

INTERPRETING THE BÖKSTEDT SMASH PRODUCT AS THE NORM

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ABSTRACT. This paper compares two models of the equivariant homotopy type of the smash powers of a spectrum, namely the “Bökstedt smash product” and the Hill-Hopkins-Ravenel norm.

1. INTRODUCTION

In any symmetric monoidal category \mathcal{C} , a basic construction of a C_n -equivariant object arises from the smash power

$$X \mapsto X^{\otimes n} = \underbrace{X \otimes X \otimes \dots \otimes X}_n.$$

The homotopical analysis of such constructions in the category of spaces and spectra is of classical importance. For instance, Steenrod operations arise from the homotopy coinvariants $C^*(X^{\times n})_{h\Sigma_n}$ of $X^{\times n}$ with respect to the Σ_n -action. More general power operations arise from the analysis of this kind of construction in the category of spectra.

Our focus in this paper is the analysis of the smash power construction in the equivariant stable category. In contrast to the setting of spaces, there are two distinct fixed-point functors in the category of equivariant spectra: for a G -equivariant X and subgroup $H \subset G$, we can construct “categorical fixed points” X^H and “geometric fixed points” $\Phi^H X$. The interplay and contrast between these two functors encodes much of the complexity of the equivariant stable category.

An important application of the smash power construction of spectra arises in the study of trace methods for computing algebraic K -theory. Algebraic K -theory has been revolutionized in the last 20 years by the development of trace methods. Following ideas of Goodwillie [7], first Bökstedt [2] and then Bökstedt, Hsiang, and Madsen [3] constructed topological analogues of Hochschild homology and negative cyclic homology, called topological Hochschild homology, $\mathrm{THH}(R)$, and topological cyclic homology, $\mathrm{TC}(R)$. They also constructed a “cyclotomic” trace map

$$K(R) \longrightarrow \mathrm{TC}(R) \longrightarrow \mathrm{THH}(R)$$

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lifting the classical Dennis trace $K(R) \rightarrow \mathrm{HH}(R)$. The relationship between relative K -theory and relative TC is well understood (after p -completion) in many circumstances [16, 4]. The target of the trace, $\mathrm{TC}(R)$, is constructed from an equivariant structure that arises on $\mathrm{THH}(R)$; exploiting the tools of equivariant homotopy theory makes $\mathrm{TC}(R)$ relatively computable.

The equivariant structure on $\mathrm{THH}(R)$ arises from the interpretation of (topological) Hochschild homology as the cyclic bar construction. In the category of spaces, for a group-like topological monoid M there is an equivalence $N^{\mathrm{cyc}}M \simeq LBM$, where for a space X , $LX = \mathrm{Map}(S^1, X)$ is the free (unbased) loop space on X . The S^1 -action on LX has an unusual property. For a finite subgroup $H \subset S^1$, $S^1/H \cong S^1$, and pulling back along this isomorphism induces a homeomorphism of S^1 -spaces

$$\mathrm{Map}(S^1, X)^H \cong \mathrm{Map}(S^1, X).$$

In the category of S^1 -spectra, the analogue of this structure is called a cyclotomic structure. In this context, THH being a cyclotomic spectrum boils down to having (suitably coherent) “diagonal” equivalences of geometric fixed points

$$\Phi^{C_n}(X^{\wedge n}) \simeq X.$$

Bökstedt introduced $\mathrm{THH}(R)$ before the invention of symmetric monoidal categories of spectra; he invented coherence machinery (which anticipated the development of symmetric spectra, as explained in [18]) to handle the smash product and proved that his model of smash powers had the right homotopy type. After the invention of modern categories of spectra, it became possible to give definitions of $\mathrm{THH}(R)$ that simply computed the cyclic bar construction in the usual way, circumventing the complexity of Bökstedt’s coherence machinery. However, it was believed that the smash-power in this context did not have the right fixed points, and so a direct construction of $\mathrm{THH}(R)$ as an equivariant spectrum using the modern categories of spectra was thought to be out of reach (e.g., see [12, 2.5.9] and [6, §IX.3.9]).

Very recently, the solution to the Kervaire invariant one problem by the fourth author, Hopkins, and Ravenel involved development of a “norm” functor N_H^G from H -spectra to G -spectra which has the correct diagonal fixed points [9]. In particular, there is an equivalence

$$R \longrightarrow \Phi^G N_e^G R$$

for any finite group G and cofibrant R . When $G = C_k$, the underlying spectrum of $N_e^{C_k} R$ is precisely the smash power $R^{\wedge k}$.

This behavior strongly suggests that the norm should agree with Bökstedt’s version of the smash powers. The purpose of this paper is to make this precise, by constructing an explicit comparison between the two as equivariant spectra. We build on earlier analysis by Shipley [18] which interpreted the Bökstedt construction in terms of a “detection functor” in symmetric spectra, and is somewhat related to analysis done by Lunøe-Nielsen and Rognes [11].

In the sequel to this paper, we use this comparison to show that the norm construction $N_e^{S^1} R$ for a cofibrant ring spectrum R directly yields a model of $\mathrm{THH}(R)$ as a cyclotomic spectrum. This new model of THH in terms of the norm allows us to define new “relative” versions of TC; in one, we work with the smash product over a commutative ring spectrum A , and in the other, we start with ring orthogonal C_n -spectra as input. The computational machinery developed in [9] then

can be applied to provide new approaches for computation, ultimately leading to calculations in algebraic K -theory.

