periodicity theorem. The situation is briefly as follows. One has the Bott map

$$\beta: K(X) \rightarrow K^{-2}(X)$$

which one wants to prove is an isomorphism. One first constructs (by elliptic operators or otherwise) a map

$$\alpha: K^{-2}(X) \rightarrow K(X)$$

and then has to show that  $\alpha$  is a 2-sided inverse of  $\beta$ . Now comes our novel trick: by using formal properties of  $\alpha$  and  $\beta$  we show that

$$\alpha\beta = 1 \Rightarrow \beta\alpha = 1.$$

Thus we need only prove  $\alpha\beta = 1$  and this is much the easier half.

The formal trick just described can be used to shorten substantially the elementary proof of the periodicity theorem given in (5). In fact,

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as we shall see, the proof of (5) is very closely connected with the proof presented here. The precise relationship will be thoroughly explored in § 7.

The layout of the paper is as follows. In § 1 we examine the formal structure of the periodicity theorem and show that all that we need is the construction of a map  $\alpha: K^{-2}(X) \rightarrow K(X)$  with certain simple formal properties. It is in this section that we meet the formal trick mentioned earlier. In § 2 we discuss the basic facts about indices of elliptic operators and elliptic families. This discussion will not enter into the technical analytic details which are by now fairly standard and for which further references will be given. In § 3 we show how to construct the required map  $\alpha$  by use of suitable elliptic families and we thus complete the proof of the complex periodicity theorem. The various generalizations are now treated similarly in §§ 4–6, with appropriate modifications and refinements. In § 7 we describe a number of variants on our construction of  $\alpha$  and show how one of these leads essentially to the elementary proof of (5). The reader who is interested in extracting the quickest elementary proof of the ordinary complex periodicity theorem will find all that he needs in § 1 and the end of § 7. Finally in § 8 we discuss the higher-dimensional analogues of the various alternative methods sketched in § 7. We make a few brief remarks about the possibility of deriving an elementary proof of the real and equivariant periodicity theorems—i.e. a proof which does not use Hilbert space but only algebraic properties of representations.

There is quite naturally a considerable overlap between this paper and the paper (9) written jointly with I. M. Singer on the index theorem. The main difference is that here the analysis is used to prove a theorem in topology whereas in (9) the situation is reversed.

As far as equivariant  $K$ -theory is concerned I should also point out that this theory was worked out jointly with G. B. Segal in (7).

### 1. Formal structure of periodicity theorem

For a compact space  $X$  we have the Grothendieck group  $K(X)$  of complex vector bundles on  $X$  [see for example (2)]. It is a commutative ring with identity. For locally compact  $X$  we introduce  $K$  with compact supports:

$$K(X) = \text{Ker}\{K(X^+) \rightarrow K(+)\}$$

where  $X^+ = X \cup \{+\}$  is the one-point compactification of  $X$ . Alternatively  $K(X)$  can be defined by complexes of vector bundles 'acyclic at  $\infty$ ' modulo a suitable equivalence relation [see (17) for details]. If we define

$K^{-a}(X) = K(\mathbb{R}^a \times X)$  then the tensor product of complexes turns  $\sum_{a \geq 0} K^{-a}(X)$  into an anti-commutative graded ring.

If  $V$  is a vector bundle over the compact space  $X$  then the exterior algebra  $\Lambda^*(V)$  defines in a natural manner a complex of vector bundles on  $V$  acyclic at  $\infty$  [cf. (2)]. The corresponding element of  $K(V)$  will be denoted by  $\lambda_V$ . In particular taking  $X = \text{point}$ ,  $V = \mathbb{C}$ , we have a basic element  $\lambda_{\mathbb{C}} \in K(\mathbb{C})$ . Actually the usual convention is to take its dual†

$$b = \lambda_{\mathbb{C}}^* \in K(\mathbb{C}) = K(\mathbb{R}^2)$$

as basic:  $b$  is called the Bott class. Multiplication (externally) by  $b$  then defines a homomorphism

$$\beta: K(X) \rightarrow K^{-2}(X)$$

called the Bott homomorphism. The *periodicity theorem*—which we want to prove—is

**THEOREM (1.1).**  $\beta: K(X) \rightarrow K^{-2}(X)$  is an isomorphism.

To prove the theorem we will, in later sections, construct a map  $\alpha: K^{-2}(X) \rightarrow K(X)$  which will be the 2-sided inverse of  $\beta$ . For the moment let us just assume that  $\alpha$  is defined† for all compact  $X$  and satisfies the following axioms

- (A1)  $\alpha$  is functorial in  $X$ ,
- (A2)  $\alpha$  is a  $K(X)$ -module homomorphism,
- (A3)  $\alpha(b) = 1$ .

In (A3)  $X = \text{point}$ ,  $b \in K^{-2}(\text{point})$  is the Bott class and  $K(\text{point})$  is identified with the integers in the usual way.

In a rather routine way we can now extend  $\alpha$  in the following manner:

**LEMMA (1.2).** *Let  $\alpha$  satisfy axioms (A1), (A2), and (A3). Then  $\alpha$  can be extended to a functorial homomorphism  $\alpha: K^{-a-2}(X) \rightarrow K^{-a}(X)$  which commutes with right multiplication by elements of  $K^{-p}(X)$ .*

*Proof.* We first extend  $\alpha$  to locally compact  $X$  by the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K^{-2}(X) & \rightarrow & K^{-2}(X^+) & \rightarrow & K^{-2}(+) \\ & & & & \downarrow \alpha & & \downarrow \alpha \\ 0 & \rightarrow & K(X) & \rightarrow & K(X^+) & \rightarrow & K(+) \end{array}$$

(the square commutes by (A1)). Replacing  $X$  by  $\mathbb{R}^a \times X$  we then get a map

$$\alpha: K^{-a-2}(X) \rightarrow K^{-a}(X)$$

† Passage to the dual bundle induces an involution  $*$  on  $K(X)$ . In the present case the dual also turns out to be the negative:  $b^* = -b$ .

‡ When necessary to make the space  $X$  explicit we write  $\alpha_X$ .

which is clearly functorial. To examine its multiplicative properties we observe first that (A2) can (using (A1)) be replaced by its 'external' form, i.e. the commutativity of the diagram

$$\begin{array}{ccc} K^{-2}(X) \otimes K(Y) & \xrightarrow{\phi} & K^{-2}(X \times Y) \\ \downarrow \alpha_X \otimes 1 & & \downarrow \alpha_{X \times Y} \\ K(X) \otimes K(Y) & \xrightarrow{\psi} & K(X \times Y) \end{array}$$

where  $X, Y$  are compact. To see this, note that all arrows are  $K(Y)$ -module homomorphisms, so it is enough to show that

$$\psi((\alpha_X \otimes 1)(u \otimes 1)) = \alpha_{X \times Y}(\phi(u \otimes 1)) \quad (u \in K^{-2}(X), 1 \in K(Y))$$

i.e. that

$$\pi^* \alpha_X(u) = \alpha_{X \times Y}(\pi^* u)$$

where  $\pi: X \times Y \rightarrow X$  is the projection. But this follows from the functoriality (A1) of  $\alpha$ . The commutativity of the corresponding diagram for locally compact  $X, Y$  follows now by passage to  $X^+, Y^+$ . Replacing  $X, Y$  by  $\mathbb{R}^q \times X, \mathbb{R}^p \times X$  and using the diagonal map we get a commutative diagram

$$\begin{array}{ccc} K^{-q-2}(X) \otimes K^{-p}(X) & \rightarrow & K^{-q-p-2}(X) \\ \downarrow \alpha \otimes 1 & & \downarrow \alpha \\ K^{-q}(X) \otimes K^{-p}(X) & \rightarrow & K^{-q-p}(X) \end{array}$$

or  $\alpha(xy) = \alpha(x)y$  as required.

