SIMPLICIAL SETS

Wednesday, April 7, 2021 1:58 PM

Δ = CATEGORY OF ORDERED FINITE SETS AND ORDER PRESERVING MAPS

ITS OBJECTS ARE SETS

$[n] := \{0, 1, \ldots n\}$

ELEMENTARY FACE OPERATORS $[n-1] \xrightarrow[\text{1-1}]{\delta^i} [n]$

" DEGENERACY " $[n+1] \xrightarrow[\text{ONTO}]{\sigma^i} [n]$

$0 \le i \le n$

THE MAPS WHICH MISS / DOUBLE UP ON $i \in [n]$

THESE SATISFY THE SIMPLICIAL IDENTITIES ON PAGE 240 HHP

DEF A SIMPLICIAL SET IS A FUNCTOR $\Delta^{op} \xrightarrow{X} Sets$

$X_n :=$ IMAGE OF $[n]$

FACE MAPS $X_{n-1} \xleftarrow{d_i} X_n$

$0 \le i \le n$

DEGENERACY MAPS $X_{n+1} \xleftarrow{s_i} X_n$

EXAMPLES

$\Delta[n]$ = STANDARD n-SIMPLEX WITH $\Delta[n]_k = \Delta([k], [n])$

$$= \mathcal{L}_{[n]}$$

LET Δ^n (NOT THE SAME) BE THE TOP. SPACE

$$\{(t_0, t_1, \cdots t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \textstyle\sum t_i = 1\}$$

Δ^2 = TRIANGLE $\subset \mathbb{R}^3$

Δ^3 = TETRAHEDRON $\subset \mathbb{R}^4$

ETC.

THE GEOMETRIC REALIZATION $|X|$ FOR A SIMP. SET X IS THE SPACE

$$\coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

$$= \int_{\mathbf{\Delta}} X_n \times \Delta^n$$

$[m], [n] \longmapsto X_m \times \Delta^n$

IS A FUNCTOR $\mathbf{\Delta}^{op} \times \mathbf{\Delta} \to \mathrm{Top}$

$$|\Delta[n]| = \Delta^n \approx D^n$$

OTHER EXAMPLES

OTHER EXAMPLES

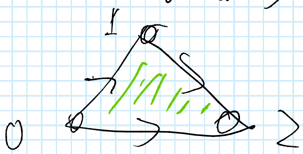
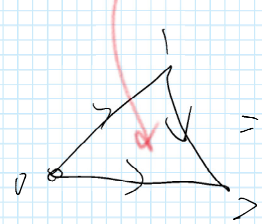
$$|\partial\Delta[n]| = \text{BOUNDARY OF } \Delta^n$$
BOUNDARY
$$\approx S^{n-1}$$

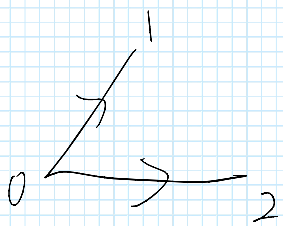
HORNS $\Lambda^k[n]$ = ALL OF $\partial\Delta[n]$ EXCEPT THE kth FACE

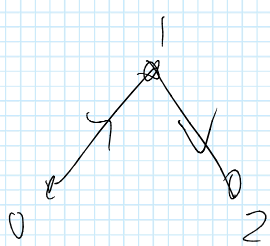
$0 \leq k \leq n$

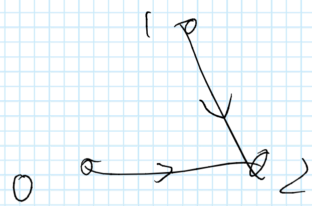
PICTURES FOR $n=2$

NO INTERIOR

 $= \Delta[2]$  $= \partial\Delta[2]$


$\Lambda^0[2]$


$\Lambda^1[2]$


$\Lambda^2[2]$

EXAMPLE LET J BE A SMALL CATEGORY. ITS NERVE $N(J)$ IS THE SIMPLICIAL SET WHERE

$$N(J)_n = \{x_0 \to x_1 \to \cdots \to x_n\}$$
DIAGRAMS IN J

FACE MAPS $\quad N(J)_{n-1} \xleftarrow{d_i} N(J)_n$

COMPOSE TWO ADJACENT MORPHISMS

DEGENERACY MAPS $N(J)_{n+1} \xleftarrow{s_i} N(J)_n$

INSERT IDENTITY MORPHISM IN iTH PLACE

ITS GEOMETRIC REALIZATION $|N(J)| = BJ$ IS THE CLASSIFYING SPACE OF J.

REMARK: THE CATEGORY J CAN BE RECOVERED FROM THE SIMPLICIAL SET $N(J)$.

SPECIAL PROPERTY OF $N(J)$:

FOR ANY DIAGRAM

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & N(J) \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \quad \text{FOR } 0<k<n$$

UNIQUE EXTENSION

TERMINOLOGY: AN <u>INNER HORN</u> IS ONE OF THE FORM $\Lambda^k[n]$ FOR $0<k<n$. $\Lambda^0[n]$ AND $\Lambda^n[n]$ ARE <u>OUTER HORNS</u>.