To state our main results, we begin by fixing some notation and definitions for our indexing categories. Let \mathcal{I} denote Bökstedt's indexing category, i.e., the category with objects $\mathbf{n} = \{1, 2, \dots, n\}$ and morphisms all injections. Let \mathcal{J} denote the subcategory of \mathcal{I} with the same objects but maps the subset inclusions. Notice that \mathcal{J} has a unique map from \mathbf{m} to \mathbf{n} for $m < n$.

Recall the following definition of the Bökstedt smash product:

Definition 1.1. Let $X(1), X(2), \dots, X(k)$ be symmetric spectra in spaces. We define the orthogonal Bökstedt smash product to be the orthogonal spectrum with W^{th} space

$$(X(1) \overset{B}{\wedge} \dots \overset{B}{\wedge} X(k))_W = \operatorname{hocolim}_{(\mathbf{n}_1, \dots, \mathbf{n}_k) \in \mathcal{I}^k} \Omega^{n_1 + \dots + n_k} (X(1)_{n_1} \wedge \dots \wedge X(k)_{n_k} \wedge S^W).$$

Note that more generally we get a continuous functor from finite based CW -complexes to spaces by plugging in a space A in place of S^W ; by restriction, we can extract an orthogonal spectrum or symmetric spectrum.

Fix a complete C_k -universe \mathcal{U} . For our model of the equivariant stable category, we use the category of orthogonal C_k -spectra [13]. Specializing to the case of the smash-power of a single spectrum X , observe that $X \overset{B}{\wedge} k$ becomes a C_k -equivariant orthogonal spectrum indexed on \mathcal{U} as we let W vary through the finite-dimensional C_k -inner product spaces that embed in \mathcal{U} . Here C_k acts by conjugation on the mapping space, permutation on the indexing category, and the action on S^W .

The following is the main theorem of this paper.

Theorem 1.2. *Let X be a cofibrant orthogonal spectrum, and let \tilde{X} denote the underlying symmetric spectrum (of topological spaces). Then there is an isomorphism in the homotopy category of C_k -equivariant orthogonal spectra*

$$\tilde{X} \overset{B}{\wedge} k \cong N_e^{C_k} X,$$

where $N_e^{C_k} X$ is the Hill-Hopkins-Ravenel norm.

Moreover, this isomorphism is induced by a zig-zag of maps natural in X .

To explain the proof, we introduce some further notation. Let \mathbb{J} denote the category of finite dimensional real inner product spaces V in \mathbb{R}^∞ , with morphisms the inclusions $V \rightarrow W$ in \mathbb{R}^∞ . We regard this as a discrete category. We will denote by $N_e^{C_k}$ both the norm from spectra to C_k -spectra as well as the norm from spaces to C_k -spaces; the usage will be clear from context.

Let ρ denote the regular representation of C_k . We now have the following main comparison diagram in the category of C_k -spaces:

$$(1.3) \quad \begin{array}{c} (\tilde{X}^{\otimes k})_W \\ \downarrow = \\ \operatorname{hocolim}_{\mathcal{I}^k} \Omega^{n_1+\dots+n_k}(\tilde{X}_{n_1} \wedge \dots \wedge \tilde{X}_{n_k} \wedge S^W) \\ \uparrow \Delta \\ \operatorname{hocolim}_{\mathcal{I}} \Omega^{kn}(\tilde{X}_n \wedge \dots \wedge \tilde{X}_n \wedge S^W) = \operatorname{hocolim}_{\mathcal{I}} \Omega^{kn}(N_e^{C_k}(\tilde{X}_n) \wedge S^W) \\ \uparrow D_1 \\ \operatorname{hocolim}_{\mathcal{J}} \Omega^{kn}(N_e^{C_k}(\tilde{X}_n) \wedge S^W) \\ \downarrow D_2 \\ \operatorname{hocolim}_{\mathbb{J}} \Omega^{V \otimes \rho}(N_e^{C_k}(X_V) \wedge S^W) \end{array}$$

The map labeled Δ is the diagonal inclusion. The maps labeled D_i are induced from the natural inclusion $\mathcal{J} \rightarrow \mathcal{I}$ and the functor $\mathcal{J} \rightarrow \mathbb{J}$ given (on objects) by $\mathbf{m} \mapsto \mathbb{R}^m$.

The proof of the main theorem amounts to showing that all of the vertical maps assemble to weak equivalences of equivariant orthogonal spectra. There are essentially three parts of the argument:

- (1) establishing a stable equivalence of equivariant orthogonal spectra

$$\left\{ W \mapsto \operatorname{hocolim}_{\mathbb{J}} \Omega^{V \otimes \rho}(N_e^{C_k}(X_V) \wedge S^W) \right\} \simeq (N_e^{C_k} X)$$

(Section 2.2),

- (2) establishing the comparisons associated to changing the indexing category (Section 3), and
- (3) studying the diagonal map Δ (Section 4).

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2. A BRIEF REVIEW OF EQUIVARIANT ORTHOGONAL SPECTRA AND THE NORM

We begin with a very rapid review of the details of the theory of equivariant orthogonal spectra that we need. The authoritative source on this subject is [13]; we also require a number of the refinements of the foundations developed in [9]. See also [1, §2] for a concise review.

2.1. Basic definitions. Fix a finite group G . A G -universe \mathcal{U} is a G -inner product space isomorphic to $\bigoplus_{n \in \mathbb{N}} V$ for some finite-dimensional representation V which contains a trivial representation [13, II.1.1]. We say \mathcal{U} is complete when it contains all irreducible representations of G . Denote by $\mathcal{V} = \mathcal{V}(\mathcal{U})$ the collection of all finite-dimensional G -inner product spaces that are isomorphic to sub- G -inner product spaces of \mathcal{U} [13, II.2.1].

