The Steenrod Algebra and Its Dual

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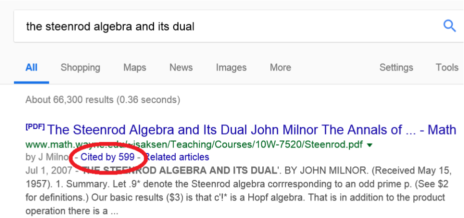

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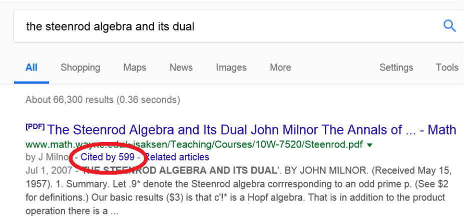

Goal : What was Milnor's work and its importance.

Motivation and Summary [9] : A **cohomology operation** is a natural transformation between cohomology functors.

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$$H^n(X;R) \longrightarrow H^{2n}(X;R)$$
$$x \mapsto x \cup x$$

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But, cohomology operations need not be homomorphisms of graded rings. Moreover, these operations do not commute with suspension. (It is called **unstable.**)

Norman Steenrod constructed stable operations

$$Sq^i : H^n(X; \mathbb{Z}_2) \longrightarrow H^{n+i}(X; \mathbb{Z}_2)$$

for all i greater than zero.

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- The properties of these operations were studied by Henri Cartan and Jose Adem. Also, these relations lead to the existence of the Serre-Cartan basis for $\mathcal{A}$.
- However, it is still complicated to know what the Steenrod algebra is.
- Milnor employed a more global view of the Steenrod algebra, recognizing the structure theorems of Cartan and Adem as aspects of the structure of a Hopf algebra.

Milnor's work

1. $\mathcal{A}$ has the structure of Hopf algebra.

2. Furthermore, Milnor has a beautiful description of its dual, giving to a construction of the Milnor basis for $\mathcal{A}$.

Goal :

1. Review the Steenrod algebra $\mathcal{A}$ over $p = 2$ and study Hopf algebra and Dual Steenrod algebra $\mathcal{A}^*$.
2. Show that $\mathcal{A}$ has the structure of Hopf algebra.
3. Obtain a beautiful description of $\mathcal{A}^*$:

$$\mathcal{A}^* \cong \mathbb{Z}_2 \left[\xi_1, \xi_2, \cdots, \xi_j, \cdots\right],$$

where deg$\xi_j = 2^j - 1$.

4. Describe explicitly the comultiplication ϕ^* for $\mathcal{A}^*$:

$$\phi^*(\xi_k) = \sum_{i=0}^{k} (\xi_{k-i})^{2^i} \otimes \xi_i$$

5. Study some properties of $\mathcal{A}, \mathcal{A}^*$.

Outline

1. The Structure of the Steenrod Algebra
   - The Steenrod Algebra $\mathcal{A}$
   - Hopf Algebras
2. The Structure of the Dual Steenrod Algebra
   - The Dual Steenrod Algebra $\mathcal{A}^*$
   - Comultiplication φ^* for $\mathcal{A}^*$
3. More properties of the Steenrod algebra $\mathcal{A}$
   - Revisited Primitive Elements
   - Milnor Basis for $\mathcal{A}$
   - Other Remarks

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Review[9] the mod 2 Steenrod algebra with the operations Sq^i.

Let K be the chain complex of a simplicial complex. Then the operations Sq^i is the natural homomorphisms

$$Sq^i : H^p(K;\mathbb{Z}_2) \longrightarrow H^{p+i}(K;\mathbb{Z}_2)$$

satisfying the following properties:

1. Sq^i is an additive homomorphism and is functorial with respect to any $f : X \longrightarrow Y$, so $f^*(Sq^i(x)) = Sq^i(f^*(x))$.
2. Sq^0 is the identity homomorphism.
3. $Sq^i(x) = x \cup x$ for $x \in H^i(X;\mathbb{Z}_2)$.
4. If $i > p$, $Sq^i(x) = 0$.
5. Cartan Formula:

$$Sq^i(x \cup y) = \sum_j (Sq^j x) \cup (Sq^{i-j} y)$$

Sq^i have more properties.

