CLOSED MONOIDAL CATEGORIES

Wednesday, March 17, 2021 1:56 PM

RECALL THAT FOR SETS X, Y AND Z,

$$Set(X \times Y, Z) \xrightarrow[\cong]{\phi} Set(X, Set(Y, Z))$$

$$\left(X \times Y \xrightarrow{f} Z\right) \longmapsto x \longrightarrow f(x,-) : Y \to Z$$

THE FUNCTOR

$$- \times Y : Set \longrightarrow Set$$

HAS A RIGHT ADJOINT

$$Set(Y, -) : Set \longrightarrow Set$$

DEF 2.6.33 LET $(\mathcal{C}, \otimes, 1)$ BE A MONOIDAL CATEGORY. FOR AN OBJECT Y, DOES

$$- \otimes Y : \mathcal{C} \longrightarrow \mathcal{C}$$

HAVE A RIGHT ADJOINT? IF IT DOES, WE SAY $(\mathcal{C}, \otimes, 1)$ IS A <u>CLOSED</u> MONOIDAL CATEGORY.

WE WILL DENOTE THE RIGHT

WE WILL DENOTE THE RIGHT ADJOINT BY $\underline{\mathcal{C}}(Y, -)$, NOT A SET

THE INTERNAL HOM FUNCTOR

$$\mathcal{C}(X \otimes Y, Z) \xrightarrow{\cong} \mathcal{C}(X, \underline{\mathcal{C}}(Y, Z))$$

WHERE $\underline{\mathcal{C}}(Y, Z)$ IS AN OBJECT IN $\mathcal{C}$, NOT TO BE CONFUSED WITH $\mathcal{C}(Y, Z)$, THE SET OF MORPHISMS $Y \to Z$.

RECALL $\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(X, \underline{\mathcal{C}}(Y, Z))$

PROP 2.6.36 (HHR)

$$\underline{\mathcal{C}}(X \otimes Y, Z) \cong \underline{\mathcal{C}}(X, \underline{\mathcal{C}}(Y, Z))$$

AND $\underline{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ IS A FUNCTOR.

QUESTION IS THERE A "CATEGORY"

$\underline{C}$ HAVING THE SAME OBJECTS AS C AND "MORPHISM SET" $\underline{C}(X,Y)$? IN PARTICULAR, IS THERE A NATURAL MORPHISM

$$\underline{C}(Y,Z) \otimes \underline{C}(X,Y) \xrightarrow{c_{X,Y,Z}} \underline{C}(X,Z) \text{ ?}$$

YES! NOTE THAT THERE IS A MAP $\underline{C}(X,Y) \otimes X \xrightarrow{\varepsilon} Y$, THE UNIT OF OUR ADJUNCTION

THE ADJUNCTION IS

$$C(X \otimes Y, Z) \xrightarrow{\cong} C(X, \underline{C}(Y,Z))$$

~~SUPPOSE $Z = Y$. THEN~~

$$\cancel{C(X \otimes Y, Y) \cong C(X, \underline{C}(Y,Y))}$$

SUPPOSE $X = \underline{C}(Y,Z)$

SUPPOSE $X = \underline{C}(Y, Z)$

$$C(\underline{C}(Y,Z) \otimes Y, Z) \xrightarrow{\cong} C(\underline{C}(Y,Z), \underline{C}(Y,Z))$$

UNIT OF ADJUNCTION

$$\varepsilon : \underline{C}(Y,Z) \otimes Y \to Z \longmapsto 1_{\underline{C}(Y,Z)}$$

WE WANT A MAP

$$\underline{C}(Y,Z) \otimes \underline{C}(X,Y) \longrightarrow \underline{C}(X,Z)$$

IT IS ADJOINT TO

$$\begin{array}{c} \underline{C}(Y,Z) \otimes C(X,Y) \otimes X \\ \big\downarrow \underline{C}(Y,Z) \otimes \varepsilon \\ \underline{C}(Y,Z) \otimes Y \\ \big\downarrow \varepsilon \\ Z \end{array}$$

WE DO GET A "CATEGORY"

$\underline{\mathcal{C}}$ WITH MORPHISM WIDGETS INSTEAD OF MORPHISM SETS.