*Remark.* Since  $\sum K^{-q}(X)$  is an anti-commutative ring there is no need to stipulate *right* multiplication in the lemma. Our reason for doing so is that in subsequent more general situations the anti-commutativity is not available. For the same reason let us consider formally the automorphism  $\theta$  of  $K^{-4}(X) = K(\mathbb{R}^2 \times \mathbb{R}^2 \times X)$  obtained by switching the two copies of  $\mathbb{R}^2$ . Then if  $x, y \in K^{-2}(X)$  we have

$$\theta(xy) = yx. \quad (1.3)$$

Now the maps  $\mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$  given by  $(u, v) \mapsto (v, -u)$  and  $(u, v) \mapsto (-v, u)$  are connected to the identity (by rotation through  $\pm \frac{1}{2}\pi$ ) and so induce the identity on  $K^{-4}(X)$ . Hence, if  $x \mapsto \tilde{x}$  denotes the involution of  $K^{-2}(X)$  induced by  $u \mapsto -u$  on  $\mathbb{R}^2$ , we deduce from (1.3)

$$xy = \tilde{y}x = y\tilde{x}. \quad (1.4)$$

Of course, in our present situation, the map  $(u, v) \mapsto (u, -v)$  is also connected to the identity so that  $x = \tilde{x}$ . This, however, uses the internal rotations of  $\mathbb{R}^2$  which we wish to avoid because they do not commute with the full group  $O(2)$  of symmetries of  $\mathbb{R}^2$ , and the situation will be even worse when  $\mathbb{R}^2$  is replaced by  $\mathbb{R}^{2n}$ . The only rotations used in

establishing (1.4), on the other hand, arose from the fact that  $\mathbf{R}^2 \times \mathbf{R}^2$  was the product of two copies of the same (linear) space: these rotations will clearly commute with the symmetries of  $\mathbf{R}^2$  and similarly for  $\mathbf{R}^{2n}$ . Thus (1.4) will generalize later on whereas the proof that  $x = \tilde{x}$  will not: in fact the equality  $x = \tilde{x}$  ultimately gets proved simultaneously with the periodicity theorem (see the proof of (1.5)).

We now come to the formal argument showing that an  $\alpha$  satisfying the axioms will yield Theorem (1.1).

**PROPOSITION (1.5).** *Suppose there exists an  $\alpha$  satisfying axioms (A1), (A2), and (A3). Then Theorem (1.1) holds and  $\alpha$  is the inverse of  $\beta$ .*

*Proof.* It is enough to prove (1.1) for compact  $X$ —the locally compact case follows by compactification as usual. Now (A1), (A2), and (A3) imply at once

$$\alpha(bx) = \alpha(b)x = x$$

for  $x \in K(X)$ . By (1.2), (1.4), and (A3) we have

$$\alpha(y)b = \alpha(yb) = \alpha(b\tilde{y}) = \alpha(b)\tilde{y} = \tilde{y},$$

for  $y \in K^{-2}(X)$ . Thus we have

$$\alpha\beta x = x, \quad \beta\alpha y = \tilde{y}.$$

Since  $y \mapsto \tilde{y}$  is an automorphism these equations imply that  $\beta$  and  $\alpha$  are isomorphisms, inverses of each other, and that  $y \mapsto \tilde{y}$  must be the identity.

*Remark.* Roughly speaking, and ignoring the involution  $y \mapsto \tilde{y}$ , the essence of (1.5) is the following. The axioms trivially imply that  $\alpha\beta = 1$ . On the other hand they also imply that, in the diagram,

$$K(X) \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\alpha} \end{array} K^{-2}(X) \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\alpha} \end{array} K^{-4}(X)$$

$\beta\alpha = \alpha\beta$  as endomorphisms of  $K^{-2}(X)$ . Thus we also have  $\beta\alpha = 1$ . In other words the identity  $\beta\alpha = 1$  for  $X$  is a consequence of  $\alpha\beta = 1$  for  $\mathbf{R}^2 \times X$ .

In § 3 we shall construct an  $\alpha$  satisfying the axioms. In fact a number of alternative constructions are possible—as we shall see in § 7—and it is partly for this reason that we have chosen to axiomatize the formal situation. It is also a help, when we come to the generalizations in §§ 4–6, to have the formal aspects divorced from the differential analysis involved in the construction of  $\alpha$ .

## 2. Index of elliptic families

In this section we shall review some relevant facts about elliptic differential operators. A general reference here is (15). For families of elliptic operators the details of what we need can be found in (18) or † (16).

Let  $M$  be a compact smooth manifold,  $E, F$  two smooth vector bundles over  $M$  and let  $d: \mathcal{D}(E) \rightarrow \mathcal{D}(F)$  be a linear elliptic differential operator ( $\mathcal{D}(E)$  denotes the space of smooth sections of  $E$ ). Then  $d$  is a Fredholm operator, i.e. it has closed range and

$$\dim \text{Ker } d < \infty, \quad \dim \text{Coker } d < \infty.$$

The index is defined by

$$\text{index } d = \dim \text{Ker } d - \dim \text{Coker } d.$$

It has the important property of being invariant under perturbation of  $d$ , and in particular depends only on the highest order terms of  $d$ .

Suppose now that  $Q$  is another smooth vector bundle over  $M$ . If  $Q$  were trivial then  $d$  would extend in an obvious way to an elliptic operator

$$d_Q: \mathcal{D}(E \otimes Q) \rightarrow \mathcal{D}(F \otimes Q).$$

If  $Q$  is not trivial we can construct extensions  $d_Q$  locally and then piece these together by partitions of unity. The resulting operator is not unique but its highest order terms are and so any two choices for  $d_Q$  have the same index. Thus

$$Q \mapsto \text{index } d_Q$$

is well defined and extends by linearity to give a homomorphism

$$K(M) \rightarrow \mathbf{Z}$$

which will be denoted by  $\text{index}_x$ . Actually  $K$  here refers to the category of smooth vector bundles but the usual kind of approximation implies that this is isomorphic to the ordinary  $K$  of continuous vector bundles.

We want now to consider families of operators. Thus let  $X$  be a compact space and let  $E$  be a family of vector bundles over  $M$  parametrized by  $X$ . This really means that  $E$  is a vector bundle over  $M \times X$  which is smooth in the  $M$ -direction: we shall not give the precise details here. If  $F$  is another such family then a family  $d$  of differential operators from  $E$  to  $F$  is a family

$$d_x: \mathcal{D}(E_x) \rightarrow \mathcal{D}(F_x)$$

with suitable continuity in  $x$  ( $E_x$  denotes the restriction of  $E$  to  $M \times \{x\}$ ). If all the  $d_x$  are elliptic (of the same order) we shall say that  $d$  is an elliptic family. It can be proved † [cf. (18) or (16)] that an elliptic family

† See also: Atiyah and Singer, 'The index of elliptic operators IV', *Ann. of Math.* (to appear).

has an index  $\in K(X)$ . If  $\dim \text{Ker } d_x$  is constant then the family  $\text{Ker } d_x$  forms a vector bundle  $\text{Ker } d$  over  $X$ , similarly for  $\text{Coker } d$  and we have

$$\text{index } d = \text{Ker } d - \text{Coker } d \in K(X).$$

In the general case we have to modify  $d$  in some way before this definition makes sense. The simplest is perhaps the method adopted in (18) in which  $d$  is regarded as a homomorphism of bundles† (of Fréchet spaces) over  $X$

$$d: \mathcal{D}(E) \rightarrow \mathcal{D}(F).$$

One shows that there is a trivial finite-dimensional vector bundle  $P$  over  $X$  and a bundle map

$$\phi: P \rightarrow \mathcal{D}(F)$$

so that

$$T = d + \phi: \mathcal{D}(E) \oplus P \rightarrow \mathcal{D}(F)$$

is an epimorphism.‡ This implies that  $\text{Ker } T_x$  is of constant dimension so that  $\text{Ker } T$  is a vector bundle. It is easy to verify that

$$\text{Ker } T - P \in K(X)$$

does not depend on the choice of  $P$ ,  $\phi$  and so we can take it as our definition of index  $d$ . An alternative method using Hilbert space is developed in (16).

The index of an elliptic family is a homotopy invariant and so depends only on the highest order terms. Thus if  $Q$  is a family of vector bundles over  $M$  (parametrized also by  $X$ ) then, just as before, we can form  $d_Q$  which will be an elliptic family from  $E \otimes Q$  to  $F \otimes Q$  and index  $d_Q \in K(X)$  will be independent of choices made. Then

$$Q \mapsto \text{index } d_Q$$

extends by linearity to give a homomorphism

$$\text{index}_d: K(M \times X) \rightarrow K(X).$$

Actually  $K(M \times X)$  should stand for the Grothendieck group of bundles smooth in the  $M$ -direction, but the usual kind of approximation shows that this coincides with the ordinary Grothendieck group.

If  $Q$  is a family of trivial bundles over  $M$ , i.e. if  $Q$  is a bundle on  $M \times X$  induced from a bundle  $Q_0$  on  $X$  then there is an obvious choice for  $d_Q$  and (when  $\dim \text{Ker } d_x$  is constant) we clearly have

$$\begin{aligned} \text{Ker } d_Q &\cong \text{Ker } d \otimes Q_0, \\ \text{Coker } d_Q &\cong \text{Coker } d \otimes Q_0. \end{aligned}$$

†  $\mathcal{D}(E)$  stands for the bundle over  $X$  whose fibre at  $x \in X$  is  $\mathcal{D}(E_x)$ .