Given two representations $V, W \in \mathcal{V}$, we define $\mathcal{I}_G(V, W)$ to be the G -space of non-equivariant isometric isomorphisms $V \rightarrow W$. This construction allows us to regard \mathcal{V} as the objects of a category \mathcal{I}_G enriched in G -spaces. An orthogonal G -spectrum is now defined to be an enriched functor X from \mathcal{I}_G to the category of G -spaces equipped with an equivariant natural transformation

$$X_V \wedge S^W \longrightarrow X_{V \oplus W}$$

that satisfies the obvious transitivity property [13, §II.2].

Remark 2.1. A convenient aspect of this definition is that any continuous (i.e., enriched) functor from G -spaces of the homeomorphism type of finite G -CW complexes to G -spaces gives rise to an orthogonal G -spectrum by restriction of the domain. This observation is relevant to our work because many of the orthogonal G -spectra we study arise in this fashion.

The category of orthogonal G -spectra has a symmetric monoidal structure induced by the Day convolution. In order to describe this, we recall an alternate description of the category of orthogonal G -spectra [13, §II.4]. For representations V and W , let $\mathcal{J}_G(V, W)$ denote the Thom space of the orthogonal complement G -bundle of linear isometries $V \rightarrow W$. We now have a category \mathcal{J}_G with the same objects as \mathcal{I}_G , and an orthogonal G -spectrum is precisely an enriched functor from \mathcal{J}_G to G -spaces. Now the Day convolution defines the symmetric monoidal structure, with unit the sphere spectrum S .

This description of orthogonal G -spectra is also useful because it provides an explicit description of the adjoint of the evaluation functors. For a G -representation V and a based G -space X , we define the shift-desuspension spectrum $S^{-V} \wedge X$ to have W 'th space $\mathcal{J}_G(V, W) \wedge X$ [13, II.4.6]. Of particular importance are the negative V -spheres S^{-V} .

We now turn to the homotopy theory of orthogonal G -spectra. The category of orthogonal G -spectra admits a model structure such that the homotopy category is equivalent to the equivariant stable category. To describe this model structure, we need to review the notion of equivariant stable homotopy groups.

The homotopy groups of an orthogonal G -spectrum X are defined for a subgroup $H \subset G$ and an integer q by

$$\pi_q^H(X) = \begin{cases} \operatorname{colim}_{V \in \mathbb{J}} \pi_q((\Omega^V X_V)^H) & \text{if } q \geq 0, \\ \operatorname{colim}_{\mathbb{R}^{-q} \subset V \in \mathbb{J}} \pi_0((\Omega^{V - \mathbb{R}^{-q}} X_V)^H) & \text{if } q < 0 \end{cases}$$

(see [13, §III.3.2]). We define the stable equivalences to be the maps $X \rightarrow Y$ that induce isomorphisms for all homotopy groups. There is a cofibrantly-generated monoidal model structure on orthogonal G -spectra in which the weak equivalences are the stable equivalences [13, §IV.2].

The stable equivalences are chosen so that the canonical map

$$S^{-V} \wedge S^V \longrightarrow S^0$$

is a stable equivalence for all V [13, III.4.5]. In particular, this means that for any G -space X , we have a stable equivalence

$$(2.2) \quad S^{-V} \wedge S^V \wedge X \longrightarrow \Sigma^\infty X.$$

We now recall an elementary properties of shifts of orthogonal spectra. For an orthogonal G -spectrum X , recall that $\Omega^V X$ denotes the orthogonal G -spectrum with

$$(\Omega^V X)_W = \Omega^V(X_W).$$

Proposition 2.3. *If X is a G -space, then we have a natural stable equivalence*

$$S^{-V} \wedge X \longrightarrow \Omega^V \Sigma^\infty X$$

of orthogonal G -spectra.

Proof. The map in question is the adjoint of the stable equivalence given by Equation 2.2:

$$S^{-V} \wedge X \longrightarrow \Omega^V \Sigma^\infty X.$$

By [13, III.3.6], a map f of orthogonal G -spectra is a stable equivalence if and only if $\Sigma^V f$ is a stable equivalence. Therefore, it suffices to check that

$$S^V \wedge S^{-V} \wedge X \longrightarrow \Sigma^V \Omega^V \Sigma^\infty X.$$

But using Equation 2.2 again, we can reduce to the question of whether the unit

$$\Sigma^\infty X \longrightarrow \Sigma^V \Omega^V \Sigma^\infty X$$

is a weak equivalence, which follows by [13, III.3.8]. \square

2.2. Canonical Homotopy Presentations. Up to stable equivalence, there is a convenient way of expressing the homotopy type of an orthogonal G -spectrum in terms of its spaces; this is referred to as the ‘‘canonical homotopy presentation’’ in [9, B.1.5]. Specifically, let

$$\dots \subset V_n \subset V_{n+1} \subset \dots$$

be an exhausting sequence of orthogonal G -representations (by which we mean that every finite dimensional G -representation V in $\mathcal{V}(\mathcal{U})$ can be embedded in some V_n) and write W_n for the orthogonal complement of V_n in V_{n+1} . We obtain zig-zags of the form

$$S^{-V_n} \wedge X_{V_n} \xleftarrow{\cong} S^{-(V_n \oplus W_n)} \wedge S^{W_n} \wedge X_{V_n} \longrightarrow S^{-V_{n+1}} \wedge X_{V_{n+1}}$$

and taking the homotopy colimit over the first m stages we get something equivalent to $S^{-V_m} \wedge X_{V_m}$. Therefore, we write

$$\operatorname{hocolim}_{V_n} S^{-V_n} \wedge X_{V_n}$$

to denote the homotopy colimit over this zig-zag. The result is stably equivalent to X .

