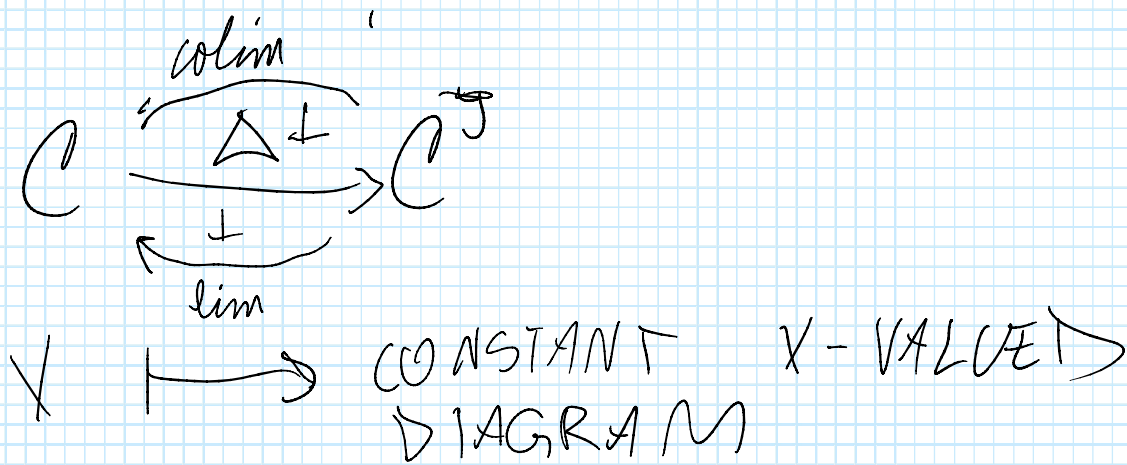
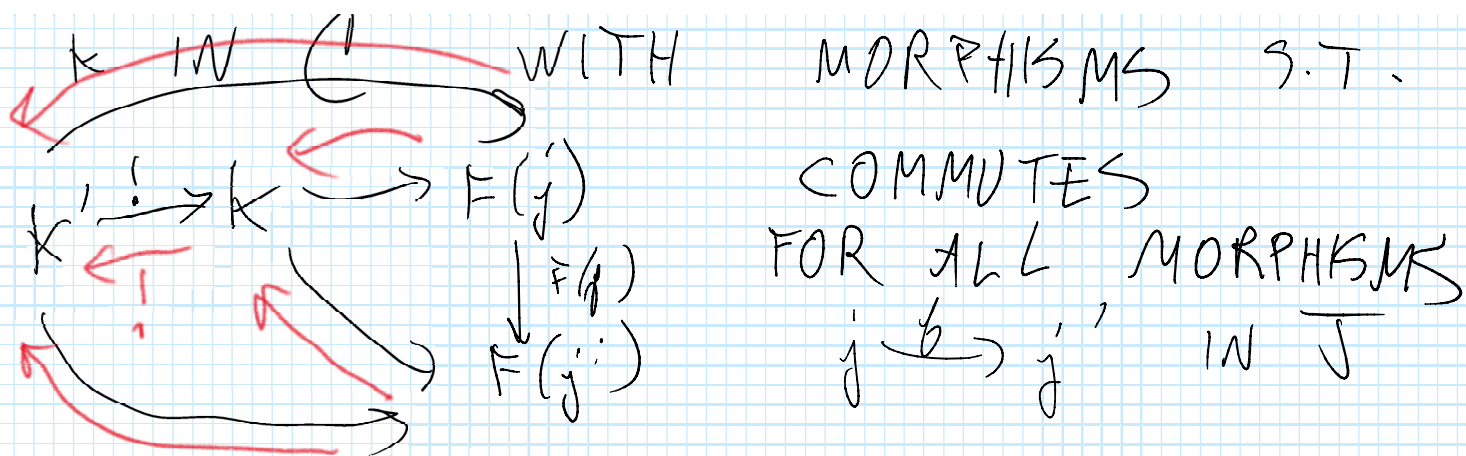


MORE ABOUT LIMITS AND COLIMITS
 LET \mathcal{C} BE A SUITABLE (LARGE) CATEGORY, SUCH AS Set , Ab , Top , ETC.
 - LET J BE A SMALL CATEGORY AND LET \mathcal{C}^J DENOTE THE CATEGORY OF FUNCTORS $J \rightarrow \mathcal{C}$, I.E. OF J -SHAPED DIAGRAMS IN \mathcal{C} . WE HAVE THE DIAGONAL FUNCTOR



ITS LEFT (RIGHT) ADJOINT, IF IT EXISTS, IS THE COLIMIT (LIMIT) OF THE FUNCTOR $J \rightarrow \mathcal{C}$.

