KS-Entropy from the View Pt. of Category Theory.

Monday, April 19, 2021 2:02 PM

Mostly Based on: "Category Theory for Autonomous and Networked Dynamical Systems" by Jean-Charles Delvenne (Entropy).

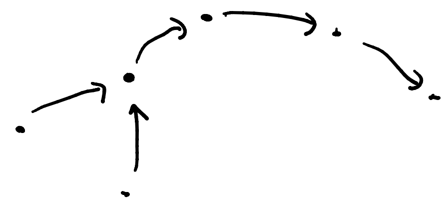
Today: §1. What's a Dynamical System? Def. and Examples.
§2. Some Categories of Dynamical Systems and Monoidal Structure.
§3. Ornstein Theorem: Interpretation in Category Theory, and Definition of Entropy.

§1. A Dynamical System is a pair (X,T), $T: X \to X$.

We look at $\quad \mathcal{O}_T(x) := \{T^n(x) : n \in \mathbb{Z}\} \qquad T^n = \underbrace{T \circ T \circ \cdots \circ T}_{n\text{-times}}$.

as well as $\quad \mathcal{O}_T^{\pm}(x) := \{T^n(x) : n \in \mathbb{Z}^{\pm}\}$

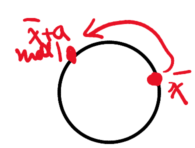
Let's spice things up: put a structure on X and let's require T to be compatible with it.



Topological Dynamics. (X,τ) - topological space
T - continuous

Chaos (or the lack of it) can be determined by:
- Dense orbit? (Minimal)
- Recurrent
- Limit sets
- Topological Mixing ($\forall U, V \in \tau$, $\exists N \in \mathbb{N}$, s.t. $n > N \Rightarrow T^n(U) \cap V \neq \emptyset$).
- Fixed / Periodic Pts. / orbits.

If metric space:
- Lyapunov Stability
- Entropy.



Ex: $a \in \mathbb{R}$, $T_a : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$
$\bar{x} \mapsto \bar{x} + a \bmod 1$.

Differentiable Dyn. Systems. $(X, (U_\alpha, \phi_\alpha)_\alpha)$ - Diff. Manifold
T - Diff.

- Stable and unstable manifolds

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Complex Dyn. $X = \mathbb{C}, \hat{\mathbb{C}}, \ldots$
T - holomorphic

T ~ holomorphic

- Fatou Set $\mathcal{F}(T) = \{ z \in X \mid \exists\, \mathcal{U} \ni z \text{ s.t. } \{T^n|_{\mathcal{U}}\} \text{ normal} \}$
- Julia Set $\mathcal{J}(T) = X \setminus \mathcal{F}(T)$

Arithmetic Dynamics.

Geometric Dyn.

Ergodic Theory. $(X, \mathcal{F}, \mu)$ – m'ble space
T – measure preserving

- Strong Mixing
- Entropy ⟵ I will talk about this later.

Ex: (Bernoulli Shift)

$p = (p_1, \dots, p_n)$ ⟵ prob. vector, i.e., $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$

p defines a probability μ_p on the set $\{1, \dots, n\}$.

We take $X = \{1, \dots, n\}^{\mathbb{Z}} = \{ \cdots x_{-2} x_{-1} x_0 x_1 x_2 \cdots \mid x_i \in \{1, \dots, n\}, i \in \mathbb{Z} \}$.
We define a measure on X by

$$\mu := \prod_{\mathbb{Z}} \mu_p$$

Now we take $\sigma : X \to X$, $(x_j)_j \mapsto (x_{j+1})_j$

$$\cdots x_{-2} x_{-1} x_0 x_1 x_2 \cdots$$
$$\cdots x_{-1} x_0 x_1 x_2 x_3 \cdots$$

$(X, \mathcal{B}, \mu, \sigma)$ - Bernoulli shift based on p.

More general: $(A, \mathcal{A}, \nu)$ prob. space, $(X, \mathcal{B}, \mu) = (A, \mathcal{A}, \nu)^{\mathbb{Z}}$, $\sigma((x_j)_j) = (x_{j+1})_j$.

§2.