1. Sq^1 is the Bockstein homomorphism β of the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

(It gives a long exact sequence

$$\cdots \longrightarrow H^n(K;\mathbb{Z}_2) \longrightarrow H^n(K;\mathbb{Z}_2) \xrightarrow{\beta} H^{n+1}(K;\mathbb{Z}_2) \longrightarrow \cdots$$

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2. $Sq^i \circ \delta^* = \delta^* \circ Sq^i$ where δ^* is the connecting homomorphism $\delta^* : H^*(L;\mathbb{Z}_2) \longrightarrow H^*(K,L;\mathbb{Z}_2)$. In particular, it commutes with the suspension isomorphism for cohomology $H^k(K;\mathbb{Z}_2) \cong H^{k+1}(K;\mathbb{Z}_2)$.

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3. Satisfy Adem's relations: For $i < 2j$,
$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$
where the binomial coefficient is taken mod 2.

(1) is used as one of the generators of the Steenrod algebra.

(2) is especially important because it says that the Steenrod squares is a stable cohomology operation, and so holds a central position in stable homotopy theory.

(3) The Adem relations allow one to write an arbitrary composition of Steenrod squares as a sum of Serre-Cartan basis elements.

Miscellaneous Algebraic Definitions.[7] Let R be a commutative ring with unit.

1. A **graded R-algebra** A is a graded R-module with a multiplication $\varphi : A \otimes A \longrightarrow A$, where φ is a homomorphism of graded R-mudules and has a two sided unit.
2. A graded R-algebra A is **associative** if $\varphi \circ (\varphi \otimes 1) = \varphi \circ (1 \otimes \varphi)$. i.e., the following diagram is commute
$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\varphi \otimes 1} & A \otimes A \\ \downarrow{\scriptstyle 1 \otimes \varphi} & & \downarrow{\scriptstyle \varphi} \\ A \otimes A & \xrightarrow{\varphi} & A \end{array} .$$
3. A graded R-algebra is **commutative** if $\varphi \circ T = \varphi$, where $T : M \otimes N \longrightarrow N \otimes M$ by $T(m \otimes n) = (-1)^{\deg n \deg m}(n \otimes m)$.

1. A graded R-algebra is **augmented** if there is an algebra homomorphism $\varepsilon : A \longrightarrow R$.
2. An augmented R-algebra is **connected** if $\varepsilon : A_0 \longrightarrow R$ is isomorphic.
3. Let M be an R-module. Write $M^0 = R$ and $M^r = M \otimes \cdots \otimes M$, r times. Then the **tensor algebra** $T(M)$ is the graded R-algebra defined by $T(M)_r = M^r$.

Remark. $T(M)$ is associative, but not commutative.

Let $R = \mathbb{Z}_2$, M be the graded $\mathbb{Z}_2$-module such that $M_i = \mathbb{Z}_2$ generated by Sq^i. Then $T(M)$ is graded.

Let Q be the ideal generated by all $R(a, b)$, where

$$R(a, b) = Sq^a \otimes Sq^b + \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} \otimes Sq^c.$$

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Definition.[7] **The mod 2 Steenrod algebra** $\mathcal{A}$ is the quotient algebra $T(M)/Q$.

Simply, we can say that **the mod 2 Steenrod algebra** $\mathcal{A}$ is a graded algebra over $\mathbb{Z}_2$ generated by Sq^i, subject to the Adem relations.

Let us look at the properties of the mod 2 Steenrod algebra.

Note that $I = (i_1, i_2, \cdots, i_r)$ is called **admissible** if $i_s \geq 2i_{s+1}$ for $s < r$. We write $Sq^I = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_r}$.

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Theorem. (Serre-Cartan basis) Sq^I form a basis for $\mathcal{A}$ as a $\mathbb{Z}_2$ module, where I runs through all admissible sequences.

For example, $\mathcal{A}_7$ has as basis $Sq^7, Sq^6Sq^1, Sq^5Sq^2, Sq^4Sq^2Sq^1$.

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Theorem. Sq^{2^i} generate $\mathcal{A}$ as an algebra, where $i \geq 0$.

