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THE mod-p LOWER CENTRAL SERIES AND THE ADAMS SPECTRAL SEQUENCE†

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§1. INTRODUCTION

USING A mod-p version of the lower central series one of us recently [10] constructed a mod-p version of the spectral sequence of one of the others [3] and showed that for every topological space X there exists a spectral sequence $\{E^iX, d^iX\}$ with the following properties:

(*i*) E^1X depends only on $H_*(X; Z_p)$,

(*ii*) for simply connected X the spectral sequence converges in the same sense as the Adams spectral sequence [1].

Our concern here is mainly with a stable version of this sequence, for which we

I. *explicitly calculate* E^1 *and* d^1

II. *show that, from* E^2 *on, this stable spectral sequence coincides with the Adams spectral sequence.*

We will prove these results in detail for $p = 2$ and indicate in an appendix the changes that have to be made for p an odd prime.

1.1. CONVENTIONS. With the following exceptions the notation and terminology of [8] and [9] will be used:

(*i*) For a spectrum (or set complex with base point) X we will denote by H_*X and H^*X the (reduced) homology and cohomology *with coefficients in* Z_p.

(*ii*) For a set with base point Y we will denote by AY the Z_p-*module* generated by Y with the base point put equal to 0. For every spectrum (or set complex with base point) X we then have

$$\pi_*AX = H_*X$$

(*iii*) A spectrum X will always be assumed to be of *finite type*, i.e. the groups π_nX are finitely generated and vanish from some degree down. This implies that H_nX and H^nX are also finitely generated and that $H_nX = \mathrm{Hom}(H^nX, Z_p)$ for all n. Hence every element T of the mod-p Steenrod algebra [12] *operates on the right on* H_nX as follows

$$H_nX = \mathrm{Hom}(H^nX, Z_p) \xrightarrow{\mathrm{Hom}(T, Z_p)} \mathrm{Hom}(H^{n-i}X, Z_p) = H_{n-i}X$$

where $i =$ *degree* T.

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§2. THE MAIN RESULTS (FOR $p = 2$)

2.1. THE STABLE SPECTRAL SEQUENCE. We recall from [10] that the *lower 2-central series* of a group G is the filtration

$$\cdots \subset \Gamma_{r+1}G \subset \Gamma_r G \subset \cdots \subset \Gamma_1 G = G$$

in which each $\Gamma_r G$ is the subgroup of G generated by the elements

$$[\gamma_1, \ldots, \gamma_s]^{2^t}$$

where $[\ ,\ \ldots,\ ] = [\ \ldots [\ ,\ ],\ \ldots,\ ]$, the $\gamma_i \in G$ and $s \cdot 2^t \geq r$. For a spectrum X we then denote by $\{E^i X, d^i X\}$ the spectral sequence associated with the homotopy exact couple of the following filtration of the free group spectrum ([9], §4) FX

$$\cdots \subset \Gamma_{2^{j+1}}FX \subset \Gamma_{2^j}FX \subset \cdots \subset \Gamma_1 FX = FX.$$

2.2. COMPARISON WITH THE NON-STABLE SPECTRAL SEQUENCE. Note that, unlike in the non-stable case, we do not use the whole lower 2-central series. This is because (as will be proved later)

2.3. *For every spectrum* X

$$\pi_*(\Gamma_r/\Gamma_{r+1})FX = 0 \qquad \textit{if} \quad r \neq 2^j \textit{ for some } j.$$

As a result of this reindexing our E^i for $1 < i < \infty$ and our d^i are not the same as those obtained by using the whole lower 2-central series. Of course our E^1 and E^∞ differ from the other only in their indexing.

We now state our main results.

2.4. THE STRUCTURE OF $(E^1 S, d^1 S)$. *Let S denote the sphere spectrum. Then*

(*i*) *$(E^1 S, d^1 S)$ is the graded associative differential algebra with unit (over Z_2) with*

(*ii*) *a generator λ_i (of degree i) for every integer $i \geq 0$*

(*iii*) *for every $m \geq 1$ and $n \geq 0$ a relation*

$$\Sigma_{i+j=n}\binom{i+j}{i}\lambda_{i-1+m}\lambda_{j-1+2m} = 0$$

(*iv*) *a differential given by*

$$\lambda_{n-1} \to \Sigma_{i+j=n}\binom{i+j}{i}\lambda_{i-1}\lambda_{j-1} \qquad n \geq 2$$

2.5. THE STRUCTURE OF $(E^1 X, d^1 X)$. *Let X be a spectrum. Then*

(*i*) *$(E^1 X, d^1 X)$ is a differential right $(E^1 S, d^1 S)$-module, where*

(*ii*) *the module structure is given by*

$$E^1 X = H_* X \otimes E^1 S$$

(*iii*) *the differential is determined by*

$$a \otimes 1 \to \Sigma_{i>0}(aSq^i \otimes \lambda_{i-1})$$

for all $a \in H_* X$.

2.6. COMPARISON WITH THE ADAMS SPECTRAL SEQUENCE. *Let X be a spectrum. Then*

$$\{E^iX, d^iX\} \qquad i \geq 2$$

is exactly the Adams spectral sequence of X [1].