Combining this with Proposition 2.3 gives a convenient description of orthogonal spectra.

Lemma 2.4. *If X is an orthogonal G -spectrum and $\{V_n\}$ is an exhausting sequence of orthogonal G -representations, then we have a stable equivalence*

$$X \simeq \left(W \mapsto \operatorname{hocolim}_{V_n} \Omega^{V_n} (X_{V_n} \wedge S^W) \right).$$

The exhausting sequences also provide a way to prove that the map D_2 of (1.3) is an equivalence. Any choice of exhausting sequence V_n in \mathbb{R}^∞ defines a functor $\mathcal{J} \rightarrow \mathbb{J}$ by

$$\mathbf{n} \mapsto V_n.$$

Stated in this way, the sequence being exhausting is the same thing as the homotopy cofinality of this functor (e.g., see [10, A.5]).

Lemma 2.5. *Let X be an orthogonal G spectrum. For any exhausting sequence V_n in \mathbb{R}^∞ , the map*

$$\operatorname{hocolim}_{n \in \mathcal{J}} \Omega^{V_n \otimes \rho}(X_{V_n \otimes \rho} \wedge S^W) \xrightarrow{D_2} \operatorname{hocolim}_{V \in \mathbb{J}} \Omega^{V \otimes \rho}(X_{V \otimes \rho} \wedge S^W)$$

is an equivariant equivalence.

In particular, the map D_2 is always an equivalence.

Proof. The map D_2 is an equivariant equivalence because the inclusion $\mathcal{J} \rightarrow \mathbb{J}$ is homotopy cofinal and passage to fixed points commutes with filtered homotopy colimits. \square

2.3. The Hill-Hopkins-Ravenel norm. We now review the special case of the norm construction that we need. Given a non-equivariant orthogonal spectrum X , the smash power $X^{\wedge n}$ can be given a C_n -action and regarded as an orthogonal C_n -spectrum indexed on the universe containing only trivial representations (which we denote by \mathbb{R}^∞). In order to obtain an orthogonal C_n -spectrum indexed on the complete C_n -universe \mathcal{U} , we simply apply the point-set change-of-universe functor $\mathcal{I}_{\mathbb{R}^\infty}^{\mathcal{U}}$ [13, §V.1]. This composite defines the norm as

$$N_e^{C_n} X = \mathcal{I}_{\mathbb{R}^\infty}^{\mathcal{U}} X^{\wedge n},$$

and more generally the analogous construction can be used to define $N_e^G X$ for any finite group G . There is a more complicated general construction of the norm $N_H^G X$ which takes orthogonal H -spectra to orthogonal G -spectra for arbitrary subgroups $H \subset G$, but since we do not need that herein we refer the interested reader to [9, §A.4] for details.

A key technical insight of Hill-Hopkins-Ravenel is that the norm N_e^G preserves weak equivalences between cofibrant orthogonal G -spectra and, in fact, cofibrant (commutative) ring orthogonal G -spectra [9, §B.2]. That is, cofibrant replacement allows us to compute the derived functor of the norm.

We need one further fact about the norm, which relates the norm to a description in terms of the canonical homotopy presentation and the norm functor on spaces [9, §2.3.2]. For a space A , define

$$N_e^G(A) = \bigwedge_G A,$$

regarded as a G -space via the permutation action. Then we have the following description of the norm:

$$N_e^G(X) = \operatorname{hocolim}_{V_n} S^{-V_n \otimes \rho} \wedge N_e^G(X_{V_n}),$$

where ρ is the regular representation and V_n is an increasing sequence of subspaces of \mathbb{R}^∞ . Note that as above, the hocolim notation hides the fact that the diagram has backwards weak equivalences.

Since for any exhausting sequence V_n for \mathbb{R}^∞ , the sequence $V_n \otimes \rho$ is exhausting for a complete universe, the canonical homotopy presentation allows us another description of the norm.

Corollary 2.6. *If X is a cofibrant orthogonal spectrum, then there is a stable equivalence of orthogonal G -spectra*

$$N_e^G(X) \simeq \left(W \mapsto \operatorname{hocolim}_{V \in \mathbb{J}} \Omega^{V \otimes \rho} (N_e^G(X_V) \wedge S^W) \right).$$

3. CHANGING DIAGRAMS

In this section, we show that the “change of indexing diagram” maps are equivariant stable equivalences.

To study D_1 , we need to establish an equivariant version of Bökstedt’s telescope lemma comparing homotopy colimits over \mathcal{I} to homotopy colimits over \mathcal{J} (i.e., telescopes). For this, we need to recall that in equivariant homotopy, there is a refined notion of “connectedness” which records the behavior of fixed points for subgroups (as opposed to the coarser notion of “inducing an isomorphism of homotopy groups through a range”). Let ν be a function from conjugacy classes of subgroups of G to $\mathbb{N} \cup \{\infty\}$. A G -equivariant map $f: X \rightarrow Y$ is said to be “ ν -connected” if

$$f^H: X^H \longrightarrow Y^H$$

is $\nu(H)$ -connected for all subgroups $H \subset G$. With this language, we can give an equivariant refinement of the telescope lemma. We use the formulation of the telescope lemma due to Schlichtkrull [17, 2.2]. In the following, we will refer to a functor $X: \mathcal{I} \rightarrow \operatorname{Top}_G$ as an $\mathcal{I} - G$ -space.