‡ Locally the existence of  $P$  follows from standard semi-continuity properties of  $\text{Coker } d$ . We then use a partition of unity to construct a global  $P$ .

so that  $\text{index } d_Q = (\text{index } d) \otimes Q_0$ . In fact the definition of  $\text{index } d_Q$  in the general case shows that this formula always holds. Thus

$$\text{index}_d: K(M \times X) \rightarrow K(X) \quad (2.1)$$

is a  $K(X)$ -module homomorphism.

If  $Y$  is another compact space and  $f: Y \rightarrow X$  a continuous map we can consider the induced family  $f^*(d)$  of elliptic operators parametrized by  $Y$ . The definition of  $\text{index}_d$  shows that it is functorial, that is we have a commutative diagram

$$\begin{array}{ccc} K(M \times X) & \xrightarrow{\text{index}_d} & K(X) \\ \downarrow (1 \times f)^* & & \downarrow f^* \\ K(M \times Y) & \xrightarrow{\text{index}_{f^*(d)}} & K(Y). \end{array}$$

In particular if  $X$  is a point (so that  $M \times X = M$  and  $d$  is just an elliptic operator on  $M$ ) we can consider the *constant family*  $f^*(d)$  parametrized by any compact space  $Y$ ,  $f: Y \rightarrow \text{point}$  being the constant map. By a slight abuse of notation we shall omit the symbol  $f^*$  and write

$$\text{index}_d: K(M \times Y) \rightarrow K(Y)$$

instead of  $\text{index}_{f^*(d)}$ .

Summarizing we may state our results in the form of a proposition:

**PROPOSITION (2.2).** *Let  $d$  be an elliptic differential operator on a compact manifold  $M$ . For any smooth vector bundle  $Q$  on  $M$  let  $d_Q$  be an extension of  $d$  to  $Q$  (i.e. symbolically  $\sigma(d_Q) = \sigma(d) \otimes Id_Q$ ). Then*

$$Q \mapsto \text{index } d_Q$$

*defines a homomorphism*

$$\text{index}_d: K(M) \rightarrow \mathbb{Z}.$$

*Moreover there is a functorial extension of this to compact spaces  $X$ , so that for each  $X$*

$$\text{index}_d: K(M \times X) \rightarrow K(X)$$

*is a  $K(X)$ -module homomorphism.*

This proposition is the key result on indices of operators which we need. What remains is to choose appropriately the manifold  $M$  and the operator  $d$  for the various applications.

In fact we shall choose very classical operators on spheres and projective spaces and, as we shall show in § 7 and § 8, it is possible in these cases to define  $\text{index}_d$  in a number of alternative ways. One variation in particular (method (2)) uses only the definition of the index of an (abstract) family of Fredholm operators (on a fixed Hilbert space) as



developed in the Appendix to (2). It is therefore not essential to use the index of more general families of elliptic operators as in (16) or (18), but we have presented things in this context because it seems conceptually clearer.

*Remark.* This proposition extends quite straightforwardly to  $K_G$ -theory, provided we use a  $G$ -invariant operator  $d$ , and to  $KR$ -theory, provided we use a Real operator  $d$ . These extensions are covered by the treatment in (18) and we shall use them in the later sections.

### 3. Proof of periodicity

We shall apply Proposition (2.2) with  $M$  being the complex projective line  $P_1(\mathbb{C})$  and  $d$  being the  $\bar{\partial}$  operator from functions to forms of type  $(0, 1)$ :

$$f \mapsto \frac{\partial f}{\partial \bar{z}} d\bar{z}.$$

For any holomorphic vector bundle  $Q$  over  $P_1(\mathbb{C})$  the operator  $\bar{\partial}$  has a natural extension  $\bar{\partial}_Q$  and it is well known [cf. (10)] that

$$\begin{aligned} \text{Ker } \bar{\partial}_Q &\cong H^0(P_1, \mathcal{O}(Q)), \\ \text{Coker } \bar{\partial}_Q &\cong H^1(P_1, \mathcal{O}(Q)), \end{aligned}$$

where  $H^0, H^1$  denote the cohomology groups of the sheaf  $\mathcal{O}(Q)$  of germs of holomorphic sections of  $Q$ . Now for  $Q = 1$  the trivial line-bundle we have

$$H^0 \cong \mathbb{C}, \quad H^1 = 0,$$

while for the dual†  $H^{-1}$  of the standard line bundle  $H$  over  $P_1$  we have

$$H^0 = H^1 = 0.$$

Thus for the homomorphism

$$\text{index}_{\bar{\partial}}: K(P_1) \rightarrow \mathbb{Z}$$

we have (3.1)

$$\text{index}_{\bar{\partial}}(1-H^{-1}) = 1.$$

We now identify  $P_1$  with the 2-sphere  $S^2$  and so with the one-point compactification of  $\mathbb{R}^2$ . Thus we have the exact sequence

$$0 \rightarrow K(\mathbb{R}^2) \rightarrow K(P_1) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0,$$

where  $\epsilon$  is the augmentation. The elements  $1-H$  and  $1-H^{-1}$  are in the kernel of  $\epsilon$  and hence are elements of  $K(\mathbb{R}^2)$ . In fact  $1-H$  is the element denoted in § 1 by  $\lambda_{\mathbb{C}}$  and so

$$1-H^{-1} = 1-H^* = \lambda_{\mathbb{C}}^* = b.$$

† Standard notation for cohomology and for Hopf bundles leads to an unfortunate, but unavoidable, clash of notation involving  $H$ .

Thus equation (3.1) asserts that

$$\text{index}_{\bar{g}}(b) = 1. \quad (3.2)$$

Returning to § 1, we recall (Proposition (1.5)) that to prove the periodicity theorem we have only to define a homomorphism

$$\alpha: K^{-2}(X) \rightarrow X$$

for compact  $X$  satisfying axioms (A1), (A2), (A3). We are now in a position to construct this  $\alpha$ . We define  $\alpha$  as the composition

$$K^{-2}(X) = K(\mathbb{R}^2 \times X) \rightarrow K(S^2 \times X) \xrightarrow{\text{index } \bar{g}} K(X).$$

The functoriality (A1) and  $K(X)$ -module property (A2) follow from (2.2), and (A3) is just equation (3.2). The periodicity theorem (1.1) is therefore proved.

#### 4. Equivariant case

Let  $G$  be a compact Lie group,  $X$  a compact  $G$ -space, then we have the group  $K_G(X)$  [see (17)]—the Grothendieck group of  $G$ -vector bundles over  $X$ . Let  $V$  be a complex  $G$ -module, then, just as in § 1, the exterior algebra  $\Lambda^*(V)$  defines an element

$$\lambda_V \in K_G(V).$$

If  $1$  denotes the trivial 1-dimensional  $G$ -module then the projective space  $P(V \oplus 1)$  is a compactification of  $V$  and so we have a natural homomorphism

$$j: K_G(V) \rightarrow K_G(P(V \oplus 1)).$$

Now  $\Lambda^*(V)$ , regarded as a complex of vector bundles over  $V$ , has a natural extension to  $P(V \oplus 1)$  [see (2) 100] and this shows that

$$j(\lambda_V) = \sum (-1)^i H^i \lambda^i(V), \quad (4.1)$$

where  $H$  denotes the standard line-bundle on  $P(V \oplus 1)$ . Taking duals we get

$$j(\lambda_V^*) = \sum (-1)^i H^{-i} \lambda^i(V^*). \quad (4.2)$$

We are now in a position to formulate the main theorem of equivariant  $K$ -theory:

**THEOREM (4.3).** *For any compact  $G$ -space  $X$  and any complex  $G$ -module  $V$ , multiplication by  $\lambda_V^*$  induces an isomorphism*

$$K_G(X) \rightarrow K_G(V \times X).$$

*Remark.* Since products are compatible with duality ( $a \mapsto a^*$ ) it follows of course that we can replace  $\lambda_V^*$  by  $\lambda_V$  in this theorem. In fact  $\lambda_V^*$  and  $\lambda_V$  differ by a unit of  $K_G(X)$  and it is not hard to show that

$$\lambda_V^* = (-1)^n (\lambda^n(V))^{-1} \lambda_V \quad (n = \dim V). \quad (4.4)$$

To prove the theorem we proceed on the same formal lines as in § 1. Proposition (1.5) extends to the present more general situation—simply replace  $R^*$  by  $V$  throughout—so that we have only to construct a map

$$d: K_G(V \times X) \rightarrow K_G(X)$$

which is functorial in (compact)  $X$ , a  $K_G(X)$ -module homomorphism and satisfies the analogue of (A3), namely

$$\alpha(\lambda_V^*) = 1 \in R(G), \tag{4.5}$$

where  $R(G) = K_G(\text{point})$  is the representation ring of  $G$ .

