

ENDS AND COENDS

INPUT : $J = \text{SMALL CATEGORY}$

$\hookrightarrow \text{Ob } J = \text{SET OF OBJECTS IN } J$
 $\hookrightarrow \text{Mor } J = \text{MORPHISMS}$

$\text{Dom}(f) = \text{SOURCE OR DOMAIN OF } f$

$\text{Cod}(f) = \text{TARGET OR CODOMAIN}$

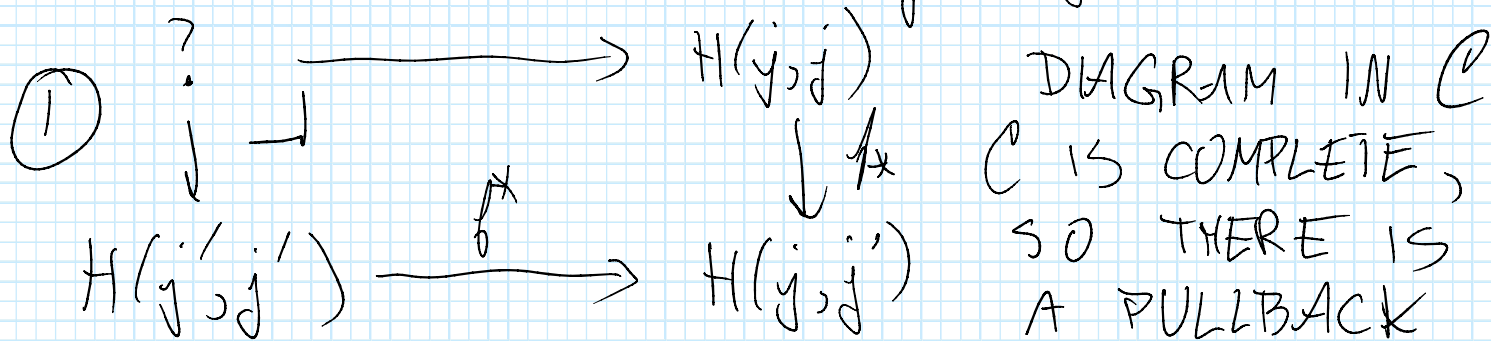
$H: J^{\text{op}} \times J \rightarrow \mathcal{C} = \text{(co) COMPLETE CATEGORY}$

EXAMPLE $J(x, y) = \text{SET OF MORPHISMS } x \rightarrow y$
 FOR (co) COMPLETE \mathcal{C} WE HAVE AN OBJECT

$\int_J H(j, j)$ END = TYPE OF LIMIT

$\int_J H(j, j)$ COEND = " COLIMIT

GIVEN A MORPHISM $j \xrightarrow{f} j'$ IN J



WANT TO DO THIS FOR ALL f

... ..

$$H(\text{Dom } f, \text{Dom } f) \xrightarrow{f^*} H(\text{Dom } f, \text{Cod } f)$$

$$H(j, j) \xrightarrow{f^*} \prod_{\substack{f \in \text{Arr } \mathcal{J} \\ \text{Dom}(f) = j}} H(j, \text{Cod } f)$$

$$\prod_{j \in \text{Obj } \mathcal{J}} H(j, j) \xrightarrow{f^*} \prod_{f \in \text{Arr } \mathcal{J}} H(\text{Dom } f, \text{Cod } f)$$

SIMILARLY FOR EACH MORPHISM f IN \mathcal{J}

$$H(\text{Cod } f, \text{Cod } f) \xrightarrow{f^*} H(\text{Dom } f, \text{Cod } f)$$



$$\prod_{j' \in \text{Obj } \mathcal{J}} H(j', j') \xrightarrow{f^*} \prod_{f \in \text{Arr } \mathcal{J}} H(\text{Dom } f, \text{Cod } f)$$