These results will be proved in §§3–7. Part (*i*) of Proposition 2.4 and parts (*i*) and (*ii*) of Proposition 2.5 are proved (modulo 2.3) in §3; §4 deals with a Whitehead type lemma which is used in §5, where we prove 2.3 and calculate the additive structure of E^1S. Up to this point we need, apart from general semisimplicial homotopy theory, only a connection between the lower 2-central series and restricted Lie algebras and some elementary facts about free restricted and unrestricted Lie algebras. In the remainder of the proofs of Propositions 2.4 and 2.5 (§6) and in the proof of Proposition 2.6 (§7), however, we make use of the mod-2 Steenrod algebra and its properties.

§3. PROOF [MODULO 2.3] OF 2.4 [i] AND 2.5 [i] AND [ii]

First we observe that in the notation of 1.1 (*ii*)

$$(\Gamma_1/\Gamma_2)F = A$$

and recall [10], [13] that if

$$L = \Sigma_{r>0} L_r$$

is the functor which assigns to a Z_2-module M the *free restricted Lie algebra* on M, then

3.1. *There is a natural equivalence*

$$(\Gamma_r/\Gamma_{r+1})F \approx L_rA \qquad r > 0$$

and hence

$$\pi_*(\Gamma_r/\Gamma_{r+1})FX \approx \pi_*L_rAX \qquad r > 0.$$

Now let *composition* $LL \to L$ be the natural transformation which assigns to a Z_2-module M the unique restricted Lie map $LLM \to LM$ which is the identity on $L_1LM = LM$. As AS is the Z_2-module spectrum freely generated by a single simplex in degree 0, this composition $LL \to L$ induces a *composition operation*

$$\pi_sL_qAX \times \pi_tL_rAS \to \pi_{s+t}L_{qr}AX$$

and one readily verifies

3.2. *The composition operation turns π_*LAS into an associative graded algebra with unit and π_*LAX into a right module over this algebra.*

And as $\pi_*L_1AX = \pi_*AX = H_*X$ part (*ii*) of Proposition 2.5 follows from 2.3 and

3.3. *The right π_*LAS map*

$$H_*X \otimes \pi_*LAS \to \pi_*LAX$$

given by $a \otimes 1 \to a$ for all a, is an isomorphism.

Proof. This is trivial if X has the homotopy type of the q-fold suspension S^qS of the sphere spectrum S for some integer $-\infty < q < \infty$. For arbitrary X the proposition then follows from the fact that AX has the same homotopy type as $\Sigma_i AS^{q_i}S$ for suitable values of q_i.

It remains to prove the statements about the differential in part (*i*) of Proposition 2.4 and 2.5. Assuming 2.3 let

$$\alpha \in \pi_s L_{2^i} AX = \pi_s(\Gamma_{2^i}/\Gamma_{2^{i+1}})FX$$

$$\beta \in \pi_t L_{2^j} AS = \pi_t(\Gamma_{2^j}/\Gamma_{2^{j+1}})FS$$

Let $b \in \Gamma_{2^j}FS$ be such that $d_0 b \in \Gamma_{2^{j+1}}FS$, $d_i b = *$ for $i \neq 0$ and *proj* $b \in \beta$. Then *proj* $d_0 b \in d^1\beta$. Write b in the form $B(Fi)$ where B is a formula involving only degeneracy operators and the operations product and inverse and where $i \in S$ is the only non-degenerate simplex. Let m be the largest integer for which B involves the operator s_m and let k be an integer such that $2k > t + m$. Then there exists a simplex $a \in \Gamma_{2^i}FX$ such that $d_{2k}a \in \Gamma_{2^{i+1}}FX$, $d_i a = *$ for $i \neq 2k$ and *proj* $a \in \alpha$ and hence *proj* $d_{2k}a \in d^1\alpha$. This implies that the simplex $B(a) \in \Gamma_{2^{i+j}}FX$ is such that *proj* $B(a) \in \alpha\beta$, *proj* $d_0 B(a) \in \alpha(d^1\beta)$, *proj* $d_{2k+t}B(a) \in (d^1\alpha)B$ and $d_i B(a) = *$ for $i \neq 0, 2k + t$. Hence

$$d^1(\alpha\beta) = (d^1\alpha)\beta + \alpha(d^1\beta).$$

§4. A WHITEHEAD LEMMA FOR SEMISIMPLICIAL LIE ALGEBRAS

J. H. C. Whitehead's result that a map between simply connected spaces induces isomorphisms of the homotopy groups if it induces isomorphisms of the homology groups, translates into group complex language as follows [7].

A homomorphism between connected free group complexes induces isomorphisms of the homotopy groups if its abelianization does so.

Here we will derive a similar statement for semisimplicial Lie algebras, which will be needed in the computation of the additive structure of E^1S (§5). In what follows *all algebras will be over* Z_2. First we define

4.1. FREE S.S. LIE ALGEBRAS. A *free s.s. Lie algebra* Y is an s.s. Lie algebra for which there exist submodules $B_n \subset Y_n$ with the properties

(*i*) Y_n is the free unrestricted Lie algebra on B_n, for all n.