To construct  $\alpha$  we consider the Dolbeault complex on the projective space  $P(V \oplus 1)$ . Using a  $G$ -invariant hermitian metric we construct the elliptic operator

$$D = \bar{\partial} + \bar{\partial}^* : \Omega^+ \rightarrow \Omega^-$$

where  $\Omega^+$  denotes the direct sum of all forms of type  $(0, 2k)$  and  $\Omega^-$  the direct sum of forms of type  $(0, 2k+1)$ . For details concerning this construction we refer to (15) 325. For any holomorphic  $G$ -vector bundle  $Q$  on  $P(V \oplus 1)$ ,  $D$  has a natural extension  $D_Q$  and we have natural ( $G$ -module) isomorphisms:

$$\begin{aligned} \text{Ker } D_Q &\cong \sum_{k \geq 0} H^{2k}(P, \mathcal{O}(Q)), \\ \text{Coker } D_Q &\cong \sum_{k \geq 0} H^{2k+1}(P, \mathcal{O}(Q)). \end{aligned}$$

Thus  $\text{index } D_Q = \sum (-1)^i H^i(P, \mathcal{O}(Q)) \in R(G)$ .

Now for the powers of the standard bundle  $H$  one has [cf. (11) (18.2.1) and (18.2.2)]:

$$H^i(P, \mathcal{O}(H^{-r})) = 0 \quad (1 \leq r \leq n)$$

for all  $i$  ( $n = \dim V = \dim P(V \oplus 1)$ ), and

$$\begin{aligned} H^i(P, \mathcal{O}) &= 0 \quad \text{for } i \geq 1 \\ &\cong \mathbf{C} \quad \text{for } i = 0. \end{aligned}$$

Hence, by (4.2), we have

$$\text{index } D_Q = 1 \quad \text{for } Q = j(\lambda_V^*). \tag{4.6}$$

Finally, then, we define the required map  $\alpha$  as the composition

$$K_G(V \times X) \xrightarrow{j} K_G(P(V \oplus 1) \times X) \xrightarrow{\text{index}_D} K_G(X).$$

The functoriality and module properties follow from the  $K_G$ -version of (2.2), and (4.5) follows from (4.6). This completes the proof of Theorem (4.3).

Suppose now that  $G = U(n)$ ,  $V = \mathbf{C}^n$  and that  $X$  is a free  $G$ -space with  $X/G = Y$ . Then

$$K_G(X) \cong K(Y), \quad K_G(V \times X) \cong K(E),$$

where  $E = (V \times X)/G$  is the vector bundle over  $Y$  associated to the principal bundle  $X$ . Moreover the element  $\lambda_V \in K_G(V \times X)$  corresponds by the above isomorphism with the element  $\lambda_E \in K(E)$ . Hence as a special case of the equivariant periodicity theorem we get the *Thom isomorphism theorem*:

**THEOREM (4.7).** *Let  $E$  be a complex vector bundle over the compact space  $Y$ . Then multiplication by  $\lambda_E$  induces an isomorphism*

$$K(Y) \rightarrow K(E).$$

More generally if  $H$  is another group and  $Y$  is an  $H$ -space with  $E$  an  $H$ -vector bundle then we apply (4.3) with  $G = U(n) \times H$ . In this case we have [see (17) (2.1)]:

$$K_G(X) \cong K_H(Y), \quad K_G(V \times X) \cong K_H(E),$$

and so exactly as above we get:

**THEOREM (4.8).** *Let  $E$  be an  $H$ -vector bundle over the compact  $H$ -space  $Y$ . Then multiplication by  $\lambda_E$  induces an isomorphism*

$$K_H(Y) \rightarrow K_H(E).$$

*Remark.* The reasoning above (due to G. B. Segal) shows that, once one has passed to equivariant theory, the periodicity theorem in the form (4.3) really includes the apparent generalization of the Thom isomorphism theorem.

Taking  $X = \text{point}$  in (4.3) and using the exact sequence for the pair  $B(V), S(V)$ —the unit ball and sphere of  $V$ —we obtain (as in (2) (2.7.6)) a formula for  $K_G^*(S(V))$ . This gives essentially the Theorem stated without proof in (1), § 3. The details will be developed elsewhere.

Besides its application to the periodicity theorem (4.3) the Dolbeault complex of projective space can also be used to establish the ‘splitting principle’ for  $K_G$ -theory as we shall now explain. Just as above, replacing  $P(V \oplus 1)$  by  $P(V)$ , the elliptic operator  $D$  associated to the Dolbeault complex defines a functorial  $K_G(X)$ -homomorphism

$$\text{index}_D: K_G(P(V) \times X) \rightarrow K_G(X).$$

Since for the sheaf  $\mathcal{O}$  on  $P(V)$  we have

$$H^0(P(V), \mathcal{O}) \cong \mathbf{C}, \quad H^q(P(V), \mathcal{O}) = 0 \quad (q \geq 1)$$

it follows that  $\text{index}_D(1) = 1 \in K_G(X)$ .

This implies

PROPOSITION (4.8). *If  $X$  is a  $G$ -space and  $V$  a complex  $G$ -module then the  $K_G(X)$ -homomorphism*

$$K_G(X) \rightarrow K_G(P(V) \times X)$$

*has a functorial left inverse. In particular it is injective.*

Suppose in particular we take

$$V = \mathbb{C}^n, \quad G = U(n) \times H,$$

where  $H$  acts trivially on  $\mathbb{C}^n$ . Since  $P(\mathbb{C}^n) = U(n)/(U(n-1) \times U(1))$  we have

$$K_G(P(\mathbb{C}^n) \times X) = K_{U(n-1) \times U(1) \times H}(X)$$

and the homomorphism of (4.8) becomes just the map

$$K_{U(n) \times H}(X) \rightarrow K_{U(n-1) \times U(1) \times H}(X)$$

induced by the inclusion  $U(n-1) \times U(1) \rightarrow U(n)$ . By iteration this finally gives

PROPOSITION (4.9). *Let  $j: T \rightarrow U$  be the inclusion of the maximal torus in the unitary group  $U = U(n)$ . For any compact  $U$ -space  $X$  let*

$$j^*: K_U(X) \rightarrow K_T(X)$$

*be the map induced by  $j$ . Then there is a functorial homomorphism of  $K_U(X)$ -modules:*

$$j_*: K_T(X) \rightarrow K_U(X)$$

*which is a left inverse of  $j^*$ . In particular  $j^*$  is injective.*

*Remarks.* (1) We have obtained  $j_*$  by iterating the construction of (4.8) using projective spaces. However we can equally well define it at one go by using the Dolbeault complex of the flag manifold  $U/T$ . The important point is that the sheaf cohomology of  $U/T$ —like that of any rational variety—has the same properties as for projective space. Thus, more generally, we can replace  $U$  in (4.9) by any compact connected Lie group: it being well known that  $G/T$  has the structure of a homogeneous rational algebraic variety.

(2) Proposition (4.9) amounts to a ‘splitting principle’ because it enables us, in many problems, to pass from the unitary group to the torus. The first proof of equivariant periodicity, given in (7), was on these lines, and other applications, similar to those in (2), have been given in (17). A particularly striking application of a rather different kind will be given in (8).

(3) An alternative approach to (4.8) and (4.9) is to use the isomorphism

$$K_G(P(V)) \cong K_{G \times S^1}(S(V))$$

where  $S^1 = \{\lambda \in \mathbb{C}; |\lambda| = 1\}$  acts on  $S(V) \subset V$  by scalar multiplication. We can now calculate  $K_{G \times S^1}(S(V))$  using (4.8) as indicated above, and we obtain in fact the complete structure of  $K_G(P(V))$ , showing that it is a free module of finite rank ( $= \dim V$ ) over  $R(G)$  with canonical generators. This approach will be developed on a future occasion.

### 5. Real case

In (3)† we introduced a functor  $KR(X)$  defined for spaces with involution (also called *real spaces*). To avoid possible confusion with the ordinary use of real (e.g. for vector spaces or vector bundles) we shall write *Real* (with a capital R) for the category with involution. Thus a *Real* vector space is a complex vector space which is the complexification of a real vector space. In this section we shall show how to extend the results of § 4 to  $KR$ .

First we introduce the equivariant form of  $KR$ . Thus let  $X$  be a *Real space*,  $G$  a *Real Lie group* and let  $X$  be a *Real  $G$ -space*. This means  $G$  has an involutory automorphism  $g \mapsto \bar{g}$  and that  $\overline{g\bar{x}} = \bar{g}x$ . A *Real  $G$ -vector bundle* over  $X$  means a complex  $G$ -vector bundle  $E$  with a compatible *Real structure*, so that  $E$  is both a *Real vector bundle* and a *Real  $G$ -space*. The Grothendieck group of *Real  $G$ -vector bundles* over  $X$  is denoted by  $KR_G(X)$ .