Lemma 3.1. *Let X be an $\mathcal{I} - G$ -space and suppose that each morphism $\mathbf{n}_1 \rightarrow \mathbf{n}_2$ in \mathcal{I} with $n_1 \geq n$ induces a ν_n -connected map*

$$X(n_1) \longrightarrow X(n_2).$$

Then given any $m \geq n$, the natural map

$$X(m) \longrightarrow \operatorname{hocolim}_{\mathcal{I}} X(i)$$

is at least $(\nu_n - 1)$ -connected.

In particular, if for each H , $\nu_n(H)$ goes to infinity, then

$$\operatorname{hocolim}_{\mathcal{J}} X(i) \longrightarrow \operatorname{hocolim}_{\mathcal{I}} X(i)$$

is an equivariant equivalence.

Proof. Since the group G does not act on \mathcal{I} , we know that the fixed points commute with the homotopy colimit: there is a natural equivalence

$$\operatorname{hocolim}_{\mathcal{I}} (X(i)^H) \xrightarrow{\simeq} \left(\operatorname{hocolim}_{\mathcal{I}} X(i) \right)^H,$$

where the map is induced from the natural inclusion $X(i)^H \rightarrow X(i)$.

By assumption, for all $\mathbf{n}_1 \rightarrow \mathbf{n}_2$ and $n_1 \geq n$ and for all H , we have

$$X(n_1)^H \longrightarrow X(n_2)^H$$

is $\nu_n(H)$ -connected. By the usual telescope lemma [17, 2.2], we conclude that

$$X(m)^H \longrightarrow \operatorname{hocolim}_{\mathcal{I}} X(i)^H \xrightarrow{\simeq} \left(\operatorname{hocolim}_{\mathcal{I}} X(i) \right)^H$$

is $(\nu_n(H) - 1)$ -connected.

The proof of the second part is immediate from the first, since by elementary cofinality arguments in \mathcal{J} , we see that under the given assumptions, the map

$$\operatorname{hocolim}_{\mathcal{J}} X(j) \longrightarrow \operatorname{hocolim}_{\mathcal{I}} X(i)$$

is infinitely equivariantly connected. \square

We will need some simple observations about the connectivity of a norm in spaces or of the norm of a map between spaces.

Lemma 3.2. *The equivariant connectivity of $N_e^G(X)$ is given by*

$$\nu_X(H) = \frac{|G|}{|H|}(\operatorname{conn}(X) + 1) - 1 \geq \frac{|G|}{|H|} \operatorname{conn}(X),$$

while for a map $f: X \rightarrow Y$,

$$\nu_f(H) = \operatorname{conn}(f) + \left(\frac{|G|}{|H|} - 1\right)(\operatorname{conn}(X) + 1).$$

Proof. Both results rely on the diagonal: if Y has the trivial action, the map

$$Y^{|G|/|H|} = N_H^G(Y) \longrightarrow N_e^G(Y)$$

is a homeomorphism on H -fixed points for any normal subgroup H . Standard arguments about the connectivity of non-equivariant smash powers of a map then show that the equivariant connectivity of $N_e^G(X)$ and $N_e^G(f)$ is given by the listed formulas. (See also [3, 3.12] for discussion of this phenomenon.) \square

Similarly, we recall a small result about the connectivity of an equivariant mapping space (e.g., see [8, 2.5]).

Lemma 3.3. *Suppose X and Y are G -spaces with X having the structure of an H -CW complex. Then*

$$(3.4) \quad \operatorname{conn}(\operatorname{Map}(X, Y)^H) \geq \min_{K \subset H} (\operatorname{conn}(Y^K) - \dim(X^K)).$$

This will allow us to easily get very coarse lower bounds.

Proposition 3.5. *Let $G = C_k$. For a space X , the suspension map*

$$E^{m\rho}: N_e^G X \longrightarrow \Omega^{m\rho} \Sigma^{m\rho} N_e^G X$$

has connectivity at least

$$\nu(H) = 2 \frac{k}{|H|} \operatorname{conn}(X).$$

Proof. We use the equivariant Freudenthal suspension theorem¹. Recall that if V is a representation of G , then the suspension map

$$E^V: Y \longrightarrow \Omega^V \Sigma^V Y$$

is ν_V -connected for any ν_V satisfying

- (1) For all H such that $V^H \neq \{0\}$, we have $\nu_V(H) \leq 2 \operatorname{conn}(Y^H) + 1$.
- (2) For all $K \subset H$ with $V^K \neq V^H$, we have $\nu_V(H) \leq \operatorname{conn}(Y^K)$.

¹For the interested reader, the first condition following is perhaps what one would expect: the connectivity is exactly what the usual Freudenthal suspension theorem would predict, just for each of the fixed points. The second condition is somewhat less obvious: this condition guarantees that the connectivity of H -equivariant maps between two spaces is the same as the ordinary maps between their H -fixed points.

We now consider $V = m\rho$ and $Y = N_e^G(X)$. For all subgroups H of G , $V^H \neq \{0\}$, and if $K \subsetneq H$, then $V^K \neq V^H$. If we let ν be as in the proposition statement, then the essential condition to check is the second one. So we must show that for all $K \subsetneq H$,

$$\nu(H) = 2\frac{k}{|H|} \text{conn}(X) \leq \text{conn}(Y^K) = \frac{k}{|K|} (\text{conn}(X) + 1) - 1.$$

However, since $K \neq H$, we know that this holds. Thus by the equivariant Freudenthal suspension theorem, the map $E^{m\rho}$ is ν -connected. \square

To prove that the map D_1 is an equivariant equivalence, we will directly verify the needed connectivity hypotheses to apply Lemma 3.1.