(*ii*) if $b \in B_n$ and $0 \leq i \leq n$, then $s_i b \in B_{n+1}$.

If Ab denotes the functor which assigns to every Lie algebra R its *abelianization*

$$AbR = R/[R,R]$$

then we can now state

4.2. THE WHITEHEAD LEMMA FOR S.S. LIE ALGEBRAS. *Let* $f\colon Y \to Y'$ *be a Lie map between connected free s.s. Lie algebras. If* $\pi_* Abf$ *is an isomorphism, then so is* $\pi_* f$.

Proof. For a Lie algebra R let $\Gamma_1 R = R$ and $\Gamma_s R = [\Gamma_{s-1}R, R]$ for $s > 1$. The lemma then follows by iterated application of the five lemma from the following two propositions.

4.3. *Let Y be a connected free s.s. Lie algebra. Then $\Gamma_s Y$ is* $\log_2 s$ *connected for all* $s \geq 1$.

Proof. Let $L^u = \Sigma_{r>0} L_r^u$ be the functor which assigns to every Z_2-module M the *free unrestricted Lie algebra* on M. By the argument of [2], §5 it then suffices to prove proposition 4.3. for the case that $Y = L^u B$ for some connected s.s. Z_2-module B. But in that case

$$\Gamma_s Y = \Sigma_{r \geq s} L_r^u B$$

and the proposition follows from [2], §7.

4.4. *Let $f\colon Y \to Y'$ be a Lie map between free s.s. Lie algebras. If $\pi_* Abf$ is an isomorphism then so is $\pi_*(\Gamma_s/\Gamma_{s+1})f$ for all $s \geq 1$.*

Proof. This follows from Dold's lemma ([4], §1) and the fact that for a free unrestricted Lie algebra R

(*i*) AbR is a (free) Z_2-module,

(*ii*) there are natural isomorphisms

$$(\Gamma_s/\Gamma_{s+1})R \approx L_s^u AbR.$$

§5. THE ADDITIVE STRUCTURE OF E^1S

In this section we prove 2.3 and compute the additive structure of E^1S by calculating the groups $\pi_* LAS$. The latter is done non-stably by investigating the groups $\pi_* LAS_n$ (where S_n denotes the semi-simplicial n-sphere [9, §2]) and their behaviour under suspension and composition.

5.1. SUSPENSION AND COMPOSITION FOR THE GROUPS $\pi_* LAS_n$. The groups $\pi_* LAS_n$ are connected by the *suspension homomorphisms* [4]

$$\pi_* LAS_n \xrightarrow{\text{Susp}} \pi_{*+1} LAS_{n+1} \qquad n \geq 0$$

and for each element $\alpha \in \pi_q L_r AS_n$ by the *composition homomorphism* (defined as in §3)

$$\pi_* L_s AS_q \xrightarrow{\alpha} \pi_* L_{rs} AS_n.$$

We will often *use the same symbol for an element $\alpha \in \pi_q LAS_n$ and its suspensions as well as for the corresponding element of $\pi_{q-n} LAS$.* No confusion will arise as *composition is compatible with suspension;* i.e. the diagram

$$\begin{array}{ccc} \pi_{*+1} LAS_{q+1} & \xrightarrow{\text{Susp}\,\alpha} & \pi_{*+1} LAS_{n+1} \\ \uparrow{\scriptstyle \text{Susp}} & & \uparrow{\scriptstyle \text{Susp}} \\ \pi_* LAS_q & \xrightarrow{\quad \alpha \quad} & \pi_* LAS_n \end{array}$$

is commutative for every $\alpha \in \pi_q LAS_n$.

Of course all this also holds for the groups $\pi_* L^u AS_n$ where L^u is the free unrestricted Lie algebra functor.

5.2. THE ELEMENTS λ_n. For an s.s. Lie algebra Y (restricted or not) and simplices

$a \in Y_m$, $b \in Y_n$ define in the universal enveloping algebra of Y a simplex $a \otimes b$ by

$$a \otimes b = \Sigma(s_{\beta_n} \cdots s_{\beta_1} a \otimes s_{\alpha_m} \cdots s_{\alpha_1} b)$$

where the sum is taken over all permutations $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\}$ of $\{0, \ldots, m+n-1\}$ for which $\alpha_1 < \cdots < \alpha_m$ and $\beta_1 < \cdots < \beta_n$ (if $m = n = 0$ this should be interpreted as $a \otimes b$) and put, for later reference, $[a, b] = a \otimes b + b \otimes a$. For the n-dimensional generator $i_n \in L_1AS_n = AS_n$ we then have $i_n \otimes i_n \in L_2AS_n$ and in fact, for $n > 0$, $i_n \otimes i_n \in L_2^u AS_n$. We now denote by $1 \in \pi_n L_1 AS_n$ and $\lambda_n \in \pi_{2n} L_2 AS_n$ the homotopy classes of i_n and $i_n \otimes i_n$ and formulate:

5.3. THE ADDITIVE STRUCTURE OF $\pi_* LAS$. *The compositions*

$$\lambda_{i_1} \cdots \lambda_{i_k} 1$$

for which $k \geq 0$ and $i_{j+1} \leq 2i_j$ for $j > 0$, form a basis for $\pi_* LAS$.