We can study categories with

Objects = ______ Dyn. System
Morphisms =

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\phi"] \\ Y \arrow[r, "S"'] & Y \end{tikzcd}$$

$\phi \circ T = S \circ \phi$ (*)
↳ factor map.

s.t. ϕ compatible with _____.

If ____ is m'ble ⟶ we ask for (*) to hold a.e.

s.t. φ compatible with ____.

If ____ is m'ble $\Longrightarrow$ we ask for $(*)$ to hold a.e.

We have

- Products $(X\times Y$, product structure, $f\times g)$
  $\tau\times\tau'$
  $(\mathcal{F}\times\mathcal{G}, \mu\times\nu)$

- Coproducts $(X\sqcup Y$, disjoint union structure, $f\sqcup g)$ $\hookrightarrow$ agrees with f on X and with g on Y.

Some Relevant Categories for this talk

① Prob = Category of Prob. spaces, with measure preserving maps as morphisms.
(the inverse image of an event is an event with the same prob.)

② Erg = Category of endomorphisms of Prob.
= cat. of m'ble dyn. systems on prob. spaces, with factor maps as morphisms.

Rmk: * Identity functor $\text{Prob} \longrightarrow \text{Erg}$
$(X,\Omega,\mathbb{P}) \longrightarrow (X,\Omega,\mathbb{P},\text{Id})$

* Bernoulli functor $\text{Prob} \longrightarrow \text{Erg}$
$(X,\Omega,\mathbb{P}) \longrightarrow ((X,\Omega,\mathbb{P})^{\mathbb{Z}},\sigma) = (X^{\mathbb{Z}},\sigma)$.

§3.

GOAL:

> **Thm.** (Ornstein)
> Any two Bernoulli shifts with the same Kolmogorov–Sinai Entropy are conjugate.

The entropy of a partition $\mathbb{Q}$ is defined

$$H(\mathbb{Q}) = -\sum_{Q\in\mathbb{Q}} \mu(Q)\log(\mu(Q)).$$

The measure-theoretic entropy of a dyn. system $(X,\mathcal{B},\mu,T)$ with respect to a partition $\mathbb{Q}=\{Q_1,\dots,Q_k\}$ is then

$$h_\mu(T,\mathbb{Q}) = \lim_{N\to+\infty} \frac{1}{N} H\left(\bigvee_{n=0}^{N} T^{-n}\mathbb{Q}\right)$$

The Kolmogorov–Sinai Entropy (KS-entropy) of $(X,\mathcal{B},\mu,T)$ as

$$h_\mu(T) = \sup_{|\mathbb{Q}|<+\infty} h_\mu(T,\mathbb{Q}).$$

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FACT: The entropy of a Bernoulli shift $(X^{\mathbb{Z}},\sigma)$ with prob. vector $p=(p_1,\dots,p_n)$ is

$$h_\mu(\sigma) = -\sum_{i=1}^{N} p_i \log(p_i)$$

$\hookrightarrow$ $p=(1/n, 1/n, \dots, 1/n) \rightsquigarrow h_\mu(\sigma) = \log(n).$

Let's rewrite the statement:

Ber = category of Bernoulli shifts and factor maps
= Image of the Bernoulli functor

$\hookrightarrow$ Subcategory of Erg $\Rightarrow$ Monoidal, with $\otimes$

$([0,+\infty], \geq)$ = Category of extended non-neg. reals, with single-arrow
$x \to y \iff x \geq y$.

$\hookrightarrow$ Monoidal with $+$.

Thm. (Ornstein)
The Monoid $(\mathrm{Ber}, \otimes)$ is $([0,+\infty], \geq, +)$.

Build a functor $\mathrm{Ber} \longrightarrow [0,+\infty]$ (KS-entropy functor).
$(X^{\mathbb{Z}},\sigma) \longrightarrow h(\sigma) = h_\mu(\sigma).$

sending every arrow $(X^{\mathbb{Z}},\sigma_X) \longrightarrow (Y^{\mathbb{Z}},\sigma_Y)$ to the unique arrow $h(\sigma_X) \to h(\sigma_Y)$.

it exist $\iff h(\sigma_X) \geq h(\sigma_Y)$.

