Monday, March 8, 2021 1:54 PM

# ENDS AND COENDS

INPUT : $J$ = SMALL CATEGORY

$Ob\, J$ = SET OF OBJECTS IN $J$

$arr\, J$ = " MORPHISMS

$Dom(f)$ = SOURCE OR DOMAIN OF $f$

$Cod(f)$ = TARGET OR CODOMAIN

$H : J^{op} \times J \longrightarrow \mathcal{C}$ = (CO) COMPLETE CATEGORY

EXAMPLE $J(x,y)$ = SET OF MORPHISMS $x \to y$

FOR (CO) COMPLETE $\mathcal{C}$ WE HAVE AN OBJECT

$$\int^{J} H(j,j) \quad \text{END} = \text{TYPE OF LIMIT}$$

$$\int_{J} H(j,j) \quad \text{COEND} = \text{" COLIMIT}$$

GIVEN A MORPHISM $j \xrightarrow{f} j'$ IN $J$

1

$$\begin{tikzcd} ? \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H(j,j) \arrow[d, "f_*"] \\ H(j',j') \arrow[r, "f^*"] & H(j,j') \end{tikzcd}$$

DIAGRAM IN $\mathcal{C}$

$\mathcal{C}$ IS COMPLETE, SO THERE IS A PULLBACK

WANT TO DO THIS FOR ALL $f$

WANT TO DO THIS FOR ALL $f$

$$H(\mathrm{Dom}\, f, \mathrm{Dom}\, f) \xrightarrow{f_*} H(\mathrm{Dom}\, f, \mathrm{Cod}\, f)$$

$$H(j, j) \xrightarrow{\phi_*} \prod_{\substack{f \in \mathrm{Arr}\, J \\ \mathrm{Dom}(f) = j}} H(j, \mathrm{Cod}\, f)$$

$$\prod_{j \in \mathrm{Ob}\, J} H(j, j) \xrightarrow{\phi_*} \prod_{f \in \mathrm{Arr}\, J} H(\mathrm{Dom}\, f, \mathrm{Cod}\, f)$$

SIMILARLY FOR EACH MORPHISM $f$ IN $J$

$$H(\mathrm{Cod}\, f, \mathrm{Cod}\, f) \xrightarrow{f^*} H(\mathrm{Dom}\, f, \mathrm{Cod}\, f)$$

$$\prod_{j' \in \mathrm{Ob}\, J} H(j', j') \xrightarrow{\phi^*} \prod_{f \in \mathrm{Arr}\, J} H(\mathrm{Dom}\, f, \mathrm{Cod}\, f)$$

THUS WE HAVE TWO MORPHISMS IN $\mathcal{C}$ BETWEEN CERTAIN PRODUCTS

DEF THE END $\int^J$

<u>DEF</u> THE <u>END</u> $\int^{J} H(j,j)$ IS THE EQUALIZER OF $\phi_*$ AND $\phi^*$.

EQUIVALENTLY, IT IS THE PULLBACK OF

$$\begin{tikzcd}
\int^{J} H(j,j) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod\limits_{j \in Ob J} H(j,j) \arrow[d, "\phi_*"] \\
\prod\limits_{j' \in Ob J} H(j',j') \arrow[r, "\phi^*"] & \prod\limits_{f: j \to j'} H(j,j')
\end{tikzcd}$$

DUALLY FOR COCOMPLETE $\mathcal{C}$

WE HAVE FOR EACH $f: j \to j'$ IN $J$

NOTE REVERSAL

$$\begin{tikzcd}
H(j',j) \arrow[r, "f_*"] \arrow[d, "f^*"'] & H(j',j') \\
H(j,j) &
\end{tikzcd}$$

(2) IT HAS A PUSHOUT

$\rightsquigarrow \coprod H(j',j) \xrightarrow{\varphi_*} \coprod H(j',j')$

$$
\begin{tikzcd}
\rightsquigarrow \coprod_{\substack{f \in \mathrm{Arr}(J) \\ f: j \to j'}} H(j', j) \arrow[r, "\varphi_*"] \arrow[d, "\varphi^*"] & \coprod_{j' \in \mathrm{Ob}\, J} H(j', j') \\
\coprod_{j \in \mathrm{Ob}\, J} H(j, j) &
\end{tikzcd}
$$

DEF. THE COEND $\int_J H(j,j)$ IS THE PUSHOUT ABOVE, EQUIVALENTLY THE COEQUALIZER OF $\varphi_*$ AND $\varphi^*$.