DEF A CONE OVER (UNDER) A DIAGRAM F IS AN OBJECT K IN \mathcal{C} WITH MORPHISMS S.T.



K IS A **(CO)LIMIT** OF F IF IT IS UNIVERSAL, I.E. ANY OTHER CONE K' ADMITS A UNIQUE MORPHISM TO **(FROM)** K MAKING THE DIAGRAM COMMUTE.

3 EXAMPLES FROM LAST TIME

(1) $\mathcal{J} \quad \bullet \rightarrow \bullet \leftarrow \bullet$ LIMIT IS PULLBACK
 $\mathcal{J} \quad \bullet \leftarrow \bullet \rightarrow \bullet$ COLIMIT IS PUSHOUT

(2) \mathcal{J} IS EMPTY CATEGORY
 COLIMIT IS INITIAL OBJECT
 LIMIT IS TERMINAL OBJECT

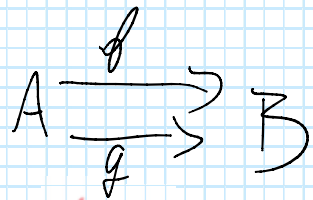
(3) \mathcal{J} IS DISCRETE: ALL MORPHISMS ARE IDENTITIES

(CO)LIMITS ARE CALLED **(CO)PRODUCTS**

EG. CARTESIAN PRODUCT
DISJOINT UNION

NEW EXAMPLES

(4) \mathcal{J} IS $\bullet \rightrightarrows \bullet$



(CO) LIMIT IS CALLED THE (CO) EQUALIZER
IN SET, THE LIMIT K

$$\{a \in A : f(a) = g(a)\} \subset A$$

COLIMIT IS A QUOTIENT OF B .

THEOREM (HHR 2.3, 8 REFERS TO MAC LANE)
(REHL 3.4, 12)

EACH (CO) LIMIT IS A (CO) EQUALIZER
OF TWO MORPHISMS BETWEEN
CERTAIN (CO) PRODUCTS.

COR A CATEGORY \mathcal{C} IS

(CO) COMPLETE (MEANING ALL (CO) LIMITS
EXIST IFF IT HAS (CO) PRODUCTS AND
(CO) EQUALIZERS.

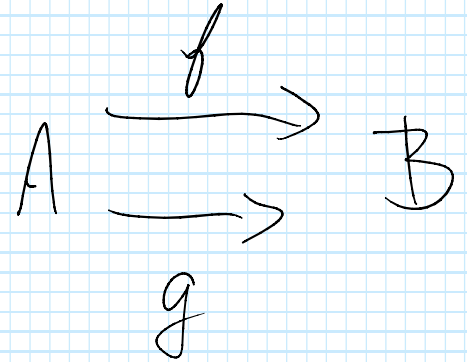
THE CONSTRUCTION FOR LIMITS:

THE CONSTRUCTION FOR LIMITS:

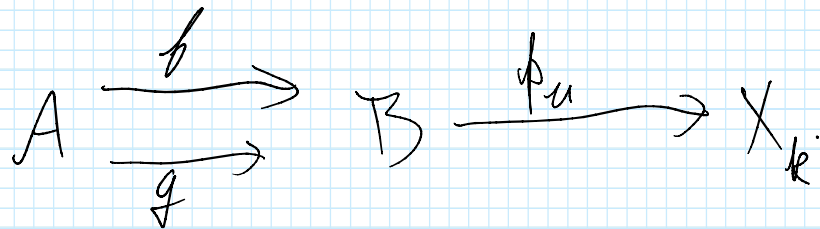
GIVEN $X: J \rightarrow \mathcal{C}$
 $j \mapsto X_j$