Theorem 3.6. *Let X be a cofibrant orthogonal spectrum with underlying symmetric spectrum \tilde{X} . Then the maps*

$$\text{hocolim}_{\mathcal{J}} \Omega^{kn} (N_e^{C_k}(\tilde{X}_n) \wedge S^W) \xrightarrow{D_1} \text{hocolim}_{\mathcal{I}} \Omega^{kn} (N_e^{C_k}(\tilde{X}_n) \wedge S^W)$$

are equivalences of C_k -spaces and therefore assemble to a stable equivalence of orthogonal C_k -spectra.

Proof. The first part of the conclusion implies the second; since we show that the map D_1 establishes level equivalences of equivariant orthogonal spectra, it gives an equivariant stable equivalence.

Since \tilde{X} is the restriction of an orthogonal spectrum, it is a semistable symmetric spectrum [18]. Thus Shipley's version of Bökstedt's Telescope Lemma ([18, Corollary 3.1.7]) shows that D_1 induces an underlying equivalence.

Since both sides of the map preserve stable equivalences of orthogonal G -spectra (i.e., see [18, 2.1.9] for the righthand side), we can simplify our analysis using the canonical homotopy presentation. Both the domain and range commute with filtered homotopy colimits in X because $S^{n\rho}$ is compact, and the canonical homotopy presentation allows us to express X as a filtered homotopy colimit of objects of the form $S^{-V} \wedge Y$. Therefore, it suffices to prove this result for orthogonal spectra of the form $X = S^{-V} \wedge Y$ with Y cofibrant and V a d -dimensional vector space. In this case, the underlying symmetric spectrum has $\tilde{X}_n = \mathcal{J}(V, \mathbb{R}^n) \wedge Y$.

To show that D_1 induces an equivariant equivalence, we verify that the $\mathcal{I} - C_k$ -space

$$\mathbf{n} \mapsto \Omega^{n\rho} (N_e^{C_k}(\tilde{X}_n) \wedge S^W)$$

has the appropriate connectivity properties for Lemma 3.1. Here we have replaced the more traditional Ω^{kn} with the visibly equal functor $\Omega^{n\rho}$ to strengthen the equivariant connection in the reader's mind.

The presence of S^W does not affect the structure of the argument; since it is constant relative to \mathcal{I} , having it will perform an affine shift of the connectivities we compute (in fact, making them increasingly connective). We can therefore get a lower bound by calculating in the case that $W = 0$.

The connectivity of \tilde{X}_n is at least the connectivity $(2n - 2d - 1)$ of the Thom space $\mathcal{J}(V, \mathbb{R}^n)$. Similarly, the connectivity of the structure map $S^m \wedge \tilde{X}_n \rightarrow \tilde{X}_{m+n}$ is at least $(n + m - 2d)$.

Let $\mathbf{n}_1 \rightarrow \mathbf{n}_2$ be a map in \mathcal{I} , and consider the following factorization of the structure map in the Bökstedt $\mathcal{I} - G$ -space:

$$\begin{array}{ccc} \Omega^{n_1\rho}(N_e^{C_k}(\tilde{X}_{n_1}) \wedge S^W) & & \\ \Omega^{n_1\rho}E^{(n_2-n_1)\rho} \downarrow & \searrow & \\ \Omega^{n_1\rho}\Omega^{\rho(n_2-n_1)}\Sigma^{(n_2-n_1)\rho}(N_e^{C_k}(\tilde{X}_{n_1}) \wedge S^W) & \xrightarrow{\Omega^{n_2\rho}N_e^{C_k}\sigma} & \Omega^{n_2\rho}(N_e^{C_k}(\tilde{X}_{n_2}) \wedge S^W) \end{array}$$

where $E^{(n_2-n_1)\rho}$ is the equivariant suspension and where σ is the structure map in $X: \Sigma^{(n_2-n_1)}X_{n_1} \rightarrow X_{n_2}$.

The structure map in the $\mathcal{I} - G$ -space is at least as connected as

$$\min(\text{conn}(\Omega^{n_1\rho}E^{(n_2-n_1)\rho}), \text{conn}(\Omega^{n_2\rho}N_e^{C_k}\sigma)),$$

and we analyze each piece individually.

We first analyze the connectivity of $\Omega^{n_1\rho}E^{(n_2-n_1)\rho}$. Applying Proposition 3.5 and Lemma 3.3, we get a lower bound for the connectivity of the map $\Omega^{n_1\rho}E^{(n_2-n_1)\rho}$:

$$\begin{aligned} \nu_1(H) &= \text{conn}((\Omega^{n_1\rho}E^{(n_2-n_1)\rho})^H) \\ &\geq \min_{K \subset H} (2\frac{k}{|K|} \text{conn}(\tilde{X}_{n_1}) - \frac{k}{|K|}n_1) \geq \frac{k}{|H|}(3n_1 - 4d - 2). \end{aligned}$$

Similarly, applying the norm $N_e^{C_k}$ to the structure map $\Sigma^{n_2-n_1}X_{n_1} \rightarrow X_{n_2}$ yields our map

$$N_e^{C_k}\sigma: \Sigma^{(n_2-n_1)\rho}N_e^{C_k}(X_{n_1}) \cong N_e^{C_k}(\Sigma^{n_2-n_1}X_{n_1}) \longrightarrow N_e^{C_k}(X_{n_2}).$$