If  $V$  is a *Real  $G$ -module* then the exterior algebra of  $V$  defines as usual an element  $\lambda_V \in KR_G(V)$  and we can formulate:

**THEOREM (5.1).** *For any Real compact  $G$ -space  $X$  and any Real  $G$ -module  $V$  multiplication by  $\lambda_V^*$  induces an isomorphism*

$$KR_G(V) \rightarrow KR_G(V \times X).$$

*Proof.* The proof proceeds exactly as in § 4 for the complex case and we shall simply mention those points which require special comment. In the first place, as observed in § 2, one has to extend Proposition (2.2) to *Real operators* (in the sense of (3)). Thus a *Real elliptic differential operator*  $d$  on a *Real manifold*  $M$  defines functorially a homomorphism

$$KR_G(M \times X) \rightarrow KR_G(X).$$

Next we have to observe that if  $V$  is a *Real  $G$ -module*  $P(V \oplus 1)$  is a *Real  $G$ -space* and the Dolbeault complex is a *Real elliptic complex*. The involution on  $P(V \oplus 1)$  can be regarded as an isomorphism of the complex manifold  $P$  with the complex conjugate manifold  $\bar{P}$  and so

† (3) is also reprinted as part of (2).

maps the Dolbeault complex of  $\bar{P}$  isomorphically onto that of  $P$ . Thus, choosing a Real  $G$ -invariant hermitian metric on  $P$ , the operator  $D = \bar{\partial} + \bar{\partial}^*$  of § 4 is Real.

As in § 4 we can also use the Real elliptic operator  $D$  to define a left inverse for

$$KR_G(X) \rightarrow KR_G(P(V) \times X),$$

where  $V$  is a Real module for the Real group  $G$ . Since  $T$  (the standard maximal torus of  $U(n)$ ) and the other intermediate groups used in the proof of (4.9) are all Real subgroups of  $U(n)$  the proof applies also in  $KR$ -theory to give

**PROPOSITION (5.2).** *Let  $j: T \rightarrow U$  be the inclusion of the standard maximal torus in  $U = U(n)$ , and let  $X$  be a Real  $U$ -space. Then*

$$j^*: KR_U(X) \rightarrow KR_T(X)$$

*has a functorial left inverse*

$$j_*: KR_T(X) \rightarrow KR_U(X)$$

*which is a homomorphism of  $KR_U(X)$ -modules.*

*Remarks.* (1) This proposition, which will be used crucially in (8), is one of the justifications for  $KR$ -theory—as opposed to  $KO$ -theory. The point is that the analogue of (5.2) for  $KO$  (i.e. when  $U$  and  $T$  are taken with trivial involutions) is false.

(2) Again, as in § 4,  $j_*$  can be defined directly using the flag manifold  $U/T$ —which is a Real algebraic variety. For a general compact connected Lie group  $G$  the same methods will apply provided the involution on  $G$  fixes a maximal torus  $T$  and interchanges positive and negative roots.

## 6. Spinor case

So far the theorems we have proved have compared  $K(X)$  and  $K(V \times X)$  with  $V$  a complex vector space. We want now to consider the case when  $V$  is real and for this it is necessary to introduce the Spinor groups. A suitable reference for the material we need is (6) and (4), § 8.

When  $V$  is a real vector space (or  $G$ -module) the appropriate compactification for our purposes is not the projective space but the one-point compactification  $V^+$ , namely the sphere. The elliptic differential operator on the sphere which we need is the Dirac operator [cf. (15) 92 or (4) § 8 for details]. More precisely, assuming  $\dim V \equiv 0 \pmod{8}$  the total Spin bundle  $S$  of the sphere decomposes into two halves

$$S = S^+ \oplus S^-$$

and the total Dirac operator maps  $S^+$  to  $S^-$  and  $S^-$  to  $S^+$ . Moreover in these dimensions all the bundles and the operator are real. We shall be interested in the operator  $D$  from  $S^+$  to  $S^-$  which is the restriction of the total Dirac operator. Since the Dirac operator is self-adjoint its restriction from  $S^-$  to  $S^+$  is  $D^*$ . If we regard  $V^+$  as the homogeneous space  $\text{Spin}(8n+1)/\text{Spin}(8n)$  then  $D$  is a homogeneous operator. Thus if  $V$  is a Spin  $G$ -module, meaning that the action on  $V$  factors through a given homomorphism  $G \rightarrow \text{Spin}(8n)$ , then  $D$  is a  $G$ -invariant real elliptic operator. Hence, by the real version of (2.2), it will induce a homomorphism

$$\text{index}_D: KO_G(V^+ \times X) \rightarrow KO_G(X).$$

The next stage in our programme is to find an element

$$u \in KO_G(V) \subset KO_G(V^+)$$

with †  $\text{index}_D u = 1 \in RO(G) = KO_G(\text{point})$ .

If we use the Riemannian connection on the sphere to extend the Dirac operator to act on  $S \otimes S$ , then it is not hard to show that we obtain the operator  $d + d^*$  acting on the exterior differential forms. Thus the kernel of this extended Dirac operator coincides with the space of harmonic forms on the sphere and so with its cohomology. Hence we can compute  $\text{index}_D S^+$  and  $\text{index}_D S^-$  in terms of the cohomology of the sphere. To avoid the work involved in identifying  $d + d^*$  with the extension of  $D$  we can however appeal to a general principle, explained in (10), according to which the index  $\in R(G)$  of a homogeneous elliptic operator on a homogeneous space  $G/H$  depends only on (the difference of) the homogeneous bundles (i.e.  $H$ -modules) where the operator acts and not on the operator itself. Now by explicit character computations [see (4) §§ 6 and 8] we have the following identities in  $RO(\text{Spin}(8n))$ :

$$\begin{aligned} (\Delta^+ - \Delta^-)^2 &= \sum (-1)^t \Lambda^t, \\ (\Delta^+ + \Delta^-)(\Delta^+ - \Delta^-) &= \Lambda_+^{4n} - \Lambda_-^{4n}, \end{aligned}$$

where  $\Delta^+$ ,  $\Delta^-$  are the two  $\frac{1}{2}$ -Spin representations,  $\Lambda^t$  are the exterior powers of  $R^{8n}$  and  $\Lambda_+^{4n}$ ,  $\Lambda_-^{4n}$  are the two irreducible components of  $\Lambda^{4n}$  given by the eigenvalues of  $*$  (the duality operator). From these it follows by the principle explained above that

$$\begin{aligned} \text{index}_D S^+ - \text{index}_D S^- &= \text{Euler characteristic of } V^+ = 2, \\ \text{index}_D S^+ + \text{index}_D S^- &= \text{Hirzebruch signature}^\ddagger \text{ of } V^+ = 0. \end{aligned}$$

†  $RO(G)$  stands for the Grothendieck group of real  $G$ -modules: it is a subgroup of  $R(G)$ .

‡ See (4), § 6 for the elliptic operator whose index is the Hirzebruch signature.



Note that the connected group  $\text{Spin}(8n)$  acts trivially on the cohomology of  $V^+$  and so the Euler characteristic is equal to 2 as an element of  $RO \text{Spin}(8n)$  and hence of  $RO(G)$ . By subtraction we now obtain†

$$\text{index}_D S^+ = 1 \in RO(G).$$

Returning to the original Dirac operator we consider the effect of the anti-podal map on the sphere. This is compatible with the Dirac operator (and with the action of  $\text{Spin}(8n)$ ) but it interchanges  $S^+$  and  $S^-$ . It therefore induces a module isomorphism

$$\text{Ker } D \cong \text{Ker } D^*,$$

showing that‡  $\text{index}_D 1 = 0 \in RO(G)$ .

Hence the required element  $u \in KO_G(V)$  can be obtained by subtracting from  $S^+$  the trivial bundle with fibre  $S^+_\infty$  (the fibre of  $S^+$  at  $\infty$ ). Thus  $u$  is just the element given by the graded Clifford module  $M$  associated to  $\Delta = \Delta^+ \oplus \Delta^-$  in the manner of (6), § 11. Note that, since  $G$  acts through  $\text{Spin}(8n)$ , it acts on  $M$  compatibly with its Clifford structure. This element  $u$  plays a fundamental role in  $KO$ -theory, as explained in (6), and to give it a name we shall call it the Bott class of the module  $V$ .

