Monday, February 22, 2021 1:45 PM

MORE ABOUT LIMITS AND COLIMITS

LET $\mathcal{C}$ BE A SUITABLE (LARGE) CATEGORY, SUCH AS $\mathcal{S}et$, $\mathcal{A}b$, $\mathcal{T}op$, ETC, LET J BE A SMALL CATEGORY AND LET $\mathcal{C}^J$ DENOTE THE CATEGORY OF FUNCTORS $J \longrightarrow \mathcal{C}$, I.E. OF J-SHAPED DIAGRAMS IN $\mathcal{C}$. WE HAVE THE DIAGONAL FUNCTOR

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "\Delta"] & & \mathcal{C}^J \arrow[ll, bend right=50, "\mathrm{colim}"', "\perp"] \arrow[ll, bend left=50, "\mathrm{lim}", "\perp"']
\end{tikzcd}
$$

$X \longmapsto$ CONSTANT X-VALUED DIAGRAM

ITS LEFT (RIGHT) ADJOINT, IF IT EXISTS, IS THE COLIMIT (LIMIT) OF THE FUNCTOR $J \xrightarrow{F} \mathcal{C}$.

DEF A <u>CONE</u> OVER (UNDER) A DIAGRAM F IS AN OBJECT ~~K IN $\mathcal{C}$~~ WITH MORPHISMS S.T.

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$$\begin{tikzcd} K' \arrow[r, dashed, "!"] \arrow[rr, bend left] \arrow[drr, bend right] & K \arrow[r] \arrow[dr] & F(j) \arrow[d, "F(f)"] \\ & & F(j') \end{tikzcd}$$

COMMUTES FOR ALL MORPHISMS $j \xrightarrow{f} j'$ IN J

K IS A (CO)LIMIT OF F IF IT IS UNIVERSAL, I.E. ANY OTHER CONE K' ADMITS A UNIQUE MORPHISM TO (FROM) K MAKING THE DIAGRAM COMMUTE.

3 EXAMPLES FROM LAST TIME

1. J $\quad \bullet \to \bullet \leftarrow \bullet$ $\quad$ LIMIT IS PULLBACK

   J^{op} $\quad \bullet \leftarrow \bullet \to \bullet$ $\quad$ COLIMIT IS PUSHOUT

2. J IS EMPTY CATEGORY

   COLIMIT IS INITIAL OBJECT

   LIMIT IS TERMINAL OBJECT

3. J IS DISCRETE: ALL MORPHISMS ARE IDENTITIES

(CO)LIMITS ARE CALLED (CO)PRODUCTS

EG. CARTESIAN PRODUCT

DISJOINT UNION

NEW EXAMPLES

(4) J IS $\bullet \rightrightarrows \bullet$ $\qquad A \underset{g}{\overset{f}{\rightrightarrows}} B$

(CO) LIMIT IS CALLED THE (CO) EQUALIZER

IN $\mathcal{Set}$, THE LIMIT K

$$\{ a \in A : f(a) = g(a) \} \subset A$$

COLIMIT IS A QUOTIENT OF B.

THEOREM (HHR 2.3.8 REFERS TO MAC LANE
RIEHL 3.4.12)

EACH (CO) LIMIT IS A (CO) EQUALIZER OF TWO MORPHISMS BETWEEN CERTAIN (CO) PRODUCTS.

COR A CATEGORY $\mathcal{C}$ IS (CO) COMPLETE (MEANING ALL (CO) LIMITS EXIST IFF IT HAS (CO) PRODUCTS AND (CO) EQUALIZERS.

THE CONSTRUCTION FOR LIMITS:

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GIVEN $X : J \longrightarrow \mathcal{C}$

$j \longmapsto X_j$

$$A = \prod_{j \in Ob(J)} X_j \qquad\qquad A \overset{f}{\underset{g}{\rightrightarrows}} B$$

$$B = \prod_{\substack{u : j \to k \\ \in Arr(J)}} X_k$$

WE SPECIFY f AND g BY DEFINING THE COMPOSITES

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{p_u} X_k$$

$p_u f = p_k : A \longrightarrow X_k$

$p_u g = X(u)\, p_j$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "p_j"'] & B \arrow[d, "p_u"] \\ X_j \arrow[r, "X(u)"] & X_k \end{tikzcd}$$

THE EQUALIZER OF f AND g IS $\lim_J X$.

IS $\lim_J X$.

THE SIMILARLY DEFINED COEQUALIZER OF TWO MAPS BETWEEN CERTAIN COPRODUCTS.

