Recall $G$-CW complexes.

For an ordinary CW $X$ we have cellular chain $C_n(X)$ defined:

\[ C_n(X) = \text{free abelian gp on } n \text{-cells}. \]

- free abelian gp on the set $S_n$.

In equiv case, $S_n$ is a $G$-set, making $C_n(X)$ a $\mathbb{Z}[G]$-algebra, the gp ring of $G$.

Def. The gp ring $\mathbb{Z}[G]$ of a gp $G$ is the free abelian gp in the set $G$. For $x \in G$, let $[x]$ be the corresponding generator of $\mathbb{Z}[G]$. 
The multiplication in $\mathbb{Z}[G]$ is given by $[g_1][g_2] = [g_1 g_2]$. For a $G$-set $S$, the free abelian gp generated by $S$ is a $\mathbb{Z}[G]$-module. Such a $\mathbb{Z}[G]$-module is called a permutation module.

Example: $G = C_n$ with generator $g$.

$\mathbb{Z}[G] = \mathbb{Z}[\{g^i\}] = \mathbb{Z}[g]/(g^n - 1)$

For a subgroup $H \leq G$,

let $G/H = \text{set of left cosets} = G \text{-set} \mathbb{Z}[G/H] = \text{free abz on } G/H
For \( H \leq K \leq G \)
\[
\mathbb{C}[G/H] \rightarrow \mathbb{C}[G/K] \quad \text{map of } G\text{-sets}
\]
\[
\mathbb{C}[G/H] \xrightarrow{\Delta} \mathbb{C}[G/K]
\]
\[
\nabla = \Delta \quad \Delta = \nabla
\]

Let \( \{ \chi_1, \chi_2, \ldots \} \) be a set of unit reps for \( K/H \). Given \( x \in G/K \),
we have \( \sum \chi(k)(x) \in G/H \).

E.g. \( H = e \) and \( K = G \).\[
\mathbb{C}[G/K] = \mathbb{C} \quad \text{with trivial } G\text{-action}\]
\[ \mathbb{Z}[G]/H \cong \mathbb{Z}[G] \]

The diagonal map \( \triangle : \mathbb{Z}[G] \to \mathbb{Z}[G] \)

Assume \( G \) is finite.

Exercise: Find the endomorphisms \( \Delta \) and \( \nabla \) above.

\( H_*(C(x)) \) is a graded \( \mathbb{Z}[G] \)-module.

More about this later.

Homotopy groups. We need a base point that is fixed by \( G \).
Given a $G$-space $X$, let $X^+$ be the disjoint union of $X$ with a base point fixed by $G$.

The $n$th entry of a pointed $G$-space is a functor $[finite \, G\text{-sets } S] \to [abelian \, g/f]$ such that $S \mapsto [S \cup S^n, X]_*$.

This represents homotopy classes of equivariant base pt preserving maps.

Call this $\Pi^G_*(X)$. 
Formal properties of this function

1) Contravariant

2) Converts disjoint unions to direct sums.

The function $\pi_i^G(x)$ is determined by its values on the G-sets $G/H$ for conjugacy classes of subgroups of G.

For $H < K < G$, we have a map of G-sets $G/H \rightarrow G/K$, giving

$$\Pi_m X(G/H) \xrightarrow{\text{Res}^K_H} \Pi_m X(G/K)$$
This is called a restriction map.

If a Mackey function \( M \) is a function

finite \( G \)-sets \( \rightarrow \) abelian \( G \)-gps

1) contravariant

2) additive on disjoint unions.

In addition to the restriction maps above, we have

\[
\begin{align*}
M(\mathbb{C}_G/H) & \xrightarrow{\text{Res}^H_K} M(\mathbb{C}_G/K) \\
\text{Tr}^K_H & \xrightarrow{\text{Tr}^K_H}
\end{align*}
\]
This is called a transfer map.

Example 1 \( M(G/H) = \mathbb{Z}[G/H] \)
with \( \text{Res}_{N}^{G} = \nabla \quad \text{Tr}_{N}^{G} = \Delta \)

2. Let \( M \) be a \( \mathbb{Z}[G] \)-module, i.e. an abelian group equipped with a \( G \)-action. Define a Mackey function \( \hat{M} \) by \( \hat{M}(G/H) = M \). Can define transfer explicitly.
Call this a fixed point Macky function.