By Lemma 3.2, this norm produces a map that, on H -fixed points, has connectivity at least

$$\begin{aligned} (n_1 + n_2 - 2d) + (\frac{k}{|H|} - 1)((2n_1 - 2d - 1) + (n_2 - n_1) + 1) \\ = \frac{k}{|H|}(n_1 + n_2 - 2d). \end{aligned}$$

By Lemma 3.3 we can also estimate the connectivity of the $(n_2\rho)$ -fold loops of this map, getting

$$\begin{aligned} \nu_2(H) &= \text{conn}((\Omega^{n_2\rho}N_e^{C_k}\sigma)^H) \\ &\geq \min_{K \subset H} (\frac{k}{|K|}(n_1 + n_2 - 2d) - \frac{k}{|K|}n_2) = \frac{k}{|H|}(n_1 - 2d). \end{aligned}$$

The factorization of the structure map above shows us that the structure map in the $\mathcal{I} - G$ -space has connectivity at least

$$\nu(H) = \min(\nu_1(H), \nu_2(H)) \geq \min(3n_1 - 4d - 2, n_1 - 2d) \geq n_1 - 4d - 2.$$

(and usually a good bit more so).

Since this term goes to infinity, Lemma 3.1 implies that D_1 is an equivariant equivalence, as required. \square

4. EQUIVARIANCE

The maps D_1 and D_2 allow us reduce from diagrams over \mathcal{I} with a trivial action to diagrams over \mathcal{J} with a trivial action of C_k . But one of the interesting aspects of the equivariant structure of the Bökstedt smash product is that C_k acts on the diagram the homotopy colimit is indexed over. In this section, we show that the “diagonal” map Δ induces an equivariant equivalence. This allows us to reduce to the case of homotopy colimits on which the group acts trivially, as analyzed in the previous section.

The map $\Delta = \Delta_1$ is really part of a compatible family of maps, Δ_i , indexed on the divisor poset of k . Analyzing the equivariant homotopy type of the Bökstedt construction is facilitated by a refinement of Hesselholt-Madsen’s determination of the fixed points of the Bökstedt construction.

Suppose $k = d \cdot s$. Then we have a C_k -equivariant diagonal map

$$(4.1) \quad \begin{array}{c} \operatorname{hocolim}_{\mathcal{I}^d} \Omega^{s(n_1+\dots+n_d)} \left(N_{C_d}^{C_k} \left(\bigwedge_{i=1}^d \tilde{X}_{n_i} \right) \wedge S^W \right) \\ \downarrow \Delta_d \\ \operatorname{hocolim}_{\mathcal{I}^k} \Omega^{n_1+\dots+n_k} \left(\bigwedge_{i=1}^k \tilde{X}_{n_i} \wedge S^W \right). \end{array}$$

The following proposition describes a key property of this generalized diagonal map. In this, we fix our model for the homotopy colimit as the two-sided bar construction.

Proposition 4.2. *Suppose $k = d \cdot s$. Then the map of equation (4.1) induces a homeomorphism on passage to C_r -fixed points for $C_s \subseteq C_r \subseteq C_k$.*

Proof. Our proof is a slight elaboration of that of Hesselholt-Madsen for the case $s = k$ [8, §2.4]. The model for the homotopy colimit identifies the simplicial space realizing the homotopy colimit over \mathcal{I}^d with a (nicely embedded) subsimplicial space of the homotopy colimit for \mathcal{I}^k , and as s varies, these spaces are appropriately nested. Passage to fixed points commutes with geometric realization and with the products making up the two-sided bar construction for the homotopy colimit. This is what allows Hesselholt-Madsen to observe that the map Δ_1 induces a homeomorphism on C_k -fixed points. Since Δ_1 factors through Δ_d for any d , and since the inclusion is inducing a homeomorphism on fixed points in this case, we learn that Δ_d also induces an homeomorphism of C_k -fixed points (since these are really statements about fixed subspaces of a big ambient space). Downward induction on the group provides the rest of the result. \square

We use this proposition to show the following:

Theorem 4.3. *The map*

$$\operatorname{hocolim}_{\mathcal{I}} \Omega^{kn} (N_e^{C_k} \tilde{X}_n \wedge S^W) \xrightarrow{\Delta_1} \operatorname{hocolim}_{\mathcal{I}^k} \Omega^{n_1+\dots+n_k} (\tilde{X}_{n_1} \wedge \dots \wedge \tilde{X}_{n_k} \wedge S^W)$$

induces a stable equivalence of equivariant orthogonal spectra.

Proof. Once again, we show that the map is a weak equivalence of C_k -spaces for each W ; the maps therefore assemble to form a level equivalence and hence a stable equivalence of equivariant orthogonal spectra.