This together with 3.3 implies 2.3 and hence yields the additive structure of E^1S.

Proposition 5.3 is an immediate consequence of

5.4. THE ADDITIVE STRUCTURE OF $\pi_* LAS_n$. *The compositions*

$$\lambda_{i_1} \cdots \lambda_{i_k} 1$$

for which $k \geq 0$, $i_1 \leq n$ and $i_{j+1} \leq 2i_j$ for $j > 0$, form a basis for $\pi_* LAS_n$.

which in turn follows from the following two propositions

5.5. *The inclusion map $L^u AS_n \to LAS_n$ and the function "composition on the right with λ_0" induce isomorphisms*

$$\pi_* L_r AS_n \approx \pi_* L_r^u AS_n \qquad r \text{ odd}$$

$$\pi_* L_r AS_n \approx \pi_* L_r^u AS_n + \pi_* L_{r/2} AS_n \qquad r \text{ even}$$

Proof. This follows from the fact that

(*i*) for a Z_2-module M the inclusion map $L^u M \to LM$ and the squaring map $LM \to LM$ (given by $m \to m \otimes m$ for all m) induce a decomposition $LM = L^u M \times LM$ of the underlying sets, and

(*ii*) the map on homotopy groups induced by the squaring map $LAS_n \to LAS_n$ is "composition on the right with λ_0".

5.6. *Let $n > 0$. Then the suspension homomorphism* Susp: $\pi_{*-1} L^u AS_{n-1} \to \pi_* L^u AS_n$ *and the composition homomorphism λ_n: $\pi_* L^u AS_{2n} \to \pi_* L^u AS_n$ induce isomorphisms*

$$\pi_* L_r^u AS_n \approx \pi_{*-1} L_r^u AS_{n-1} \qquad r \text{ odd}$$

$$\pi_* L_r^u AS_n \approx \pi_{*-1} L_r^u AS_{n-1} + \pi_* L_{r/2}^u AS_{2n} \qquad r \text{ even}$$

Proof. For $n = 1$ this follows from the results of [2], §6 and the observation that (if J^r is as in [2])

$$L^u AS_1 = L_1^u AS_1 + L^u(\Sigma_{r>1} J^r AS_1)$$

For $n > 1$ let W be the s.s. Lie algebra freely generated by simplices x, y and z in dimensions $n-1$, n and $2n$ respectively, with faces $d_n y = x$, $d_i y = *$ for $i \neq n$ and $d_i z = *$

for all i and let f: $W \to L^u AS_n$ be the Lie map given by $y \to i_n$ and $z \to i_n \otimes i_n$. The complex $T = ker\, f$ then is a free s.s. Lie algebra which in every dimension is freely generated by the simplices of the form (see [11])

$$\text{I} \quad [s_{\alpha_1}x, s_{\alpha_2}y, \dots, s_{\alpha_r}y] \qquad r \geq 1$$

$$\text{II} \quad [s_{\alpha_1}(y \otimes y - z), s_{\alpha_2}y, \dots, s_{\alpha_r}y] \qquad r \geq 1$$

where the s_{α_i} are iterated degeneracy operators, and AbT is an s.s. Z_2-module on the same generators. Let $U \subset AbT$ be the submodule generated by the simplices of the form I and let $V = (AbT)/U$. Then one readily verifies that $\pi_i U = Z_2$ with generator $[x, y, \dots, y]$ whenever $i = rn - 1$ for some $r \geq 1$ and $\pi_i U = 0$ otherwise and that $\pi_i V = Z_2$ with generator $[(y \otimes y - z), y, \dots, y]$ whenever $i = rn$ for some $r \geq 2$ and $\pi_i V = 0$ otherwise. However, in AbT the generators in dimension $> n - 1$ kill each other, i.e.

$$d_{rn}[(y \otimes y - z), y, \dots, y] = [x, y, y, \dots, y]$$

$$d_i[(y \otimes y - z), y, \dots, y] = 0 \quad \text{for } i < rn$$

and thus $\pi_{n-1} AbT = Z_2$ with generator x and $\pi_i AbT = 0$ for $i \neq n - 1$. As $n > 1$ the Whitehead Lemma (§4) implies that the Lie maps $g: L^u AS_{n-1} \to T$ and h: $W \to L^u AS_{2n}$ given by $i_{n-1} \to x$ and $z \to i_{2n}$ induce isomorphisms of the homotopy groups. As the composition

$$L^u AS_{n-1} \xrightarrow{g} T \xrightarrow{\text{incl}} W \xrightarrow{h} L^u AS_{2n}$$

is trivial it follows that $T \xrightarrow{\text{incl}} W$ is trivial on the homotopy groups. The exactness of the homotopy sequence of the fibre map f: $W \to L^u AS_n$ now yields the desired result.