We now have all the necessary data to proceed formally as in the preceding sections and prove

**THEOREM (6.1).** *Let  $X$  be a compact  $G$ -space,  $V$  a real  $\text{Spin } G$ -module of dimension  $8n$ , and let  $u \in KO_G(V)$  be the Bott class of  $V$ . Then multiplication by  $u$  induces an isomorphism*

$$KO_G(X) \rightarrow KO_G(V \times X).$$

*Remarks.* Taking  $V$  trivial of dimension 8 this gives the mod 8 periodicity of  $KO_G$ . Also taking  $G = \text{Spin}(8n)$  it gives the Thom isomorphism theorem on the lines explained in § 4, for  $\text{Spin}(8n)$ -bundles [cf. (6) (12.3) (i)].

We want now to obtain the Real version of this theorem. We suppose therefore that  $G$  is a Real group, that  $X$  is a Real  $G$ -space and that  $V$  with trivial involution is a Real  $G$ -space ( $G$  acting linearly). As before we assume  $\dim V = 8n$ . Furthermore we assume that  $V$  is a Real  $\text{Spin}^c G$ -module, i.e. the action of  $G$  on  $V$  factors through a Real homomorphism

$$G \rightarrow \text{Spin}^c(8n),$$

where  $\text{Spin}^c(8n) = \text{Spin}(8n) \times_{Z_2} U(1)$

† An alternative way to derive this equation is to apply the fixed-point formula of (4): we leave this amusing exercise to the reader.

‡ In fact by a result of Lichnerowicz [see (14)] there are no harmonic spinors on the sphere so that  $\text{Ker } D = \text{Ker } D^* = 0$ .

is the group defined in (6), § 3 and the involution on it is induced by complex conjugation of  $U(1)$ . A graded Real module for the Clifford algebra  $C_{8n} \otimes_{\mathbb{R}} \mathbb{C}$  then defines a Real representation for  $\text{Spin}^c(8n)$ . Thus, corresponding to the real representation  $\Delta$  of  $\text{Spin}(8n)$  there is a Real representation  $\Delta^c = \Delta \otimes_{\mathbb{R}} \mathbb{C}$  of  $\text{Spin}^c(8n)$ . This defines a Real vector bundle  $S^c$  over the sphere

$$V^+ = \text{Spin}^c(8n+1)/\text{Spin}^c(8n).$$

In fact  $S^c = S \otimes_{\mathbb{R}} \mathbb{C}$  and it decomposes into  $S^c_+$  and  $S^c_-$ . The (complexified) Dirac operator acts on  $S^c$  as a Real operator and its restriction  $D$  from  $S^c_+$  to  $S^c_-$  is also Real. Since  $D$  is compatible with the action of  $G$  it induces as before a homomorphism

$$\text{index}_D: KR_G(V^+ \times X) \rightarrow KR_G(X).$$

Now the graded Real Clifford module  $M^c$  associated to  $\Delta^c$  defines, by the construction of (6), § 11, an element  $u \in KR_G(V^+)$ —called again the Bott class. The calculations made before show that

$$\text{index}_D u = 1 \in KR_G(\text{point}).$$

Hence we obtain

**THEOREM (6.2).** *Let  $G$  be a Real group,  $X$  a Real  $G$ -space,  $G \rightarrow \text{Spin}^c(8n)$  a Real homomorphism. Let  $V$  denote  $\mathbb{R}^{8n}$  with induced  $G$ -action and trivial involution and let  $u \in KR_G(V)$  be the Bott class of  $V$ . Then multiplication by  $u$  induces an isomorphism*

$$KR_G(X) \rightarrow KR_G(V \times X).$$

*Remarks.* (1) Taking all involutions trivial we recover (6.1).

(2) Taking  $X = Y \times S^{1,0}$  where  $S^{1,0}$  denotes as in (3) § 3 the anti-podal 0-sphere (acted on trivially by  $G$ ) we have as in (3) (3.3)

$$KR_G(X) \cong K_G(Y), \quad KR_G(V \times X) \cong K_G(V \times Y).$$

Now take  $G = \text{Spin}^c(8n)$  and we deduce the Thom isomorphism for  $K$  for  $\text{Spin}^c(8n)$ -bundles [cf. (6), (12.3) (ii)]. Note that the restriction to  $8n$  dimensions is not significant because, by the periodicity theorem (1.1), we can alter dimensions by even integers.

We have now computed  $KR_G(V \times X)$  in the two extreme cases

- (i)  $V$  with complex structure,
- (ii)  $V$  with trivial involution.

In fact we can combine these together in one further generalization. Following (3) § 4 we let  $\text{Cliff } R^{p,q}$  denote the Clifford algebra (over  $\mathbb{R}$ ) of the quadratic form  $-\left(\sum_1^p y_i^2 + \sum_1^q x_j^2\right)$  with the involution induced by

$(y, x) \mapsto (-y, x)$ . We form the complexification  $\text{Cliff } R^{p,q} \times_{\mathbf{R}} \mathbf{C}$  and extend the involution by conjugation on  $\mathbf{C}$ . The group of units of this algebra contains the group  $\text{Spin}^c(p+q)$  and, with the induced involution, we denote it by  $\text{Spin}^c(p, q)$ . On the lines of (3) and (6) it is not difficult to show [cf. also (13)] that, if  $p \equiv q \pmod{8}$ , the representation  $\Delta^c$  of  $\text{Spin}^c(p+q)$  has a Real structure compatible with the Real structure of  $\text{Spin}^c(p, q)$ . This implies that the bundles  $S_+^c, S_-^c$  over  $(R^{p,q})^+$  are Real and that the Dirac operator from  $S_+^c$  to  $S_-^c$  is also Real. Thus finally we get

**THEOREM (6.3).** *Let  $G \rightarrow \text{Spin}^c(p, q)$  be a Real homomorphism with  $p \equiv q \pmod{8}$  and let  $u \in KR_G(R^{p,q})$  be the Bott class. Then, for any real  $G$ -space  $X$ , multiplication by  $u$  induces an isomorphism*

$$KR_G(X) \rightarrow KR_G(R^{p,q} \times X).$$

(6.2) is, of course, the special case of (6.3) with  $p = 0$ . On the other hand we can take  $p = q$  and observe that the homomorphism

$$\tilde{l}: U(p) \rightarrow \text{Spin}^c(2p)$$

of (6), § 5 is actually a Real homomorphism

$$U(p) \rightarrow \text{Spin}^c(p, p).$$

As in (6) (5.11) one can then show that this homomorphism is compatible with the basic modules used to define the Bott classes. This shows that (5.1) is also a special case of (6.3). Thus Theorem (6.3) is the most general of its type.

## 7. Comparison with elementary proof

In this section we shall examine the proof of periodicity given in § 3, discuss a number of variants of it and show how it is related to the elementary proof given in (5). Since our aim will be purely explanatory we shall only indicate proofs and many technical points will be passed over.

We return to the situation of § 3 where we used the  $\bar{\partial}$  operator on  $P_1$  to define the crucial homomorphism

$$\text{index}_{\bar{\partial}}: K(P_1 \times X) \rightarrow X.$$

There are in fact two other methods of constructing this homomorphism which amount to minor variations on the same theme. We still use the

basic notion of the index of an elliptic family but the  $\bar{\partial}$  operator is replaced by

- (1) a boundary value problem for the disc,  
or (2) a singular integral operator on the circle.

Both of these are very classical and we shall now briefly describe them.

*Alternative (1)*

We take the differential operator in the disc  $|z| \leq 1$

$$(u, v) \mapsto \left( \frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial z} \right).$$

With the boundary condition  $u - v = \phi$ . More formally we consider the operator  $T$  defined by

$$T(u, v) = \left( \frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial z}, (u - v) | S^1 \right),$$

where  $u | S^1$  denotes the boundary value of  $u$ . Then, whether considered on  $C^\infty$  functions or in suitable Hilbert spaces,  $T$  is a Fredholm operator [(12) Chapter X]. In fact it is clear that  $\text{Ker } T$  consists of the constants while  $\text{Coker } T = 0$  so that

$$\text{index } T = 1.$$

Except that we have identified the interior and exterior of the unit disc by  $z \mapsto z^{-1}$  it is clear that  $T$  is just the 'transmission operator' corresponding to  $\bar{\partial}$  on  $P_1$ .

In dealing with the  $\bar{\partial}$  operator in § 3 it was important to extend it to an operator  $\bar{\partial}_E$  on a smooth vector bundle  $E$  on  $P_1$ . Now  $E$  can always be constructed by taking two vector spaces  $E^0, E^\infty$  and a smooth map

$$f: S^1 \rightarrow \text{Iso}(E^0, E^\infty),$$

unique up to homotopy [(2) § 2.2]. For the operator  $T$  it is more convenient to define its extension  $T_f$  by

$$T_f(u, v) = \left( \frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial z}, f \cdot u | S^1 - v | S^1 \right),$$

where  $u, v$  are vector-valued functions with values in  $E^0, E^\infty$  respectively and  $f \cdot u | S^1$  denotes the function  $S^1 \rightarrow E^\infty$  given by

$$z \mapsto f(z)u(z) \quad (|z| = 1).$$

By approximation we can, if we wish, assume  $f$  is a polynomial in  $z, \bar{z}$ .