$$A = \prod_{j \in \text{Ob}(J)} X_j$$

$$B = \prod_{\substack{u: j \rightarrow k \\ \in \text{Arr}(J)}} X_k$$

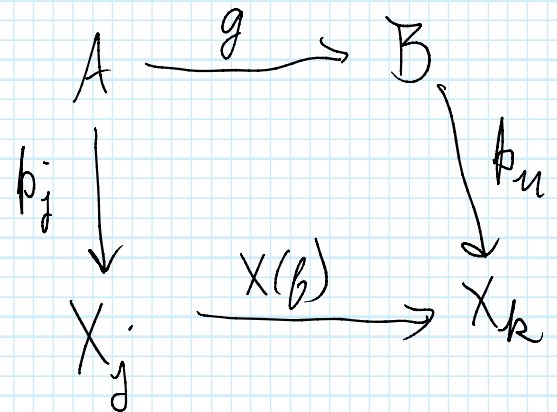


WE SPECIFY f AND g BY DEFINING THE COMPOSITES



$$p_u f = p_k: A \rightarrow X_k$$

$$p_u g = X(B) p_j$$



THE EQUALIZER OF f AND g
 IS $\lim_{\leftarrow} X$

is $\lim_J \lambda$.

THE SIMILARLY DEFINED
COEQUALIZER OF TWO MAPS
BETWEEN CERTAIN COPRODUCTS.

⑤

G = GROUP

BG = ASSOCIATED ONE OBJECT
CATEGORY

A FUNCTOR $BG \xrightarrow{\mathbb{X}} \text{Set}$

IS A SET \mathbb{X} WITH G -ACTION

EXERCISE: $\lim_{BG} X = X^G = \text{FIXED POINT SET}$

$$= \{x \in \mathbb{X} : \gamma(x) = x \forall \gamma \in G\}$$

$\text{colim}_{BG} X = X_G = X/G = \text{ORBIT SET}$

= SET OF EQUIVALENCE
CLASSES IN X

DEFINED BY THE G -ACTION.

⑥_a: $J = \mathbb{N} = \text{NATURAL NUMBERS}$

$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$

$\text{colim}_N X$ IS CALLED A SEQUENTIAL COLIMIT

$$\mathbb{Q} = \text{colim} \mathbb{Z}$$

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \rightarrow \dots$$

THE COLIMIT IS \mathbb{Q}

$$G(\mathbb{Z}) \quad \mathbb{Z} = \mathbb{N}^{op}$$

$$0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow \dots$$

IN Ab $p = \text{PRIME}$

$$0 \leftarrow \mathbb{Z}/p \xleftarrow{\text{ONTO}} \mathbb{Z}/p^2 \xleftarrow{\text{ONTO}} \mathbb{Z}/p^3 \xleftarrow{\text{ONTO}} \dots$$