§6. COMPLETION OF THE PROOFS OF 2.4 AND 2.5

Using some results on the mod-2 Steenrod algebra $\mathscr{A}$ we will prove part (*iii*) of Proposition 2.5 and then deduce from this parts (*iii*) and (*iv*) of Proposition 2.4.

Proof of 2.5. (*iii*). As $\pi_* L_2 AS$ is generated by the λ_n (5.3) and as the map

$$\pi_* L_1 AX = H_* X \xrightarrow{d_1} \pi_* L_2 AX = H_* X \otimes \pi_* L_2 AS$$

is natural it follows from the duality between $H_* X$ and $H^* X$ (1.1) that there are unique elements $T_n \in A$ (with *degree* $T_n = n$) such that

$$a \xrightarrow{d_1} \Sigma_{n>0}(aT_n \otimes \lambda_{n-1})$$

for all $a \in H_* X$ and we thus have to prove that $T_n = Sq^n$ for all $n > 0$. This we will do using the fact that [5]

6.1. *$Sq^n \in \mathscr{A}$ is the only non-zero element of degree n which vanishes on $H_{2n-1} AS_{n-1}$.*

Let X be the spectrum such that $X_0 = AS_{n-1}$ and X_q is the q-fold suspension of X_0 for $q \geq 0$. Then we have a commutative diagram

$$\begin{array}{ccc} H_{2n-1}X_0 = \pi_{2n-2}(\Gamma_1/\Gamma_2)GX_0 & \xrightarrow{d_1} & \pi_{2n-3}(\Gamma_2/\Gamma_3)GX_0 = \Sigma(H_iX_0 \otimes \pi_{2n-3}L_2AS_{i-1}) + u \\ \downarrow f \,\approx & & \downarrow g \\ H_{2n-1}X = \pi_{2n-1}(\Gamma_1/\Gamma_2)FX & \xrightarrow{d_1} & \pi_{2n-2}(\Gamma_2/\Gamma_3)FX = \Sigma(H_iX \otimes \pi_{2n-2-i}L_2AS) \end{array}$$

where G is as in [6] and f and g are induced by the "inclusion" $X_0 \subset X$. The upper left equality is proved as in [6] and implies that $f\colon H_{2n-1}X_0 \to H_{2n-1}X$ is the isomorphism induced by the "inclusion". The upper right equality is a consequence of 5.4; as cross effects are killed by suspension ([4], §5) one can choose the direct summand U in such a manner that it is killed by g. And as on $\Sigma(H_iX_0 \otimes \pi_{2n-3}L_2AS_{i-1})$ the map g is the map induced by the "inclusions" $X_0 \subset X$ and $S_{i-1} \subset S$ it follows (5.4) that an element $b \otimes \lambda_{n-1}$ is in the image of g only if $b = 0$. Thus $aT_n = 0$ for all $a \in H_{2n-1}X$ and therefore also for all $a \in H_{2n-1}X_0$ and it thus (6.1) remains to show that $T_n \neq 0$. In order to do this we take a closer look at the spectral sequence for AS. First we recall from [12].

6.2. *H_*AS is a polynomial algebra on generators ξ_1 of degree $2^i - 1 (i \geq 0)$ with one relation $\xi_0 = 1$. The Sq^n operate on the right on H_*AS according to the formulas*

$$\xi_i Sq = \xi_i + \xi_{i-1} \qquad \textit{where } Sq = \Sigma\, Sq^n$$

$$(\xi\xi')Sq = (\xi Sq)(\xi' Sq) \qquad \textit{for all } \xi,\ \xi' \in H_*AS.$$

Now $T_1 = 0$ would imply that $1 \otimes \lambda_0 \in E^\infty AS$, which would contradict the convergence of the spectral sequence [10]. Thus $T_1 = Sq^1$. Similarly $T_2 = Sq^2$ (because otherwise $1 \otimes \lambda_1$ or $1 \otimes \lambda_1 + \xi^1 \otimes \lambda_0$ is in $E^\infty AS$).

Therefore assume inductively that $T_n = Sq^n$ for $n < 2k$ ($k \geq 1$) and suppose T_{2k+1} were 0. Then a simple calculation (using 6.2) yields that $d^1d^1(\xi_1\xi_2^k)$ is a polynomial in the ξ_i with coefficients in π_*L_4AS of which the constant term is $\lambda_k\lambda_{2k-1}$. But as $d^1d^1 = 0$ this would imply $\lambda_k\lambda_{2k-1} = 0$, in contradiction to 5.3. Thus $T_{2k+1} = Sq^{2k+1}$. Applying the same argument to $\xi_1^2\xi_2^k$ we get also that $T_{2k+2} = Sq^{2k+2}$.

Proof of 2.4 (*iii*) *and* (*iv*). Using 6.2 and 2.5 (*iii*) one can write $d^1d^1(\xi_1^n\xi_2^m)(m \geq 1)$ as a polynomial in the ξ_i with coefficients in π_*L_4AS of which the constant term is

$$\Sigma_{i+j=n}\binom{i+j}{i}\lambda_{i-1+m}\lambda_{j-1+2m}.$$

As $d^1d^1 = 0$ this constant term vanishes and thus yields a relation between the λ_i. Moreover, it follows readily from 5.3 that there can be no more relations than these. Thus part (*iii*) of Proposition 2.4 is proved.