(5) G = GROUP

BG = ASSOCIATED ONE OBJECT CATEGORY

A FUNCTOR $BG \xrightarrow{X} \mathcal{S}et$

IS A SET X WITH G-ACTION

EXERCISE: $\lim_{BG} X = X^G$ = FIXED POINT SET

$= \{x \in X : \gamma(x) = x \ \forall \gamma \in G\}$

$\operatorname{colim}_{BG} X = X_G = X/G$ = ORBIT SET

= SET OF EQUIVALENCE CLASSES IN X DEFINED BY THE G-ACTION.

(6)a $J = \mathbb{N}$ = NATURAL NUMBERS

$$0 \to 1 \to 2 \to 3 \to \cdots$$

$\operatorname{colim}_{N} X$ IS CALLED A <u>SEQUENTIAL COLIMIT</u>

$\mathcal{C} = \mathcal{A}b$

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \longrightarrow \cdots$$

THE COLIMIT IS $\mathbb{Q}$

6(b) $J = N^{op}$

$$0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow \cdots$$

IN $\mathcal{A}b$ p = PRIME

$$0 \leftarrow \mathbb{Z}/p \xleftarrow{\text{ONTO}} \mathbb{Z}/p^2 \xleftarrow{\text{ONTO}} \mathbb{Z}/p^3 \xleftarrow{\text{ONTO}} \cdots$$

THE LIMIT IS THE p-ADIC INTEGERS $\mathbb{Z}_p$.

FORMAL PROPERTIES

THEOREM (2.3.30 HHR)

RIGHT (LEFT) ADJOINTS PRESERVE (CO) LIMITS.

(CO) LIMITS.

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\Delta"] \arrow[d, phantom, "\dashv"] & \mathcal{D} \arrow[l, shift left, "U"] \arrow[d, shift right, "\Delta"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{C}^J \arrow[r, shift left, "F_*"] \arrow[r, phantom, "\perp"] \arrow[u, shift left, "\mathrm{colim}"] & \mathcal{D}^J \arrow[l, shift left, "U_*"] \arrow[u, shift right, "\mathrm{colim}"'] \end{tikzcd}$$

$$\boxed{\begin{matrix} \mathrm{colim}_J F_* X \\ \| \\ F \ \mathrm{colim}_J X \end{matrix}}$$

$$J \xrightarrow{X} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

$X \in \mathcal{C}^J \qquad FX \in \mathcal{D}^J$

THEOREM (HHR 2.3.41)

LIMITS (COLIMITS) COMMUTE WITH EACH OTHER.

SUPPOSE J, J' ARE SMALL CATS

$$J \times J' \xrightarrow{F} \mathcal{C}$$

$$J' \xrightarrow{F_J} \mathcal{C}^J \qquad\qquad J \xrightarrow{F_{J'}} \mathcal{C}^{J'}$$

$$j' \longmapsto F(-, j') \qquad\qquad j \longmapsto F(j, -)$$

THEN (SHOULD BE IN RED)

$$\operatorname{colim}_{J\times J'} F \cong \operatorname{colim}_{J}\left(\operatorname{colim}_{J'} F_j\right) \cong \operatorname{colim}_{J'}\left(\operatorname{colim}_{J} F_{j'}\right)$$

$$\begin{tikzcd}
C^{J\times J'} \arrow[r, shift left, "\operatorname{colim}_J"] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "\operatorname{colim}_{J'}"'] \arrow[d, phantom, "\dashv"] & C^{J'} \arrow[l, shift left, "\Delta"] \arrow[d, shift left, "\operatorname{colim}_{J'}"] \arrow[d, phantom, "\dashv"] \\
C^{J} \arrow[u, shift right, "\Delta"'] \arrow[r, shift right, "\operatorname{colim}_J"'] \arrow[r, phantom, "\top"] & C \arrow[l, shift right, "\Delta"'] \arrow[u, shift left, "\Delta"]
\end{tikzcd}$$

COMMUTES

⚠ LIMITS + COLIMITS NEED NOT COMMUTE

EXAMPLE

$$\begin{tikzcd}
\vdots \arrow[d, "p"] & \vdots \arrow[d, "p"] & \\
\mathbb{Z}_{(p)} \arrow[r, "p"] \arrow[d, "p"] & \mathbb{Z}_{(p)} \arrow[r, "p"] \arrow[d, "p"] & \cdots \\
\mathbb{Z}_{(p)} \arrow[r, "p"] & \mathbb{Z}_{(p)} \arrow[r, "p"] & \cdots
\end{tikzcd}$$

LIMIT OF EACH COLUMN IS 0

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COLIMIT OF " ROW IS $\mathbb{Q}$

$$\operatorname{colim} \lim = 0$$

$$\lim \operatorname{colim} = \mathbb{Q}$$

NOT THE SAME