We need to show that the map is an equivalence on C_s -fixed points for each $s \mid k$. Consider the composite

$$\begin{array}{c} \text{hocolim}_{\mathcal{I}} \Omega^{kn} (X_n \wedge \dots \wedge X_n \wedge S^W) \\ \downarrow \\ \text{hocolim}_{\mathcal{I}^d} \Omega^{s(n_1+\dots+n_d)} ((X_{n_1} \wedge \dots \wedge X_{n_d})^{\wedge s} \wedge S^W) \\ \downarrow \\ \text{hocolim}_{\mathcal{I}^k} \Omega^{n_1+\dots+n_k} (X_{n_1} \wedge \dots \wedge X_{n_k} \wedge S^W), \end{array}$$

where $k = d \cdot s$ as before. The second map is an equivalence on C_s -fixed points by Proposition 4.2, and we claim that the first map is an equivalence on C_s -fixed points as well. Consider the diagram

$$\begin{array}{ccc} \text{hocolim}_{\mathcal{J}} \Omega^{kn} (\bigwedge_k X_n \wedge S^W) & \longrightarrow & \text{hocolim}_{\mathcal{J}^d} \Omega^{s(n_1+\dots+n_d)} ((\bigwedge_{i=1}^d X_{n_i})^{\wedge s} \wedge S^W) \\ \downarrow & & \downarrow \\ \text{hocolim}_{\mathcal{I}} \Omega^{kn} (\bigwedge_k X_n \wedge S^W) & \longrightarrow & \text{hocolim}_{\mathcal{I}^d} \Omega^{s(n_1+\dots+n_d)} ((\bigwedge_{i=1}^d X_{n_i})^{\wedge s} \wedge S^W) \end{array}$$

The point is that C_s acts trivially on all the indexing categories in the diagram, and the diagonal map $\mathcal{J} \rightarrow \mathcal{J}^d$ is homotopy cofinal. Since the passage to fixed points commutes with these homotopy colimits, we conclude that the top horizontal map is a C_s -equivalence.

The left hand side vertical map is an equivariant equivalence by Section 3; the right hand side vertical map is an equivariant equivalence for the same reason, using Fubini's theorem for homotopy colimits (e.g., see [5, 24.9]). \square

This completes the proof of Theorem 1.2. We have shown that the vertical maps in the main comparison diagram (Equation 1.3) from Section 1 do indeed assemble to weak equivalences of equivariant orthogonal spectra. We have also identified the orthogonal C_k -spectrum at the bottom of the comparison diagram with the spectrum $N_e^{C_k}(X)$.

APPENDIX A. ON EQUIVARIANT HOMOTOPY COLIMITS AND COFINALITY

Our proof of the equivalence of the map Δ_1 was a direct argument. A more categorical approach uses an equivariant notion of homotopy cofinality. For our purposes, we write G -category to denote an internal category in G -sets. A G -functor from a G -category \mathcal{C} to Top_G is a functor which commutes with the G -actions. The fixed points of a G -category \mathcal{C} is an ordinary category, and the restriction of a G -functor F to \mathcal{C}^G is an ordinary functor into G -spaces. We stress that the G -action is *not* assumed to consist of morphisms in the category. This is essential to ensure that the resulting homotopy colimits have a G -action. The prototypical example is $G = C_k$, $\mathcal{C} = \mathcal{I}^k$, and our functor is Bökstedt's construction.

The following lemma is essentially obvious (though we remark that it is only true in spaces), relating the fixed points of a homotopy colimit of a G -functor to the homotopy colimit of the fixed points.

Lemma A.1. *If \mathcal{C} is a G -category and $F: \mathcal{C} \rightarrow \text{Top}_G$ is a G -functor, then the natural map of G -spaces*

$$\text{hocolim}_{\mathcal{C}^G} F(c) \longrightarrow \text{hocolim}_{\mathcal{C}} F(c)$$

is an equivalence on G -fixed points, and the map of spaces

$$\text{hocolim}_{\mathcal{C}^G} F(c)^G \longrightarrow \left(\text{hocolim}_{\mathcal{C}} F(c) \right)^G$$

is an equivalence.

Proof. We choose as a model for hocolim the two-sided bar construction $B(*, \mathcal{C}, F)$. This is the geometric realization of a simplicial space, and since G -fixed point functor commutes with geometric realization, we need only show that this equality is true for the r -simplices of the bar construction for each r . Since we are in spaces, the fixed point space of a product is the product of the fixed point spaces, and we conclude

$$\left(\text{hocolim}_{\mathcal{C}} F(c) \right)^G \cong \text{hocolim}_{\mathcal{C}^G} F(c)^G.$$

In fact, since we have chosen a particular model, these spaces are actually homeomorphic. The right most term is visibly $(\text{hocolim}_{\mathcal{C}^G} F(c))^G$. \square

The following trivial generalization of the preceding result is surprisingly useful:

Corollary A.2. *Suppose that \mathcal{C} is a G -category and \mathcal{D} is a G -subcategory such that $\mathcal{C}^G \subset \mathcal{D}$. Then for any G -functor F from \mathcal{C} to Top_G , we have an equivalence of fixed points*

$$\left(\text{hocolim}_{\mathcal{D}} F(d) \right)^G \longrightarrow \left(\text{hocolim}_{\mathcal{C}} F(c) \right)^G.$$

Proof. The assumptions on \mathcal{C} and \mathcal{D} ensure that $\mathcal{C}^G = \mathcal{D}^G$. The result follows from identifying both with this common colimit. \square

To complete the proof of the equivalence of Δ_1 , we can directly appeal to an equivariant form of Quillen's Theorem A [19, Theorem 3.10].

Theorem A.3. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor such that for every $d \in \mathcal{D}$, the overcategory $F \downarrow d$ is $\text{Stab}(d)$ -equivariantly contractible. Then F induces an equivalence on homotopy colimits.*

In our context, this version of Quillen's Theorem A shows that the natural map $\mathcal{J} \rightarrow \mathcal{J}^k$ is equivariantly cofinal. By the Bökstedt Telescope Lemma and the commutativity of the obvious diagram of homotopy colimits, we conclude that Δ_1 is an equivariant equivalence.

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