Part (*iv*) of Proposition 2.4 is obtained likewise by calculating the constant term in $d^1d^1\xi_1^n$.

§7. PROOF OF PROPOSITION 2.6

In order to prove Proposition 2.6 it suffices [1] to show that the map

$$\begin{array}{c} H_*\Gamma_{2^{j+1}}FX = \pi_*A\Gamma_{2^{j+1}}FX \\ \downarrow \\ H_*\Gamma_{2^j}FX = \pi_*A\Gamma_{2^j}FX \end{array}$$

induced by the inclusion $\Gamma_{2^{j+1}} \subset \Gamma_{2^j}$, is trivial. But as A and $\Gamma_r F$ are defined dimension wise it follows from [9], 15.4 and the naturality of that result, that *there is a commutative diagram*

$$\begin{array}{ccc} \pi_*A\Gamma_{2^{j+1}}FX & \longrightarrow & \pi_*A\Gamma_{2^j}FX \\ \downarrow\approx & & \downarrow\approx \\ \pi_*\Gamma_{2^{j+1}}FAX & \longrightarrow & \pi_*\Gamma_{2^j}FAX \end{array}$$

where the horizontal maps are induced by the inclusion $\Gamma_{2^{j+1}} \subset \Gamma_{2^j}$ *and the vertical maps are isomorphisms.* We thus must show that the bottom map in 7.1 is trivial for all X, or equivalently

7.2. *The spectral sequence collapses for spectra of the form AX, i.e. if X is a spectrum, then*

$$E^2AX = E^\infty AX.$$

Proof. It suffices to prove this if $X = S$. In view of 2.4, 2.5 and 6.2 E^1AS *is freely generated by the elements*

$$\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}\lambda_{i_1} \cdots \lambda_{i_m}$$

for which $\alpha_i \geq 0$ for all i and $i_{j+1} \leq 2i_j$ for all $j > 0$. Moreover, a straightforward calculation, using 6.2, yields

7.4. $$d^1(\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k}\lambda_{i_1} \cdots \lambda_{i_m}) = \xi_1^{\alpha_2} \cdots \xi_{k-1}^{\alpha_k}\lambda_j\lambda_{i_1} \cdots \lambda_{i_m} + \Sigma\, \xi_1^{\beta_1} \cdots \xi_k^{\beta_k}\lambda_{j_0} \cdots \lambda_{j_m}$$

where $j = \alpha_k \cdot 2^{k-1} + \cdots + \alpha_1 - 1$ and where the sum is taken over certain generators with the property that

$$(0, \alpha_k, \ldots, \alpha_2) < (\beta_k, \ldots, \beta_1) \leq (\alpha_k, \ldots, \alpha_1)$$

in the lexicographical ordering.

For every integer $t > 0$ let F^t be the submodule generated by the generators of the form 7.3 for which $k + m \geq t$ where k is the largest integer for which $\alpha_k > 0$. By 7.4 $\gamma \in F^t$ implies $d^1\gamma \in F^t$ and hence $Q^t = F^t/F^{t+1}$ is again a differential module. Now for fixed t and $s \leq t$ let $G^s \subset Q^t$ be generated by the generators of the form 7.3 for which either $m > s$ or $m = s$ and $i_1 \leq \alpha_k \cdot 2^k + \cdots + \alpha_1 \cdot 2 - 2$. Then $R^s = G^s/G^{s+1}$ is again a differential module and it follows readily from 7.4 that $H_*R^s = 0$ for all s, which implies that $H_*Q^t = 0$ for all $t > 0$. Again using 7.4 a standard argument now yields that $E^2AS = Z_2$ and therefore $E^2AS = E^\infty AS$.

7.5. REMARK. Proposition 2.6 could also have been proved more directly (i.e. without the use of [9], 15.4) as follows. Let $\{\bar{E}^iX, \bar{d}^iX\}$ denote the spectral sequence associated with the *homology* exact couple of the filtration of 2.1. Then one has to show that $\bar{E}^2X = \bar{E}^\infty X$ for all X. Again it suffices to prove this for $X = S$ which is done by observing (*i*) that $(\bar{E}^1S, \bar{d}^1S)$ is a left differential module over (E^1S, d^1S)

(*ii*) that $\bar{E}^1S = E^1S \otimes H_*AS$

(*iii*) that $\bar{d}^1S$ is given by $a \to \Sigma(\lambda_{n-1} \otimes Sq^n a)$ for all $a \in H_*AS$, where the Sq^n operate on the left on H_*AS according to the formulas

$$Sq\xi_i = \xi_i + \xi_{i-1}^2$$

$$Sq(\xi\xi') = (Sq\xi)(Sq\xi') \text{ for all } \xi, \xi' \in H_*AS$$

and then applying an argument similar to the one used above.

