CATEGORY THEORY

~~DE~~ DEFINITION: A <u>CATEGORY</u> $\mathcal{C}$ CONSISTS OF A COLLECTION OF OBJECTS

tom, dick, sally, amy, ...

AND MORPHISMS, WHICH ARE ARROWS BETWEEN OBJECTS

$$\text{tom} \longrightarrow \text{dick}$$

DOMAIN CODOMAIN

FOR ANY TWO OBJECTS X AND Y IN $\mathcal{C}$, THERE A POSSIBLY EMPTY SET OF MORPHISMS $X \to Y$ DENOTED BY $\mathcal{C}(X,Y)$, SUBJECT TO TWO AXIOMS

1) FOR EACH OBJECT $X \in \mathcal{C}$ (SHORTHAND FOR X IS AN OBJECT IN $\mathcal{C}$) THERE IS AN IDENTITY MORPHISM $1_X : X \to X$.

$1_X : X \longrightarrow X.$

2) GIVEN MORPHISMS

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

THERE IS A COMPOSITE MORPHISM $gf : X \longrightarrow Z$

SUCH THAT

a)

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta\alpha"'] & X \arrow[r, "\beta"] \arrow[rr, bend left, "\gamma\beta"] & Y \arrow[r, "\gamma"] & Z \end{tikzcd}$$

$(\gamma\beta)\alpha = \gamma(\beta\alpha)$

MORPHISM COMPOSITION IS ASSOCIATIVE.

b)

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[rr, bend right, "f"'] & X \arrow[r, "f"] \arrow[rr, bend left, "f"] & Y \arrow[r, "1_Y"] & Y \end{tikzcd}$$

COMPOSITION WITH IDENTITY MORPHISM BEHAVES AS EXPECTED.

EXAMPLE OF A NONCONCRETE CATEGORY:

LET G BE GROUP.

EVERY MORPHISM IS INVERTIBLE

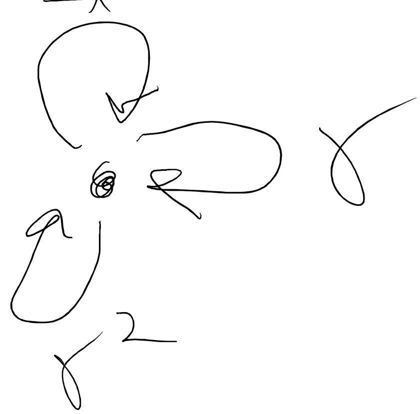
BG = CATEGORY WITH ONE OBJECT X, AND A MORPHISM $X \xrightarrow{\gamma} X$ FOR EACH $\gamma \in G$ WITH MORPHISM COMPOSITION GIVEN BY GROUP MULTIPLICATION

e.g. $G = C_3$ = CYCLIC GROUP OF ORDER 3

$= \{e, \gamma, \gamma^2\}$

$\gamma^3 = e$

$e = 1_X$



<u>DEFINITION</u> A MORPHISM $X \xrightarrow{f} Y$ IS INVERTIBLE

f^{-1}

IF THERE IS A MORPHISM
$f^{-1} : Y \to X$ SUCH THAT

SUCH A MORPHISM IS CALLED
AN ISOMORPHISM.

A GROUP IS THE SAME AS A
CATEGORY WITH ONE OBJECT
IN WHICH EACH MORPHISM IS
INVERTIBLE.

<u>DEF</u> A CATEGORY IS <u>SMALL</u>
IF ITS COLLECTION OF OBJECTS
IS A SET, I.E. IT IS "NOT TOO
BIG"

DEF A **GROUPOID** IS A SMALL CATEGORY IN WHICH EACH MORPHISM IS INVERTIBLE.

LET $\mathcal{C}$ BE A CATEGORY. ITS OPPOSITE CATEGORY $\mathcal{C}^{op}$ HAS THE SAME OBJECTS AS $\mathcal{C}$ AND MORPHISMS GOING IN THE OPPOSITE DIRECTION $\mathcal{C}^{op}(Y,X) = \mathcal{C}(X,Y)$

$\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$\mathcal{C}^{op}$

$$\begin{tikzcd} X & Y \arrow[l, "\bar{f}"'] & Z \arrow[l, "\bar{g}"'] \arrow[ll, bend left, "\bar{f}\bar{g}"] \end{tikzcd}$$

$$\bar{f}\bar{g} = \overline{(gf)}$$

NEXT BIG TOPIC: FUNCTORS

DEF A FUNCTOR $\mathcal{C} \xrightarrow{F} \mathcal{D}$ FROM CATEGORY $\mathcal{C}$ TO CATEGORY $\mathcal{D}$ CONSISTS OF

1) FOR EACH OBJECT $X \in \mathcal{C}$, WE HAVE AN OBJECT $FX \in \mathcal{D}$

2) FOR EACH MORPHISM IN $\mathcal{C}$

$$X \xrightarrow{g} Y \qquad \mathcal{C}$$

$$FX \xrightarrow{F(g)} FY \qquad \mathcal{D}$$

SUCH THAT

a) $F(1_X) = 1_{FX}$

$$F(g_2 g_1) = F(g_2)\, F(g_1)$$

$$\mathcal{C} \qquad \begin{tikzcd} X \arrow[r, "g_1"] \arrow[rr, bend right, "g_2 g_1"'] & Y \arrow[r, "g_2"] & Z \end{tikzcd}$$

COVARIANT

$$\mathcal{D} \qquad \begin{tikzcd} FX \arrow[r, "F(g_1)"] \arrow[rr, bend right] & FY \arrow[r, "F(g_2)"] & FZ \end{tikzcd}$$

$$F(g_2 g_1) = F(g_2) F(g_1)$$

FUNCTORS PRESERVE IDENTITY MORPHISMS AND COMPOSITIONS.

<u>LEMMA 1.3.8</u> (RIEHL)

FUNCTORS PRESERVES ISOMORPHISMS.

EXAMPLE

$\mathcal{C}$ = <u>GROUP</u> = CATEGORY OF GROUPS

$\mathcal{D}$ = <u>SET</u> = CATEGORY OF SETS

$U : \mathcal{C} \longrightarrow \mathcal{D}$ FORGETFUL

FUNCTOR

DEF A <u>CONTRAVARIANT</u> FUNCTOR $\mathcal{C} \longrightarrow \mathcal{D}$ IS ONE THAT REVERSES ARROW DIRECTION, I.E. A FUNCTOR

$$\mathcal{C}^{op} \longrightarrow \mathcal{D}$$