ILLUSTRATION FOR $n=2$

A MAP $\Lambda^k[2] \longrightarrow N(J)$ IS DEFINED BY A DIAGRAM IN J

IS DEFINED BY A DIAGRAM IN J

$k=0$
$$\begin{tikzcd} & x_1 \arrow[dr, dashed, "?"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & x \end{tikzcd}$$

$k=1$
$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] & & x_2 \end{tikzcd}$$

$k=2$
$$\begin{tikzcd} & x_2 \arrow[dr, "g"] & \\ x_0 \arrow[ur, dashed, "?"] \arrow[rr, "h"'] & & x_1 \end{tikzcd}$$

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "gf"'] & & x_2 \end{tikzcd} \in N(J)_2$$

THIS IS CALLED THE INNER HORN CONDITION

DEFINITION A QUASI-CATEGORY A IS A SIMPICIAL SUCH THAT FOR $n \geq 2$ AND $0<k<n$ ANY MAP

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "\lambda"] \arrow[d] & A \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

CAN BE EXTENDED TO $\Delta[n]$.
(IT NEED NOT BE UNIQUE)

THIS IS ONE WAY TO

THIS IS ONE WAY TO DEFINE AN ∞-CATEGORY.

PROP 1.1.6 RV THE NERVE OF ANY SMALL CATEGORY J IS A QUASI-CATEGORY.

WHAT CAN WE RECOVER FROM A QUASI-CATEGORY A?

WE GET A "CATEGORY" WITH OBJECT SET A_0 AND MORPHISM SET A_1

$$\begin{tikzcd} A_0 \arrow[r, "\sigma_0"] \arrow[r, bend right, "\delta_1"'] & A_1 \arrow[l, bend right, "\delta_0"'] \end{tikzcd}$$

$$\delta_0\sigma_0 = 1_{A_0} = \delta_1\sigma_1$$

σ_0 SENDS AN OBJECT (AN ELEMENT IN A_0) TO ITS IDENTITY MORPHISM

δ_0 AND δ_1 SEND A MORPHISM TO ITS SOURCE AND TARGET.

HOW DO WE COMPOSE MORPHISMS?

MORPHISMS ?

GIVEN A MAP

$$\begin{tikzcd} \Lambda^1[2] \arrow[r, "\lambda"] \arrow[d] & A \\ \Delta[2] \arrow[ur, dashed, "?"] & \end{tikzcd},$$

λ

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, dotted] & & a_2 \end{tikzcd}$$

NOT UNIQUE

COMPOSITION IS <u>NOT</u> WELL DEFINED!

NOW WHAT ???

REMARKS

IN A QUASICATEGORY A

A_0 = OBJECTS SET

A_1 = MORPHISM SET

A_2 = DIAGRAMS OF THE FORM

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd} \qquad \begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h'"'] & & a_2 \end{tikzcd}$$

HOW TO ASSOCIATE AN ACTUAL CATEGORY WITH A_1

ACTUAL CATEGORY WITH A,
ITS <u>FREE CATEGORY</u> FA

A_0 = SET OF OBJECTS

ITS MORPHISMS ARE ALL FINITE COMPOSITES OF 1-SIMPLICES

$$a_0 \xrightarrow{\alpha} a_1 \xrightarrow{\beta} a_2 \xrightarrow{\gamma} a_3$$

$\alpha, \beta, \gamma \in A_1$

BUT $\gamma\beta\alpha$ NEED NOT BE IN A_1, BUT IT IS A MORPHISM IN $F(A)$

SUCH MORPHISMS CAN BE COMPOSED BY CONCATENATION

e.g. THE COMPOSITE OF $a_0 \xrightarrow{f} a_1$ AND $a_1 \xrightarrow{g} a_2$ IS THE DIAGRAM

$$a_0 \xrightarrow{f} a_1 \xrightarrow{g} a_2$$

NO USE IS MADE HERE

a_0, a_1, a_2

NO USE IS MADE HERE OF THE SETS A_n FOR $n \geq 2$.

THE <u>HOMOTOPY CATEGORY</u> HA HAS THE SAME OBJECTS AS FA. ITS MORPHISMS ARE EQUIVALENCE CLASSES AS FOLLOWS: FOR EACH 2-SIMPLEX

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

$$h \simeq g \circ f.$$

IT FOLLOWS THAT FOR A 3-SIMPLEX

$$\begin{tikzcd} & a_1 \arrow[drr, "\beta"] \arrow[dd, "\delta"] & & \\ a_0 \arrow[ur, "\alpha"] \arrow[rrr, "\gamma"] \arrow[dr, "\varepsilon"'] & & & a_3 \\ & a_2 \arrow[urr, "\phi"'] & & \end{tikzcd}$$

4 OBJECTS
6 MORPHISMS

$\gamma \simeq \beta \circ \alpha$, $\varepsilon \simeq \delta \circ \alpha$,

$\gamma \simeq \phi \circ \varepsilon$, $\beta \simeq \phi \circ \delta$

so $\gamma \simeq \phi \circ \delta \circ \alpha$

THE HOMOTOPY CATEGORY DEPENDS ON A_n FOR ALL n.