APPENDIX

§8. CHANGES NEEDED IF p IS AN ODD PRIME

All of 1.1 and most of §2 carries over without any difficulty. The only non-trivial change is in propositions 2.4 and 2.5 which become:

2.4′. THE STRUCTURE OF (E^1S, d^1S).

(*i*) *(E^1S, d^1S) is a graded associative differential algebra with unit (over Z_p) with*

(*ii*) *a generator λ_{i-1} (of degree $2i(p-1)-1$) for every integer $i > 0$ and a generator μ_{i-1} (of degree $2i(p-1)$) for every integer $i \geq 0$,*

(*iii*) *for every $m \geq 1$ and $n \geq 0$ the relations*

$$\Sigma_{i+j=n}\binom{i+j}{i}\lambda_{i-1+m}\lambda_{j-1+pm} = 0$$

$$\Sigma_{i+j=n}\binom{i+j}{i}(\lambda_{i-1+m}\mu_{j-1+pm} - \mu_{i-1+m}\lambda_{j-1+pm}) = 0$$

and for every $m \geq 0$ and $n \geq 0$ the relations

$$\Sigma_{i+j=n}\binom{i+j}{i}\mu_{i-1+m}\lambda_{j+pm} = 0$$

$$\Sigma_{i+j=n}\binom{i+j}{i}\mu_{i-1+m}\mu_{j+pm} = 0$$

(*iv*) *a differential given by*

$$d^1\lambda_{n-1} = \Sigma_{i+j=n}\binom{i+j}{i}\lambda_{i-1}\lambda_{j-1} \qquad n \geq 2$$

$$d^1\mu_{n-1} = \Sigma_{i+j=n}\binom{i+j}{i}(\lambda_{i-1}\mu_{j-1} - \mu_{i-1}\lambda_{j-1}) \qquad n \geq 1$$

$$d^1(\sigma\tau) = (-1)^{\deg\tau}(d^1\sigma)\tau + \sigma(d^1\tau) \qquad \sigma, \tau \in E^1S$$

2.5′. THE STRUCTURE OF (E^1X, d^1X).

(*i*) *(E^1X, d^1X) is a differential right (E^1S, d^1S)-module where*

(*ii*) *the module structure is given by*

$$E^1X = H_*X \otimes E^1S$$

(*iii*) *the differential is determined by*

$$d^1(a \otimes 1) = \Sigma_{i>0}(aP^i \otimes \lambda_{i-1}) + \Sigma_{i\geq 0}(a\beta P^i \otimes \mu_{i-1})$$

for all $a \in H_*X$.

The proofs of 2.4′ (*i*) and 2.5′ (*i*) and (*ii*) are as in §3 (modulo of course the analogue of 2.3) and the Whitehead Lemma remains valid; but the additive structure of E^1S is more complicated than for $p = 2$. One has

5.4′. THE ADDITIVE STRUCTURE OF π_*LAS_{2n}.

$$\pi_{2pi-1}L_pAS_{2i} \approx Z_p \qquad \text{for } i > 0$$
$$\pi_{2pi}L_pAS_{2i} \approx Z_p \qquad \text{for } i \geq 0$$

and if

$$\lambda'_{i-1} \in \pi_{2pi-1}L_pAS_{2i} \qquad i > 0$$
$$\mu'_{i-1} \in \pi_{2pi}L_pAS_{2i} \qquad i \geq 0$$

are arbitrary but fixed non-zero elements, then the compositions

$$v_{i_1} \cdots v_{i_k}1 \qquad (v = \lambda' \text{ or } \mu')$$

for which $k \geq 0$, $i_1 \leq n-1$ *and* $i_{j+1} \leq pi_j + p - 2$ *if* $v_{i_j} = \lambda'_{i_j}$, $i_{j+1} \leq pi_j + p - 1$ *if* $v_{i_j} = \mu'_{i_j}$ *for* $j > 0$, *form a basis for* π_*LAS_{2n}

and, obviously, the corresponding result for π_*LAS.