DEF 3.4.1 LET $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbb{1})$ BE A MONOIDAL CATEGORY. A **$\mathcal{V}$-CATEGORY** $\mathcal{C}$ HAS OBJECTS AND FOR EACH PAIR OF OBJECTS X, Y A MORPHISM "WIDGET" $\mathcal{C}(X,Y) \in \mathcal{V}$ WITH COMPOSITION MORPHISMS

$$\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \xrightarrow{c_{X,Y,Z}} \mathcal{C}(X,Z)$$

IN $\mathcal{V}$ WITH SUITABLE PROPERTIES. FOR EACH OBJECT X WE HAVE

$$\mathbb{1} \xrightarrow{1_X} \mathcal{C}(X,X)$$

$1 \qquad \mathcal{C}(X,X)$

(SUBSTITUTE FOR IDENTITY MORPHISM)

$\mathcal{C}$ IS A CATEGORY <u>ENRICHED</u> OVER $\mathcal{V}$

$\mathcal{C}$ NEED NOT BE MONOIDAL.

FOR SUCH A $\mathcal{C}$, THERE IS AN UNDERLYING ORDINARY $\mathcal{C}_0$ HAVING THE SAME OBJECTS AS $\mathcal{C}$ AND MORPHISM SETS

$$\mathcal{C}_0(X,Y) := \mathcal{V}_0(1, \mathcal{C}(X,Y))$$

OBJECTS IN THE ORDINARY CATEGORY $\mathcal{V}_0$

DEF 3.1.12 COMPOSITION + PRECOMPOSITION OF "MORPHISMS" IN $\mathcal{C}$ WITH MORPHISMS IN $\mathcal{C}_0$.

GIVEN $f \in \mathcal{C}_0(X,Y)$ AND $W \in \mathcal{C}$, THE MORPHISM $f_* = \mathcal{C}(W,f)$ IS THE COMPOSITE IN $\mathcal{V}$

$$
\begin{tikzcd}
\mathcal{C}(W,X) \arrow[r, "\lambda^{-1}_{\mathcal{C}(W,X)}"] \arrow[d, "f_*"] & 1 \otimes \mathcal{C}(W,X) \arrow[d, "f \otimes \mathcal{C}(W,X)"'] \\
\mathcal{C}(W,Y) & \mathcal{C}(X,Y) \otimes \mathcal{C}(W,X) \arrow[l, "c_{W,X,Y}"']
\end{tikzcd}
$$

SIMILARLY FOR AN OBJECT Z

$$
\begin{tikzcd}
\mathcal{C}(Y,Z) \arrow[r, "\rho^{-1}_{\mathcal{C}(Y,Z)}"'] \arrow[d, dashed, "f^*"'] & \mathcal{C}(Y,Z) \otimes 1 \arrow[l, bend right, "\rho_{\mathcal{C}(Y,Z)}"'] \arrow[d, "\mathcal{C}(Y,Z) \otimes f"] \\
\mathcal{C}(X,Z) & \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \arrow[l, "c_{X,Y,Z}"']
\end{tikzcd}
$$

$$\mathcal{C}(X,Z) \xleftarrow{c_{X,Y,Z}} \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y)$$

IN $\mathcal{V}$.

DEF 3.1.13 ENRICHED FUNCTORS

LET $\mathcal{C}$ AND $\mathcal{D}$ BE $\mathcal{V}$-CATEGORIES

A $\mathcal{V}$-FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$

SENDS OBJECTS IN $\mathcal{C}$ TO ONES IN $\mathcal{D}$

AND FOR EACH PAIR OF OBJECTS

X, Y IN $\mathcal{C}$ WE HAVE A MORPHISM IN $\mathcal{V}$

$$\mathcal{C}(X,Y) \xrightarrow{F_{X,Y}} \mathcal{D}(FX, FY)$$

SUCH THAT

$$\begin{array}{ccc} \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) & \xrightarrow{c_{X,Y,Z}} & \mathcal{C}(X,Z) \\ \Big\downarrow F_{Y,Z} \otimes F_{X,Y} & & \Big\downarrow F_{X,Z} \\ \mathcal{D}(FY,FZ) \otimes \mathcal{D}(FX,FY) & \xrightarrow{D_{FX,FY,FZ}} & \mathcal{D}(FX,FZ) \end{array}$$

COMMUTES IN $\mathcal{V}$

$$\mathcal{D}(FY,FZ)\otimes\mathcal{D}(FX,FY) \xrightarrow{M_{FX,FY,FZ}} \mathcal{D}(FX,FZ)$$