THE LIMIT IS THE p -ADIC INTEGERS

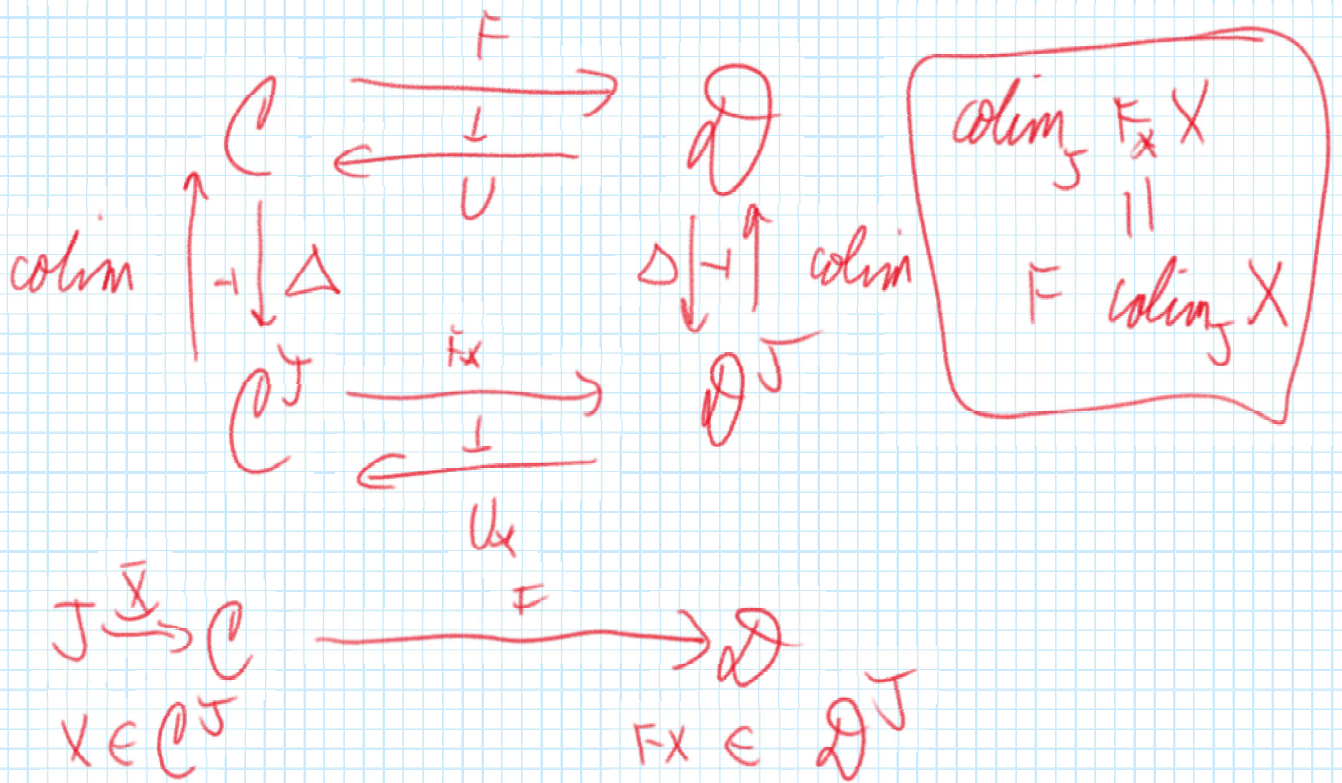
$$\mathbb{Z}_p.$$

FORMAL PROPERTIES

THEOREM (2.3.36 HHR)

RIGHT (LEFT) ADJOINTS PRESERVE
(CO) LIMITS.

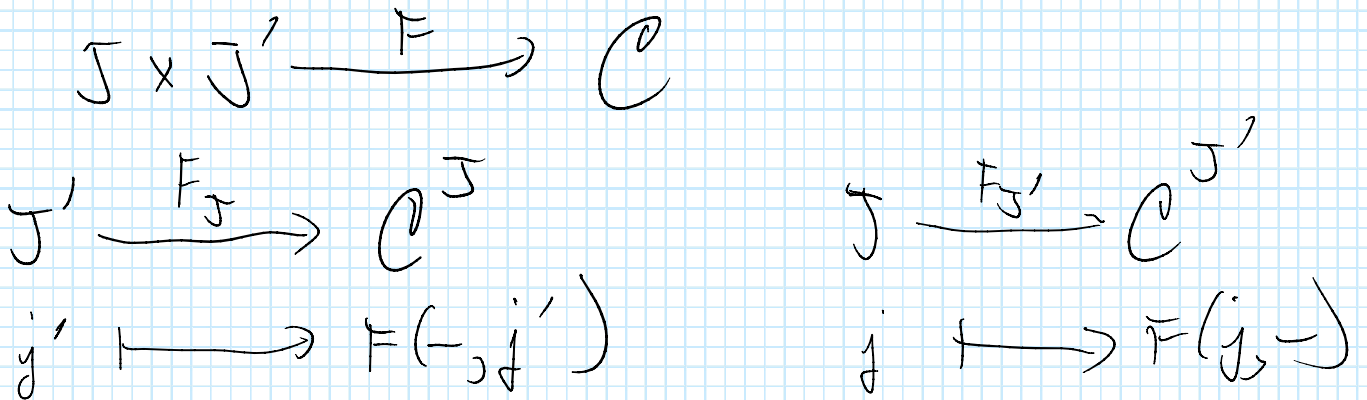
(CO) LIMITS.



THEOREM (HHR 2.3.41)