Proposition 5.4′ follows from the analogue of 5.5 and the following modification of the argument used in the proof of 5.6. Let the operations $\otimes$ and [,] be defined in the obvious manner, i.e. involving suitable signs, let W be the s.s. Lie algebra freely generated by simplices x, y and z in dimensions $2n - 1$, $2n$ and $2pn$ respectively, with faces $d_{2n}y = x$, $d_iy = *$ for $i < 2n$ and $d_iz = *$ for all i, let f: $W \to L^uAS_{2n}$ be the Lie map given by $y \to i_{2n}$ and $z \to i_{2n} \otimes \cdots \otimes i_{2n}$ and let $T = \ker f$. Let R be the s.s. Lie algebra freely generated by simplices r_i in dimensions $2ni - 1 (1 \leq i < p)$ with faces $d_{-1}\, r_i = \Sigma_{2ni}\binom{i-1}{a-1}[r_a, r_{i-a}]$ and $d_jr_i = *$ for $j < 2ni - 1$. Then the Lie maps g: $W \to L^uAS_{2pn}$ and h: $R \to T$ given by $z \to i_{2pn}$ and $r_i \to [x, y, \ldots, y]$ induce isomorphisms of the homotopy groups and hence the map $T \xrightarrow{\text{incl}} W$ is homotopically trivial. Similarly let W' be the s.s. Lie algebra freely generated by simplices u_i in dimensions $2ni - 2 (1 \leq i \leq p)$ and v_i in dimensions $2ni - 1 (1 \leq i \leq p)$ with faces $d_{2ni-1}v_i = u_i$ for $1 \leq i < p$, $d_jv_i = *$ otherwise and $d_ju_i = *$ for all j and i, let f': $W' \to R$ be the Lie map given by $v_i \to r_i$ for $1 \leq i < p$ and $u_p \to \Sigma\binom{p-1}{a-1}[r_a, r_{p-b}]$ and let $T' = \ker f'$. Then the Lie maps g': $W' \to L^uAS_{2pn-2}$ and h': $L^uAS_{2n-2} \to T'$ given by $u_p \to i_{2pn-2}$ and $i_{2n-2} \to u_1$ also induces isomorphisms of the homotopy groups and hence $T' \xrightarrow{\text{incl}} W'$ is also homotopically trivial. The proposition now readily follows.

To complete the proofs of Propositions 2.4′ and 2.5′ one needs instead of 6.1 and 6.2:

6.1′. *If* $T \in \mathscr{A}$ *has degree* $2n(p-1)$ *and vanishes on* $H_{2pn-1}AS_{2n-1}$ *then* $T = \varepsilon_nP^n$ *for some integer* ε_n. *Similarly if* $T \in \mathscr{A}$ *has degree* $2n(p-1)+1$ *and vanishes on* $H_{2pn}AS_{2n-1}$, *then* $T = \eta_n\beta P^n$ *for some integer* η_n.

6.2′. *H_*AS is the tensor product of a polynomial algebra on generators $\xi_i (i \geq 0)$ of degree $2p^i - 2$ with one relation $\xi_0 = 1$ and an exterior algebra on generators $\tau_i (i \geq 0)$ of degree $2p^i - 1$. Moreover β and the P^n operate on the right on H_*AS according to the formulas*

$$\begin{aligned} &\xi_i P = \xi_i + \xi_{i-1} && \text{where } P = \Sigma P^n \\ &\tau_i P = \tau_i + \tau_{i-1} && \\ &(\rho\rho')P = (\rho P)(\rho' P) && \text{for all } \rho, \rho' \in H_*AS \\ &\xi_i \beta = \tau_i \beta = 0 && \text{for } i > 0 \\ &\tau_0 \beta = 1 && \end{aligned}$$

$$(\rho\rho')\beta = (-1)^{\deg \rho}\rho(\rho'\beta) + (\rho\beta)\rho' \text{ for all } \rho, \rho' \in H_*AS$$

From this one derives (as in §6) formulas similar to 2.4′ (*iii*) and (*iv*) and (2.5′) (*iii*). In fact they only differ from these in that every λ_{i-1} and μ_{i-1} is replaced by $\varepsilon_i \lambda'_{i-1}$ and $\eta_i \mu'_{i-1}$, where the ε_i and η_i are suitable integers $\not\equiv 0 \pmod p$ which come from 6.1′. Thus 2.4′ and 2.5′ follow merely by putting

$$\lambda_{i-1} = (1/\varepsilon_i)\lambda'_{i-1} \qquad i > 0$$

$$\mu_{i-1} = (1/\eta_i)\mu'_{i-1} \qquad i \geq 0.$$

Finally the analogue of Proposition 2.6 is proved as in §7 using the fact that application of d^1 to a generator of E^1AS

$$\xi_1^{\alpha_1} \cdots \xi_k^{\alpha_k} \tau_0^{\phi_0} \cdots \tau_k^{\phi_k} \nu_{i_1} \cdots \nu_{i_m}$$

($\nu = \lambda$ or μ, $\phi_i = 0$ or 1) yields an expression of the form

$$\xi_1^{\alpha_2} \cdots \xi_{k-1}^{\alpha_k} \tau_0^{\phi_1} \cdots \tau_{k-1}^{\phi_k} \nu_j \nu_{i_1} \cdots \nu_{i_m} + \Sigma\, \xi_1^{\beta_1} \cdots \xi_k^{\beta_k} \tau_0^{\psi_0} \cdots \tau_k^{\psi_k} \nu_{j_0} \cdots \nu_{j_m}$$

where $j = (\alpha_k + \phi_k)p^{k-1} + \cdots + (\alpha_1 + \phi_1) - 1$, $\nu_j = \lambda_j$ if $\phi_0 = 0$, $\nu_j = \mu_j$ if $\phi_0 = 1$ and where the sum is taken over certain generators with the property that

$$(0, \phi_k, \ldots, \phi_1, 0, \alpha_k, \ldots, \alpha_2) < (\psi_k, \ldots, \psi_0, \beta_k, \ldots, \beta_1) \leq (\phi_k, \ldots, \phi_0, \alpha_k, \ldots, \alpha_1)$$

in the lexicographical ordering.

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