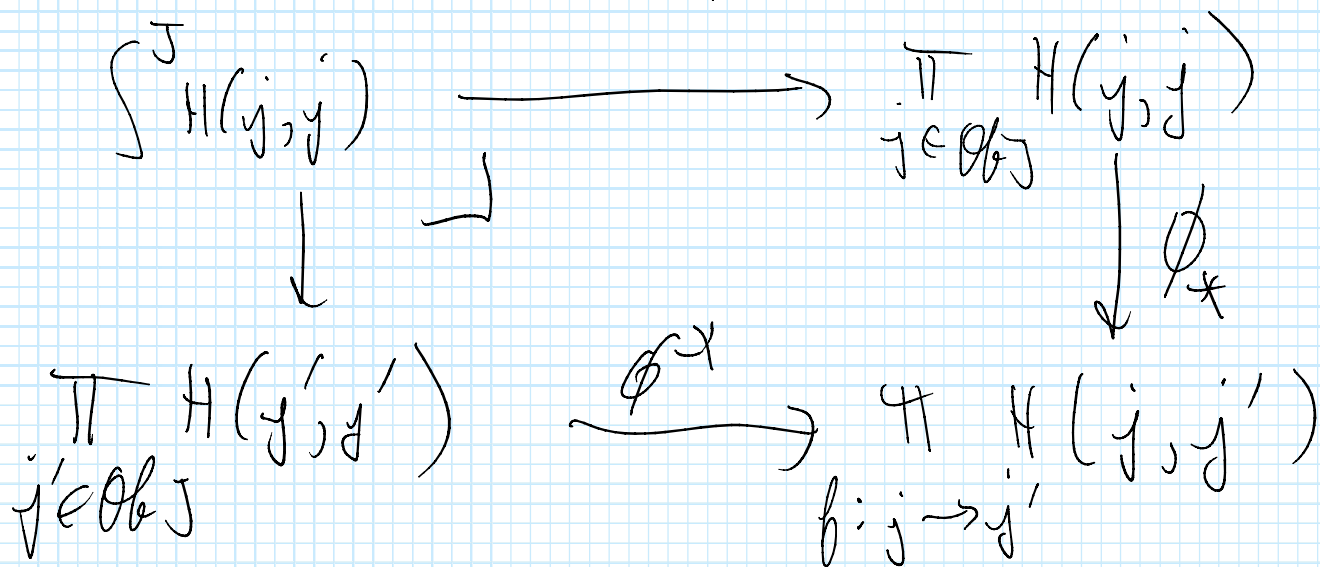
THUS WE HAVE TWO MORPHISMS IN \mathcal{C} BETWEEN CERTAIN PRODUCTS

DEF THE END \mathcal{J}

DEF THE END $\int^J H(j, j)$ IS

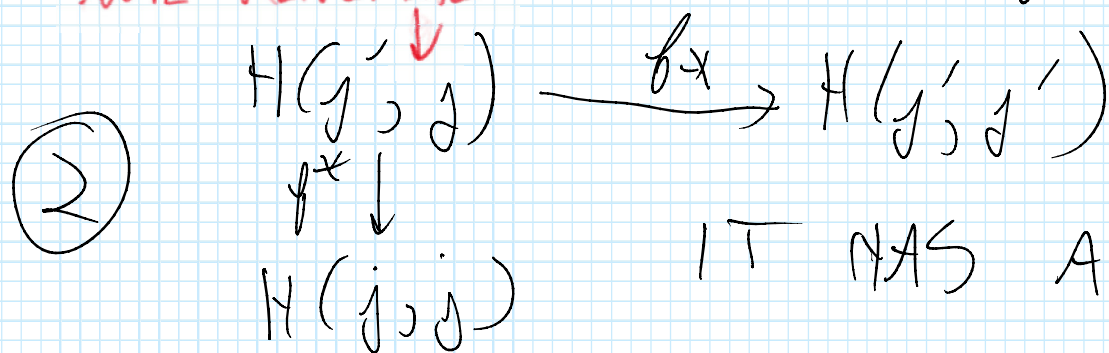
THE EQUALIZER OF ϕ_* AND ϕ^* .

EQUIVALENTLY, IT IS THE PULLBACK OF



DUALY FOR COCOMPLETE \mathcal{C}

WE HAVE FOR EACH $b: j \rightarrow j'$ IN J



IT HAS A PUSHOUT



$$\begin{array}{ccc}
 \Downarrow_{\substack{\beta \in \text{Arr}(J) \\ \beta: j \rightarrow j'}} H(j', j) & \xrightarrow{\varphi_*} & \Downarrow_{j' \in \text{Ob } J} H(j', j') \\
 \downarrow \varphi^* & \approx & \\
 \Downarrow_{j \in \text{Ob } J} H(j, j) & &
 \end{array}$$

DEF. THE COEND $\int_J H(j, j)$ IS THE PUSHOUT ABOVE, EQUIVALENTLY THE COEQUALIZER OF p_x AND φ^* .

REFERENCES ARE TO THE

PROP 2.4.11 SUPPOSE $H: J \times J \rightarrow \mathcal{C}$ IS CONSTANT ON FIRST VARIABLE THEN