REFERENCES ARE TO HHP

PROP 2.4.11 SUPPOSE $H : J^{op} \times J \to \mathcal{C}$ IS CONSTANT ON FIRST VARIABLE

THEN $\int^J H(j,j) = \lim_J H(?,-)$

$\int_J H(j,j) = \mathrm{colim}_J H(?,-)$

$$\int_J H(j,j) = \mathrm{colim}_J H(-,-)$$

PROP 2.4.8 SUPPOSE $J$ IS $0 \longrightarrow 1$, THE WALKING ARROW CATEGORY

$\int_J H(j,j)$ = PUSHOUT OF

$$\begin{tikzcd} & H(1,0) \arrow[dl] \arrow[dr] & \\ H(0,0) & & H(1,1) \end{tikzcd}$$

$\int^J H(j,j)$ = PULLBACK OF

$$\begin{tikzcd} H(0,0) \arrow[dr] & & H(1,1) \arrow[dl] \\ & H(0,1) & \end{tikzcd}$$

PROP 2.4.10 A NATURAL TRANS $\theta : H \longrightarrow H'$ INDUCES MORPHISMS

$$\int_J H(j,j) \xrightarrow{\int_J \theta} \int_J H'(j,j)$$

AND $\int^J H(j,j) \xrightarrow{\int^J \theta} \int^J H'(j,j)$

AND $\int^{j} H(j,j) \longrightarrow \int^{j} H'(j,j)$

PROP 2.4.16 "FUBINI THEOREM" FOR COENDS. GIVEN TWO SMALL CATEGORIES $J_1$ AND $J_2$ AND A FUNCTOR

$$H : J_1^{op} \times J_2^{op} \times J_1 \times J_2 \longrightarrow \mathcal{C}$$

$\mathcal{C}$ = COCOMPLETE CATEGORY

THEN FOR $a, b \in J_1$, WE HAVE

$$H(a, -, b, -) : J_2^{op} \times J_2 \longrightarrow \mathcal{C}$$

IT HAS A COEND

$$\int^{j_2 \in J_2} H(a, j_2, b, j_2) \in \mathcal{C}$$

WHICH IS A FUNCTOR $J_1^{op} \times J_1 \to \mathcal{C}$

ITS COEND IS

$$\int_{j_1 \in J_1} \int_{j_2 \in J_2} H(j_1, j_2, j_1, j_2)$$

A "DOUBLE COEND"

SIMILARLY WE HAVE

$$\int_{j_2 \in J_2} \int_{j_1 \in J_1} H(j_1, j_2, j_1, j_2)$$

THIRDLY, WE HAVE

$$H : (J_1 \times J_2)^{op} \times (J_1 \times J_2) \to \mathcal{C}$$

IT HAS A COEND

$$\int_{J_1 \times J_2} H(j_1, j_2, j_1, j_2)$$

THESE THREE OBJECTS

THESE THREE OBJECTS IN $\mathcal{C}$ ARE NATURALLY ISOMORPHIC.

PROP 2.4.18 GIVEN $F, G : J \to \mathcal{C}$ FOR $J$ SMALL AND $\mathcal{C}$ (COMPLETE), LET

$$H : J^{op} \times J \longrightarrow \mathcal{S}ets$$
$$(j, j') \longmapsto \mathcal{C}(F(j), G(j'))$$

THEN $\int^{J} H(j,j)$ IS THE SET OF NATURAL TRANSFORMATIONS $F \Rightarrow G$

EXAMPLE

SUPPOSE $\mathcal{C} = \mathcal{S}et$, $A \in J$

$$F = \text{よ}^{A} = J(A, -) : J \to \mathcal{S}et$$

THE ABOVE SAYS

THE ABOVE SAYS

$$\int^{B \in J} \mathcal{S}et(\mathcal{L}^A(B), G(B))$$

$$= \mathcal{N}at(\mathcal{L}^A, G)$$

$$= G(A)$$ BY YONEDA LEMMA

2.4.19

## YONEDA REDUCTION

$$\int^{B \in J} \mathcal{S}et(J(A,B), G(B)) = G(A)$$

YONEDA CO-REDUCTION

$$\int_{A \in J} J(A,B) \times G(A) = G(B)$$

$$\|$$

$$\int_{A \in J} \mathcal{L}^A(B) \times G(A) = G(B)$$

$$\|$$

$$\int_{A \in J}$$

$$\| $$

$$\int_{A \in J} よ_B(A) \times G(A) = G(B)$$

$よ_B(-)$ IS THE "DIRAC DELTA AT B."

$J(-, B) = よ_B(-)$ IS A CONTRAVARIANT FUNCTOR.