YONEDA KNOW SOME CATEGORY THEORY TO GET THIS JOKE

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LET A BE AN OBJECT IN A CATEGORY $\mathcal{C}$ AND CONSIDER THE FUNCTOR $\mathcal{C}(A,-) : \mathcal{C} \longrightarrow SET$

$$\mathcal{C}(A,-) = H^A$$

$$A \longrightarrow B \xrightarrow{f} B'$$

$$\mathcal{C}(A,B) \xrightarrow{f_*} \mathcal{C}(A,B') \qquad B \mapsto \mathcal{C}(A,B)$$

$\mathcal{C}(A,B)$ = SET OF MORPHISMS $A \to B$

THIS IS CALLED A REPRESENTABLE FUNCTOR.

WE ALSO HAVE $\mathcal{C}(-,A) : \mathcal{C}^{op} \longrightarrow SET$

$$\mathcal{C}(-,A) = H_A \qquad B \to \mathcal{C}(B,A)$$

THIS IS CONTRAVARIANT IN B

A MORPHISM

$$B' \xrightarrow{f} B \to A$$ INDUCES

$$\mathcal{C}(B',A) \xleftarrow{f^*} \mathcal{C}(B,A)$$

$$\mathcal{C}(B',A) \xleftarrow{f^A} \mathcal{C}(B,A)$$

CHENG CALLS BOTH FUNCTORS REPRESENTABLE

$$\text{よ}^A := \mathcal{C}(A,-) : \mathcal{C} \longrightarrow \mathbf{Set}$$

YO HIRAGANA

$$\text{よ}_B = \mathcal{C}(-,B) : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$$

<u>YONEDA LEMMA</u> SUPPOSE $F : \mathcal{C} \longrightarrow \mathbf{Set}$ IS ANOTHER FUNCTOR. THEN THERE IS A NATURAL BIJECTION BETWEEN

$$\mathrm{Nat}(\text{よ}^A, F) \xrightarrow{\quad K \quad} F(A)$$

$$\theta \longmapsto \theta_A(1_A)$$

SET(?) OF NATURAL TRANSFORMATIONS $\theta : \text{よ}^A \Longrightarrow F$

$1_A \in \underline{h}^A(A) = \mathcal{C}(A, A)$
$\shortparallel$
IDENTITY MORPHISM ON A

SINCE $\theta : \underline{h}^A \Longrightarrow F$ IS A NATURAL TRANSFORMATION,

FOR EACH OBJECT X IN $\mathcal{C}$, WE HAVE A MORPHISM $\theta_X : \underline{h}^A(X) \to F(X)$ IN $\mathcal{S}et$. e.g.

$$\begin{array}{ccc} \theta_A : \underline{h}^A(A) & \longrightarrow & F(A) \\ \cup & & \cup \\ 1_A & \longmapsto & \theta_A(1_A) \end{array}$$

PROOF: WE WILL SHOW THAT $k(\theta) = \theta_A(1_A)$ UNIQUELY DETERMINES THE NATURAL TRANSFORMATION θ, AND THAT $\theta_A(1_A)$ CAN BE ANY OBJECT IN THE SET $F(A)$.

LET $\theta : \underline{h}^A \Longrightarrow F$ BE A NATURAL TRANSFORMATION AND

LET $A \xrightarrow{f} X$ BE A MORPHISM IN $\mathcal{C}$

$$
\begin{tikzcd}
1_A \arrow[d, mapsto] \arrow[rrrr, bend left, mapsto] & \in & \mathcal{L}^A(A) \arrow[r, "\theta_A"] \arrow[d, "\mathcal{L}^A(f)"'] \arrow[dr, phantom, "\text{COMMUTES}"] & F(A) \arrow[d, "F(f)"] & K(\theta) \arrow[d, mapsto] \\
f \arrow[rrrr, bend right, mapsto] & \in & \mathcal{L}^A(X) \arrow[r, "\theta_X"] & F(X) & \theta_X(f) = F(f)(K(\theta))
\end{tikzcd}
$$

HENCE θ_X IS DETERMINED BY $K(\theta) = \theta_A(1_A)$

IF $\mathcal{C}(A, X) = \emptyset$, THEN $\mathcal{L}^A(X) = \emptyset$ SO θ_X IS UNIQUELY DETERMINED.

HENCE θ IS DETERMINED BY $K(\theta) \in F(A)$, AND K IS ONE TO ONE.