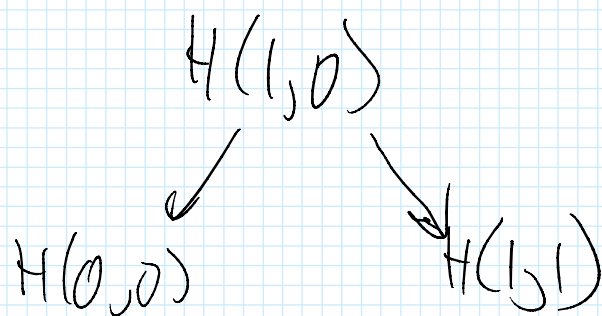
$$\int^J H(j, j) = \lim_J H(?, -)$$

$$\int_J H(j, j) = \text{colim}_J H(?, -)$$

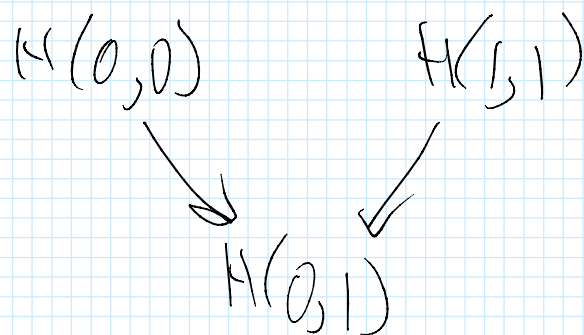
$$\int_J H(y, y) = \operatorname{colim}_J H(i, j)$$

PROP 2.4.8 SUPPOSE J IS
 $0 \rightarrow 1$, THE WALKING ARROW
 CATEGORY

$\int_J H(j, j) = \text{POSHOUT}$
 OF



$\int^J H(j, j) = \text{PULLBACK}$
 OF



PROP 2.4.10 A NATURAL TRANS
 $\theta: H \rightarrow H'$ INDUCES MORPHISMS

$$\int_J H(j, j) \xrightarrow{\int_J \theta} \int_J H'(j, j)$$

AND

$$\int^J H(i, i) \xrightarrow{\int^J \theta} \int^J H'(i, i)$$

$$\text{AND } \int H(j, j) \xrightarrow{\quad} \int H'(j, j)$$

PROP 2, 4, 16 "FUBINI THEOREM"

FOR COENDS. GIVEN TWO SMALL CATEGORIES J_1 AND J_2 AND A FUNCTOR

$$H: J_1^{\text{op}} \times J_2^{\text{op}} \times J_1 \times J_2 \longrightarrow \mathcal{C} = \begin{matrix} \text{COCOMPLETE} \\ \text{CATEGORY} \end{matrix}$$

THEN FOR $a, b \in J_1$, WE HAVE

$$H(a, -, b, -) : J_2^{\text{op}} \times J_2 \longrightarrow \mathcal{C}$$

IT HAS A COEND

$$\int_{j_2 \in J_2} H(a, j_2, b, j_2) \in \mathcal{C}$$

WHICH IS A FUNCTOR $J_1^{\text{op}} \times J_1 \rightarrow \mathcal{C}$

ITS COEND IS

$$\int_{j_1 \in J_1} \int_{j_2 \in J_2} H(j_1, j_2, j_1, j_2)$$

A "DOUBLE COUNT"

SIMILARLY WE HAVE

$$\int_{j_2 \in J_2} \int_{j_1 \in J_1} H(j_1, j_2, j_1, j_2)$$

THIRDLY, WE HAVE

$$H: (J_1 \times J_2)^{\text{op}} \times (J_1 \times J_2) \rightarrow \mathcal{O}$$

IT HAS A COUNT

$$\int_{J_1 \times J_2} H(j_1, j_2, j_1, j_2)$$

THESE THREE OBJECTS

IN \mathcal{C} ARE NATURALLY ISOMORPHIC.

PROP 2.4.18 GIVEN $F, G: \mathcal{J} \rightarrow \mathcal{C}$
FOR \mathcal{J} SMALL AND \mathcal{C} (COMPLETE),

LET $H: \mathcal{J}^{\text{op}} \times \mathcal{J} \rightarrow \text{sets}$

$$(j, j') \mapsto \mathcal{C}(F(j), G(j'))$$

THEN $\int^{\mathcal{J}} H(j, j)$ IS THE SET

OF NATURAL TRANSFORMATIONS

$$F \Rightarrow G$$

EXAMPLE

SUPPOSE $\mathcal{C} = \text{Set}$, $A \in \mathcal{J}$

$$F = \mathcal{J}^A = \mathcal{J}(A, -): \mathcal{J} \rightarrow \text{Set}$$

THE ABOVE SAYS

2.4.18

1.1.12 11 DUVE \rightarrow SAY \rightarrow

$$\int^{B \in J} \text{Set}(\mathcal{L}^A(B), G(B))$$

$$= \text{Nat}(\mathcal{L}^A, G)$$

$$= G(A) \quad \text{BY YONEDA}$$

2.4.19

LEMMA

YONEDA REDUCTION

$$\int^{B \in J} \text{Set}(J(A, B), G(B)) = G(A)$$

YONEDA CO-REDUCTION

$$\int_{A \in J} J(A, B) \times G(A) = G(B)$$

$$\int_{A \in J} \mathcal{L}^A(B) \times G(A) = G(B)$$

||

$\cup_{A \in J}$

$$\int_{A \in J} \delta_B(A) \times G(A) = G(B)$$

$\delta_B(-)$ IS THE "DIRAC DELTA AT B."

$J(-, B) = \delta_B(-)$ IS A CONTRAVARIANT FUNCTOR.