AND

$$\begin{tikzcd}
 & \mathcal{C}(X,X) \arrow[d, "F_{X,X}"] \\
I \arrow[ur, "1_X"] \arrow[r, phantom, "\text{COMMUTES IN } \mathcal{V}"] \arrow[dr, "1_{FX}"'] & \\
 & \mathcal{D}(FX,FX)
\end{tikzcd}$$

GIVEN TWO SUCH FUNCTORS F AND G, A $\mathcal{V}$-NATURAL TRANSFORMATION $\theta : F \Rightarrow G$ CONSISTS OF A MORPHISM

$$I \xrightarrow{\theta_X} \mathcal{D}(FX,GX)$$

ORDINARY MORPHISM IN $\mathcal{D}_0$

FOR EACH $X \in \mathcal{C}$ SUCH THAT

$$\begin{tikzcd}
\mathcal{C}(X,Y) \arrow[r, "F_{X,Y}"] \arrow[d, "G_{X,Y}"'] \arrow[dr, phantom, "\text{COMMUTES IN } \mathcal{V}"] & \mathcal{D}(FX,FY) \arrow[d, "(\theta_Y)_*"] \\
\mathcal{D}(GX,GY) \arrow[r, "(\theta_X)^*"'] & \mathcal{D}(FX,GY)
\end{tikzcd}$$

$$(\theta_x)^*$$

FOR $\mathcal{V}$- CATEGORIES $\mathcal{C}$ and $\mathcal{D}$, $[\mathcal{C}, \mathcal{D}]$ DENOTES THE $\mathcal{V}$-CATEGORY WHOSE OBJECTS ARE $\mathcal{V}$-FUNCTORS $\mathcal{C} \longrightarrow \mathcal{D}$ AND WHOSE MORPHISMS ARE $\mathcal{V}$-NATURAL TRANSFORMATION AS ABOVE

ONE DEFINE ENRICHED END + COENDS

$$[\mathcal{C}, \mathcal{D}](F, G) = \int^{X \in \mathcal{C}} \mathcal{D}(FX, GX)$$

WHEN $\mathcal{C}$ IS SMALL AND

$\mathcal{D}$ IS COMPLETE.
(GENERALIZES A SIMILAR ORDINARY FORMULA)

ENRICHED YONEDA LEMMA 3.1.29

$\mathcal{C} = \mathcal{V}$-CATEGORY

$K \in \mathcal{C}$

ENRICHED YONEDA FUNCTOR

$$\mathcal{L}^K = \mathcal{C}(K,-) : \mathcal{C} \longrightarrow \mathcal{V}$$

LET $F : \mathcal{C} \to \mathcal{V}$ BE ANOTHER SUCH FUNCTOR, THEN

$$[\mathcal{C}, \mathcal{V}] : (\mathcal{L}^K, F) \cong F(K)$$

IN $\mathcal{V}_0$

DEF 3.1.31 TENSORS AND COTENSORS

COTENSORS

$\mathcal{C} = \mathcal{V}$- CATEGORY , $\mathcal{V}$ = CLOSED MONOIDAL CATEGORY

$\mathcal{C}$ IS TENSORED OVER $\mathcal{V}$ IF $\forall (K, X) \in \mathcal{V} \times \mathcal{C}$ THERE ARE OBJECTS $K \otimes X$ AND $X \otimes K$ IN $\mathcal{C}$ WITH

$$\mathcal{C}(K \otimes X, Y) \cong \mathcal{C}(X \otimes K, Y)$$
$$\cong \underline{\mathcal{V}}(K, \mathcal{C}(X, Y)) \quad \text{IN } \mathcal{V}$$

A CLOSED MONOIDAL CATEGORY $\mathcal{V}$ IS ENRICHED OVER ITSELF.

$\mathcal{C}$ IS COTENSORED (POWERED) OVER $\mathcal{V}$ IF WE HAVE $Y^K \in \mathcal{C}$ FOR $K \in \mathcal{V}$, $Y \in \mathcal{C}$

IF WE HAVE Y $\in \mathcal{C}$ FOR $Y \in \mathcal{C}$

WITH

$$\mathcal{C}(X, Y^K) \cong \mathcal{C}(X \otimes K, Y)$$
$$\cong \underline{\mathcal{V}}(K, \mathcal{C}(X, Y))$$

CONJUGACY