Prop. (Delvenne, 19)
Consider an arbitrary functor $F: \mathrm{Ber} \to [0,+\infty]$ preserving the monoidal structure.
Assume F assigns a non-zero non-infinity value to at least one finite-alphabet Bernoulli shift. Then $\exists\, \lambda>0$ such that $F = \lambda \cdot h$.

Pf. Let $B_0 = (X_0^{\mathbb{Z}}, \sigma_{X_0})$ be Bernoulli shift with $0 < F(B_0) < +\infty$.

Take B any other Bernoulli shift. We will show $F(B)$ is uniquely determined by $F(B_0)$.

$\forall\, k,\ell \in \mathbb{Z}^+$, either $\exists$ factor map $B_0^k \longrightarrow B^\ell$ or $\exists$ factor map $B^\ell \to B_0^k$, or possibly both.

$\Rightarrow$ either $F(B_0^k) \geq F(B^\ell)$ or $F(B^\ell) \geq F(B_0^k)$ resp.

$\forall\, k, l \in \mathbb{Z}^+$, either $\exists$ factor map $B_0 \to B$ or $\exists$ factor map $B \to B_0$, or possibly both.

$\Rightarrow$ either $F(B_0^k) \geq F(B^l)$ or $F(B^l) \geq F(B_0^k)$, resp.

$\Rightarrow$ either $F(B)/F(B_0) \leq l/k$ or $F(B)/F(B_0) \geq k/l$.

We do this for all $k, l \in \mathbb{Z}^+$ to get upper and lower bounds for $F(B)$.

<u>Cases</u>:

- $F(B)/F(B_0)$ upper bounded by all $l/k \in \mathbb{Q}^+ \longrightarrow F(B) = 0$.
- $F(B)/F(B_0)$ lower bounded by all $k/l \in \mathbb{Q}^+ \longrightarrow F(B) = +\infty$.
- $F(B)/F(B_0)$ has both an upper and lower bound in $\mathbb{Q}^+$, we take sequence of those to assign a real value to $F(B)$.

$\Rightarrow$ F uniquely determined by $F(B_0)$.

On the other hand, the functor $\left(F(B_0)/h(B_0)\right) \cdot h$ satisfies that

$$\underbrace{\left(F(B_0)/h(B_0)\right)}_{\lambda} h(B_0) = F(B_0).$$

$\square$

We can extend this functor to Erg.

Observe that $([0,+\infty], \geq)$ is complete: A cone over $F: J \to [0,+\infty]$ (J-shaped diagram)

$$\begin{tikzcd}
\text{universal?}\ k' \arrow[r, dashed] \arrow[rr, bend left] \arrow[drr, bend right] & k \arrow[r] \arrow[dr] & F(j) \arrow[d, "F(f)"] \\
& & F(j')
\end{tikzcd}
\qquad f: j \to j'$$

$$\left.\begin{array}{l} k \geq F(j) \\ k \geq F(j') \\ F(j) \geq F(j') \end{array}\right\} k \text{ upper bound for } F(j).$$

The limit is $k = \sup\{F(j) : j \in F\}$. We allow it to be $+\infty$, so $[0,+\infty]$ is complete.

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Let's look at BerFin = category of finite-alphabet shifts, with factor maps.

$$\begin{tikzcd}
\mathrm{BerFin} \arrow[r, "h"] \arrow[dr, "\text{inclusion}"'] & {[0,+\infty]} \\
& \mathrm{Erg} \arrow[u, dashed]
\end{tikzcd}$$

$\exists\, \mathrm{Ran}_i h : \mathrm{Erg} \to [0,+\infty]$ ⟶ Corollary 6.2.6 (ii) (Riehl)

In this case, $\mathrm{Ran}_i h(X,T) = \sup \left\{ h(B) : \begin{array}{l} B \text{ finite-alphabet Bernoulli shifts} \\ \exists\, \phi . \quad \begin{tikzcd} B \arrow[r, "\sigma"] \arrow[d, "\phi"'] & B \arrow[d, "\phi"] \\ X \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & T \end{tikzcd} \end{array} \right\}$

Thm. (Sinai)
A non-atomic ergodic measure-preserving system has any Bernoulli shift factor of no greater entropy.

$\Rightarrow \mathrm{Ran}_i h$ is the KS-entropy.