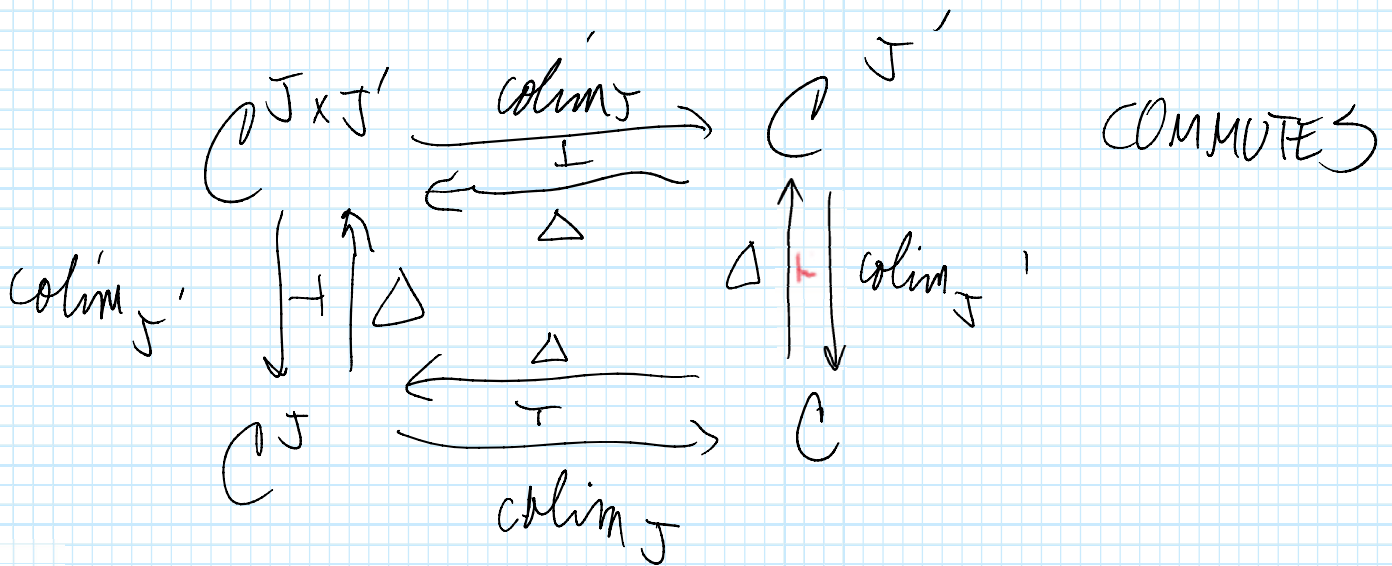
LIMITS (COLIMITS) COMMUTE WITH EACH OTHER.

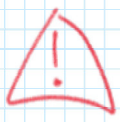
SUPPOSE J, J' ARE SMALL CATS



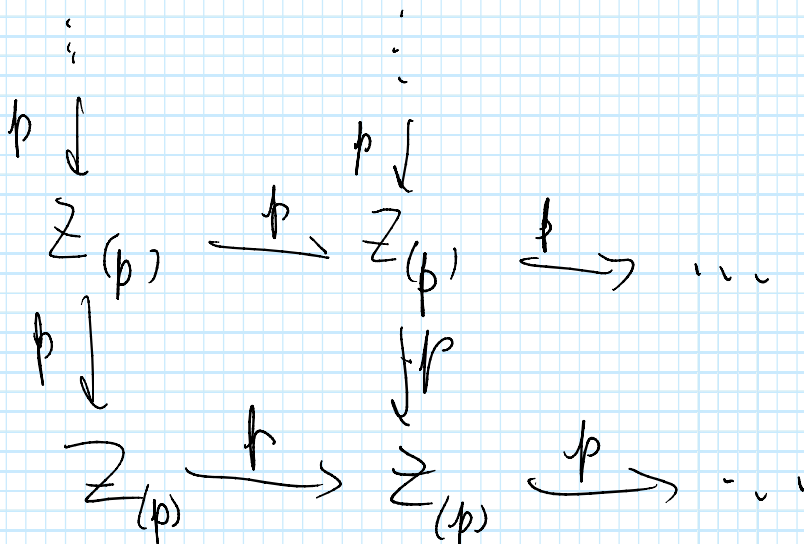
THEN (SHOULD BE IN RED)

$$\begin{aligned} \operatorname{colim}_{J \times J'} F &\cong \operatorname{colim}_J (\operatorname{colim}_{J'} F_J) \\ &\cong \operatorname{colim}_{J'} (\operatorname{colim}_J F_{J'}) \end{aligned}$$



 LIMITS + COLIMITS NEED NOT COMMUTE

EXAMPLE



LIMIT OF EACH COLUMN IS \mathcal{A}

LIMIT OF EACH COLUMN IS 0
LIMIT OF " ROW IS \mathbb{Q}

$\text{colim lim} = 0$ NOT
 $\text{lim colim} = \mathbb{Q}$ THE SAME