Then  $E$  is a holomorphic vector bundle† on  $P_1$ ,  $\bar{\partial}$  has its natural extension and it is easy to see that

$$\text{Ker } \bar{\partial}_E = H^0(P_1, \mathcal{O}(E)) \cong \text{Ker } T_f.$$

Passing to adjoints one also gets

$$\text{Coker } \bar{\partial}_E \cong H^1(P_1, \mathcal{O}(E)) \cong \text{Coker } T_f$$

so that (7.1)

$$\text{index } \bar{\partial}_E = \text{index } T_f.$$

Just as in § 3 we can define  $T_f$  for families parametrized by  $X$  and so obtain a homomorphism

$$\text{index}_{T_f}: K(P_1 \times X) \rightarrow K(X).$$

The proof of (7.1) also extends to families and shows that  $\text{index}_{T_f}$  is the same homomorphism as  $\text{index}_{\bar{\partial}_E}$ .

*Alternative (2)*

Let  $E^0$  be a vector space and let

$$f: S^1 \rightarrow \text{Aut}(E^0)$$

be a smooth map. Then we introduce an operator  $A_f$ , acting on the space of functions  $S^1 \rightarrow E^0$ , by

$$\begin{aligned} A_f(z^n e) &= z^n f(z) e & (n > 0) \\ &= z^n e & (n \leq 0) \end{aligned}$$

(here  $z^n f(z) e$  denotes of course the function  $z \mapsto z^n f(z) e$ , for  $|z| = 1$  and  $e \in E^0$ ). This is well known to be an elliptic pseudo-differential operator [(9), § 8] and so  $\text{index } A_f$  is well defined.

The operator  $A_f$  is intimately connected with the boundary value problems discussed above. Consider the operator  $S_f$  defined by

$$S_f(u, v) = \left( \frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial z}, A_f(zu \mid S^1 - v \mid S^1) \right).$$

This is an elliptic problem with a pseudo-differential boundary condition and it is the composition of  $T_{S_f}$  with the map

$$(u, v, w) \mapsto (u, v, A_f w)$$

(where  $I$  denotes the identity automorphism of  $E^0$ ).

Since†  $\text{index } T_{S_f} = 0$  we have

$$\text{index } S_f = \text{index } A_f.$$

† In fact  $E$  has a natural holomorphic structure even if  $f$  is only differentiable [see H. Rohrl, *Bull. Amer. Math. Soc.* 68 (1962), 125–60].

‡ This corresponds to the fact, used in § 3, that

$$\text{index } \bar{\partial}_{E^{-1}} = 0.$$

On the other hand it is clear that

$$\text{Ker } S_f = \text{Ker } T_{sf},$$

and by considering adjoints we get

$$\text{Coker } S_f \cong \text{Coker } T_{sf}.$$

Thus, finally, we have

$$\text{index } A_f = \text{index } S_f = \text{index } T_{sf}. \quad (7.2)$$

Just as with  $\bar{\partial}$  and  $T_f$  so we can extend  $A_f$  to families parametrized by  $X$ . Since every bundle over  $P_1 \times X$  can be constructed from a family of maps

$$f_x: S^1 \rightarrow \text{Aut}(E_x^0)$$

it follows that  $f \mapsto \text{index } A_f$  defines a homomorphism

$$\text{index}_A: K(P_1 \times X) \rightarrow K(X).$$

The proof of (7.2) extends to families and shows that

$$\text{index}_A(E) = \text{index}_T(H^{-1}E). \quad (7.3)$$

We shall now discuss the relative advantages and disadvantages of the three alternative methods—the use of  $\bar{\partial}$  we shall refer to as method (0).

It is fairly clear that methods (0) and (1) are very close. Method (1) has the advantage that we need only work with trivial bundles so that some of the technical complications of families of operators are avoided, but on the other hand we need the analysis of boundary value problems which is more delicate.

Method (2) has one drawback, as it stands, and that is we have to take the bundles  $E^0$  and  $E^\infty$  to be equal. This precludes a generalization to the equivariant case. On the other hand (2) has many advantages. It is very much the simplest to define technically. In fact  $f \mapsto A_f$  defines at once a map

$$\alpha: \Omega GL(N, \mathbb{C}) \rightarrow \mathcal{F}(H_N),$$

where  $\Omega$  denotes the space of smooth† maps

$$f: S^1 \rightarrow GL(N, \mathbb{C}) \quad (f(1) = 1),$$

$H_N$  is the Hilbert space of  $L^2$ -functions  $S^1 \rightarrow \mathbb{C}^N$ , and  $\mathcal{F}(H_N)$  denotes the space of Fredholm operators on  $H_N$ . If we topologize  $\Omega$  by  $\sup\|f\|$  (i.e. as a subspace of the continuous loop-space) then  $\alpha$  is continuous and induces, for any compact  $X$ ,

$$\alpha_X: [X, \Omega GL(N, \mathbb{C})] \rightarrow [X, \mathcal{F}],$$

where  $[X, Y]$  denotes the set of homotopy classes  $X \rightarrow Y$ . By (2), Appendix, we have the map

$$\text{index}: [X, \mathcal{F}] \rightarrow K(X)$$

† In fact we could take all continuous maps.

(which is actually an isomorphism [(2) (A1)]). From this, and letting  $N$  become large,  $\alpha_X$  gives at once the required map

$$K^{-2}(X) \rightarrow K(X).$$

Moreover, if we restrict (by approximation) to maps  $f$  which are given by finite Laurent series in  $z$ , it becomes possible to define index  $A_f$  purely algebraically, without resorting to analysis in Hilbert space. This brings us essentially back to the elementary proof of (5) where the analysis has been banished from the scene. As this point is of some interest we shall explain it in detail.

We suppose then that

$$f: X \times S^1 \rightarrow GL(N, \mathbb{C})$$

is a map of the form 
$$f(x, z) = \sum_{n=-k}^k a_n(x)z^n,$$

where each  $a_n$  is a continuous map  $X \rightarrow GL(N, \mathbb{C})$ . We want an algebraic definition of index  $A_f \in K(X)$ . According to the method of (2), Appendix, for defining the index of a family of Fredholm operators in Hilbert space  $H$  we must first choose a closed subspace  $V \subset H$  of finite co-dimension and meeting the kernels of all the operators in 0 only. For our family  $A_f$  there is an obvious choice for  $V$ , namely the space spanned by the vectors  $z^n u$  with  $u \in \mathbb{C}^N$  and  $n \leq 0$  or  $n > k$  (i.e. we exclude the powers  $z, z^2, \dots, z^k$ ). It is clear that

$$V \cap \text{Ker } A_{f_x} = 0 \quad \text{for all } x \in X$$

because the positive and negative powers of  $z$  are now kept apart. It follows that the spaces  $H/A_{f_x}(V)$  are of constant dimension and form a vector bundle over  $X$  which we denote by  $H/A_f(V)$ . According to (2), Appendix, we define index  $A_f \in K(X)$  by

$$\text{index } A_f = kN - H/A_f(V).$$

We shall now try to express  $H/A_f(V)$  in purely algebraic terms. If we introduce the polynomial  $p$  defined by

$$p(x, z) = z^k f(x, z)$$

we have an obvious isomorphism

$$H/A_f(V) \cong H/A_p(H).$$

Now  $A_{p^{-1}}: H \rightarrow H$  is clearly a left inverse of  $A_p$  so that

$$H/A_p(H) \cong \text{Ker } A_{p^{-1}}.$$

But the kernel of  $A_{p^{-1}}$  consists of vectors  $(u, v)$  with  $p_x^{-1}u + v = 0$  ( $u$  involving powers  $z^n$  with  $n > 0$  and  $v$  involving powers with  $n \leq 0$ )

and this equation implies that  $u$  is a polynomial. Thus  $\text{Ker } A_{p_x^{-1}}$  is isomorphic to the space of those polynomials  $u$  (with values in  $\mathbb{C}^N$ ) for which  $p_x^{-1}u$  has no positive powers in its Laurent expansion. Suppose now we regard  $p_x$  as a module homomorphism†

$$\mathbb{C}[z]^N \rightarrow \mathbb{C}[z]^N$$

and consider the cokernel  $M_{p_x}$ . This is a  $\mathbb{C}[z]$ -module annihilated by  $d = \det p_x$ . Since  $d(z) \neq 0$  on  $|z| = 1$  we can decompose it as  $d = d^+d^-$ , where  $d^+(z) = 0$  has all its roots inside  $|z| = 1$  while  $d^-(z) = 0$  has its roots outside  $|z| = 1$ . The module  $M_{p_x}$  can then be decomposed naturally as

$$M_{p_x} = M_{p_x}^+ \oplus M_{p_x}^-$$

where  $M_{p_x}^+$  is annihilated by  $d^+$  and  $M_{p_x}^-$  by  $d^-$ . From the description of  $\text{Ker } A_{p_x^{-1}}$ , given above it follows that we have a natural isomorphism

$$\text{Ker } A_{p_x^{-1}} \cong M_{p_x}^+$$

Hence finally we get a purely algebraic definition for index  $A_f$ , namely

$$\text{index } A_f = kN - M_p^+ \quad (7.4)$$

where  $M_p^+$  is the bundle with fibres  $M_{p_x}^+$ .