HOW TO SHOW K IS ONTO ?? (SORRY)

QED

RECALL $\mathcal{L}^A = \mathcal{C}(A, -) : \mathcal{C} \to \mathcal{S}et$

YONEDA FUNCTOR

LET $[\mathcal{C}, \mathcal{S}et]$ BE THE CATEGORY OF $\mathcal{S}et$-VALUED FUNCTORS ON $\mathcal{C}$, ITS MORPHISMS ARE NATURAL TRANSFORMATIONS

$$A \longmapsto \mathcal{L}^A$$

$$\mathcal{C}^{op} \xrightarrow{\mathcal{L}} [\mathcal{C}, \mathcal{S}et]$$

THIS IS THE <u>YONEDA EMBEDDING</u>

IT IS CONTRAVARIANT BECAUSE A MORPHISM $A \xrightarrow{g} A'$ IN $\mathcal{C}$

$$\mathcal{C}(A, -) \xleftarrow{g^*} \mathcal{C}(A', -)$$

DUAL NOTIONS

LET B BE AN OBJECT IN $\mathcal{C}$

$\mathcal{C}(-, B)$ IS A SET

$$\mathcal{C}^{op} \xrightarrow{\mathcal{L}_B} \mathcal{S}et$$

CO-YONEDA LEMMA:

FOR A FUNCTOR $G : \mathcal{C}^{op} \to \mathcal{S}et$

THEN

$$\mathcal{N}at(\mathcal{L}_B, G) \longrightarrow G(B)$$

$$\theta \longmapsto \theta_B(1_B)$$

WHERE $1_B \in \mathcal{L}_B(B) = \mathcal{C}(B, B)$

WE HAVE THE CO-YONEDA EMBEDDING

$$\mathcal{C} \xrightarrow{\mathcal{L}} [\mathcal{C}^{op}, \mathcal{S}et]$$

$$B \longmapsto \mathcal{L}_B = \mathcal{C}(-,B)$$

THE YONEDA EMBEDDING WAS

$$\mathcal{C}^{op} \xrightarrow{\mathcal{L}} [\mathcal{C}, \mathcal{S}et]$$

$$A \longmapsto \mathcal{L}^A = \mathcal{C}(A,B)$$

IF WE REPLACE $\mathcal{C}$ ABOVE BY $\mathcal{C}^{op}$, THEN WE GET THE YONEDA CO-EMBEDDING.

ADJOINT FUNCTORS

BRIEF FORMAL DEFINITION:

$$\begin{array}{ccc} X \in & \mathcal{C} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D} & \ni \; FX \\ UY & & Y \end{array}$$

WE SAY F IS THE LEFT ADJOINT OF U, U " RIGHT F

$$F \dashv G \qquad \text{KAN TURNSTILE}$$

IF THERE ARE NATURAL ISOMORPHISMS

$$\mathcal{C}(X, UY) \xrightarrow{\phi_{X,Y}} \mathcal{D}(FX, Y)$$

$\phi_{X,Y}$ IS THE ADJUNCTION ISO

EXAMPLES

$$X \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "U"] \end{tikzcd} \quad A$$

① $\mathcal{C} = \mathcal{S}et$ $\quad \mathcal{D} = \mathcal{A}b =$ ABELIAN GROUPS

F = FREE ABELIAN GROUP FUNCTOR

U = FORGETFUL FUNCTOR

$$\left.\begin{array}{l} \mathcal{C}(X, UA) = \mathcal{S}et(X, UA) \\ \mathcal{D}(FX, A) = \mathcal{A}b(FX, A) \end{array}\right\} \cong$$

= gp homs $FX \to A$

= set map $X \to UA$

WHAT DOES "NATURAL MEAN"

LET $A \xrightarrow{\alpha} A'$ BE GROUP HOM.

$$\begin{array}{ccc} \mathbf{Set}(X, UA) & \xrightarrow{\cong} & \mathbf{Ab}(FX, A) \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ \mathbf{Set}(X, UA') & \xrightarrow{\cong} & \mathbf{Ab}(FX, A') \end{array}$$

(2) LET A, B AND C BE SETS

(*) $\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, \mathbf{Set}(B, C))$

GIVEN $A \times B \xrightarrow{f} C$ and $a \in A$

$$B \xrightarrow{f_a} C$$
$$b \longmapsto f(a, b)$$

LET $F : \mathbf{Set} \to \mathbf{Set}$ $\quad$ $U : \mathbf{Set} \to \mathbf{Set}$

$X \longmapsto X \times B$ $\quad$ $X \mapsto \mathbf{Set}(B, X)$

(*) CAN BE REWRITTEN AS

$$\mathbf{Set}(FA, C) \cong \mathbf{Set}(A, UC)$$

$$F \dashv U$$

TURNSTILE POINTS TOWARD THE LEFT ADJOINT