The formula (7.4) is exactly the one which occurs in the elementary proof of (5). The quickest elementary proof of the complex periodicity theorem is therefore obtained by using Laurent maps  $f$  as in (5), defining index  $A_f$  as in (7.4), and then appealing to the formal axiomatic reasoning of § 1.

The algebraic method, obtained as we have just explained, by approximation from method (2) we shall refer to as method (2A). It has one important advantage over method (2), namely it does extend to the (one-dimensional) equivariant case, as is clear in (5). The reason for this is that approximation enables us to separate the positive and negative powers of  $z$ . Whereas in method (2) we need the identity operator on negative powers (and so require  $E^0 = E^\infty$ ), in (2A) we ignore the negative powers and define our operators only on the positive powers. Of course the same effect can be achieved, independently of approximation, by replacing the operator  $A_f$  by the operator‡

$$B_f: H^+ \rightarrow H^+$$

given by  $u \mapsto Pfu$ , where  $P$  is the projection  $H \rightarrow H^+$ , and  $H^+$  is the closed subspace of  $H$  involving only positive powers of  $z$ .

† To fit with the Hilbert space it would be better to make  $p$  act on  $z\mathbb{C}[z]^N$  but this makes no essential difference to the modules.

‡ This is a discrete (matrix) analogue of the Wiener-Hopf operator.



### 8. Higher-dimensional case

In the preceding section we discussed only the complex one-dimensional periodicity. In this section we shall give briefly a similar discussion of the more general higher-dimensional cases of §§ 4–6. For simplicity we shall restrict ourselves to the  $\text{Spin}(8n)$ -case of § 6. This is typical of the other cases.

The basic operator of § 6 is the Dirac operator on  $S^{8n}$  from  $S^+$  to  $S^-$  ( $S^\pm$  denoting the two halves of the Spin bundle over  $S^{8n}$ ). Just as in § 7 we can replace this by a boundary value problem on the unit ball  $B^{8n}$  in  $\mathbb{R}^{8n}$  or by pseudo-differential operators on  $S^{8n-1}$ .

#### *Method (1)*

We consider the total Dirac operator  $D$  of  $\mathbb{R}^{8n}$ . If  $\Delta = \Delta^+ \oplus \Delta^-$  is the Spin representation of  $\text{Spin } 8n$  and if  $u, v$  denote functions on  $\mathbb{R}^{8n}$  with values in  $\Delta^+, \Delta^-$  respectively, then the Dirac operator switches factors:  $(u, v) \mapsto (Dv, Du)$ . On the boundary  $S^{8n-1}$ , of the unit ball  $B^{8n}$ , Clifford multiplication can be used to identify  $\Delta^+$  with  $\Delta^-$ . We can therefore consider the boundary condition

$$u(x) - v(x) = \phi(x) \quad (x \in S^{8n-1})$$

for the Dirac operator. This is very similar to the boundary value problem of § 7 except that here our problem has index 0 instead of index 1: it corresponds more to the operator  $T_f$  than to the operator  $T$ .

$$\text{More generally let } f: S^{8n-1} \rightarrow \text{Iso}(E^0, E^\infty) \quad (8.1)$$

be a smooth map and define the operator  $T_f$  by

$$T_f(u, v) = (Dv, Du, f \cdot u |_{S^{8n-1}} - v |_{S^{8n-1}}),$$

where  $u, v$  are functions on  $B^{8n}$  with values in  $\Delta^+ \otimes E^0$  and  $\Delta^- \otimes E^\infty$  respectively. One can show that  $T_f$  is coercive and so gives a Fredholm operator. Moreover taking  $E^0 = \Delta^+, E^\infty = \Delta^-$  and  $f$  to be given by Clifford multiplication one can show that  $\text{index } T_f = 1$ .

#### *Method (2)*

We consider now the (real) Hilbert space  $H$  of  $L^2$  sections of the Spin bundle of  $S^{8n-1}$ . This decomposes naturally as

$$H = H^+ \oplus H^-, \quad (8.2)$$

where  $H^\pm$  consists of those  $u$  which are boundary values of harmonic sections of  $\Delta^\pm$  over  $B^{8n}$  (harmonic means satisfying  $Du = 0$ ). Let  $P$

† We could also take  $B^{8n}$  with the curved metric, i.e. the upper hemisphere.

denote the projection  $H \rightarrow H^+$ : it is a pseudo-differential operator on  $S^{8n-1}$ . For any map  $f$  as in (8.1) we consider the operator

$$B_f: H^+ \otimes E^0 \rightarrow H^+ \otimes E^\infty$$

given by

$$B_f(u) = (P \otimes I)fu,$$

where  $I$  is the identity on  $E^\infty$ . Then  $B_f \in \mathcal{F}R$  the space of real Fredholm operators and  $f \mapsto B_f$  induces a map

$$\Omega^{8n-1}GL(N, R) \rightarrow \mathcal{F}R$$

and so a homomorphism

$$KO^{-8n}(X) \rightarrow KO(X).$$

Both methods (1) and (2) apply as they stand to the equivariant case. Moreover it is clear that in method (2) we can approximate  $f$  by finite sums  $\sum f_p$  where  $p$  runs over the irreducible representations of  $\text{Spin}(8k)$  (we decompose  $H$  under the action of  $\text{Spin}(8k)$ ). Since the decomposition (8.2) is compatible with the action of  $\text{Spin}(8k)$ , and can presumably be described purely algebraically, it seems plausible that  $B_f$  can be defined purely algebraically (when  $f$  is a finite sum  $\sum f_p$ ). Further examination of the representation theory of  $\text{Spin}(8k)$  might then lead to an elementary proof of periodicity in full generality, but this remains an open question. In connection with (8.2) one might conjecture, in analogy with the complex case of § 7, that  $H^+$  and  $H^-$  are respectively the positive (negative) spaces of the Dirac operator on  $S^{8n-1}$  (recall that the Dirac operator is self-adjoint).

In conclusion we should point out that there is a somewhat different way of defining the basic homomorphism

$$K(\mathbb{R}^{2k} \times X) \rightarrow K(X)$$

from the ones discussed so far. To explain this let us recall [cf. (9)] that an elliptic operator  $P$  on a compact manifold  $M$  defines, via its symbol  $\sigma(P)$ , an element  $[\sigma(P)] \in K(TM)$ , where  $TM$  denotes the tangent bundle of  $M$ , and that  $\text{index } P$  depends only on  $[\sigma(P)]$ . Thus the index is essentially a homomorphism

$$\text{index}: K(TM) \rightarrow \mathbb{Z}. \quad (8.3)$$

In § 2 our approach was to pick a basic element  $[\sigma(d)] \in K(TM)$  corresponding to the operator  $d$  and to define a homomorphism

$$\text{index}_d: K(M) \rightarrow \mathbb{Z} \quad (8.4)$$

by  $\text{index}_d(a) = \text{index}(a \cdot [\sigma(d)])$ , where  $a \cdot [\sigma(d)]$  is the module multiplication of  $K(M)$  on  $K(TM)$ . In some ways it is more natural to work with (8.3) rather than (8.4). Extending (8.3) to families and taking  $M$  to be the sphere  $S^k$  we obtain a homomorphism

$$K(TR^k \times X) \rightarrow K(TS^k \times X) \rightarrow K(X).$$

Since  $TR^k = \mathbb{R}^{2k}$ , this is a homomorphism of the right type for the formal proof of periodicity and it can be used for this purpose. This method is the one closest to the remarks in (1) and also to the general theory of the index in (9). Its drawback is that it only gives the generalized periodicity

$$K_G(X) \cong K_G(V \times X)$$

for  $G$ -modules  $V$  which are of the form

$$V \cong TW$$

for some real  $G$ -module  $W$ .

When  $k = 1$  this method is essentially the same as method (2) of § 7— modulo a conformal transformation taking the unit circle to the real axis. When  $k > 1$ , however, these two methods are quite different; one involves operators on  $S^{2k-1}$  and the other operators on  $S^k$ .

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