Remark. These elements do not generate $\mathcal{A}$ freely since it is subjected by Adem's relations.

For example, $Sq^2 Sq^2 = Sq^3 Sq^1 = Sq^1 Sq^2 Sq^1$ and $Sq^1 Sq^1 = 0$.

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Now we are done with reviewing the contents that we learned in Doug's class.

Furthemore, $\mathcal{A}$ has one more additional structure.

Let M be the graded $\mathbb{Z}_2$-module generated by Sq^i. Define an algebra homomorphism $\psi : T(M) \longrightarrow T(M) \otimes T(M)$ by

$$\psi(Sq^i) = \sum_j Sq^j \otimes Sq^{i-j}.$$

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Lemma. The map ψ extends to an algebra homomorphism

$$\psi : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}.$$

Sketch of Proof. Let $p : T(M) \longrightarrow \mathcal{A}$ be the projection. It suffices to show that ker$p \subset$kerψ. Then we can extend ψ as follows.

$$\begin{array}{ccc} T(M) & \xrightarrow{\ p\ } & \mathcal{A} := T(M)/Q \\ \downarrow \psi & \swarrow \psi & \\ \mathcal{A} \otimes \mathcal{A} & & \end{array}$$

Denote K_n be the n-fold cartesian product of $K(\mathbb{Z}_2, 1)$.

- Define a map $w : \mathcal{A} \longrightarrow H^*(K_n; \mathbb{Z}_2)$ by $w(\theta) = \theta(\sigma_n)$.
- Define a map $w' : \mathcal{A} \longrightarrow H^*(K_{2n}; \mathbb{Z}_2)$ by $w(\theta) = \theta(\sigma_{2n})$.
- To show the following diagram commutes.

$$\begin{array}{ccccc} T(M) & \xrightarrow{\quad p \quad} & \mathcal{A} & & \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle w \times w} & \searrow^{w'} & \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{w \otimes w} & H^*(K_n) \otimes H^*(K_n) & \xrightarrow{\alpha} & H^*(K_n \times K_n = K_{2n}) \end{array}$$

- Let $z \in T(M)$ with $p(z) = 0$. By the diagram, we get

$$0 = w'(p(z)) = \alpha(w \otimes w)(\psi)(z)$$

Since $w \otimes w$ is 1-1 for some n, we have $\psi(z) = 0$. $\square$

Example. Let us calculate some elements of the Steenrod algebra of ψ.

- $\psi(Sq^3) = 1 \otimes Sq^3 + Sq^1 \otimes Sq^2 + Sq^2 \otimes Sq^1 + Sq^3 \otimes 1$.
- $\psi(Sq^2Sq^1) = Sq^2Sq^1 \otimes 1 + Sq^1 \otimes Sq^2 + Sq^2 \otimes Sq^1 + 1 \otimes Sq^2Sq^1$.
- $\psi(Sq^3 + Sq^2Sq^1) = (Sq^3 + Sq^2Sq^1) \otimes 1 + 1 \otimes (Sq^3 + Sq^2Sq^1)$.

$$\begin{aligned}
Sq^2Sq^1(yz) &= Sq^2(Sq^1(yz)) \\
&= Sq^2((Sq^1y)z + ySq^1z) \\
&= Sq^2(Sq^1yz) + Sq^2(ySq^1z) \\
&= Sq^2Sq^1yz + Sq^1Sq^1ySq^1z + Sq^1ySq^2z + Sq^2ySq^1z \\
&\quad + Sq^1ySq^1Sq^1z + ySq^2Sq^1z \\
&= Sq^2Sq^1 \otimes 1 + Sq^1 \otimes Sq^2 + Sq^2 \otimes Sq^1 + 1 \otimes Sq^2Sq^1
\end{aligned}$$

by $Sq^1Sq^1 = 0$ from Adem's relation

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We already have the Steenrod algebra $(\mathcal{A}, \varphi)$ where φ is a multiplication in $\mathcal{A}$. We can see

$$\mathcal{A} \xrightarrow{\psi} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\varphi} \mathcal{A}.$$

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Question. What is Hopf algebra?

Answer. Roughly speaking, a Hopf algebra is a bigraded algebra with a multiplication and comultiplication.

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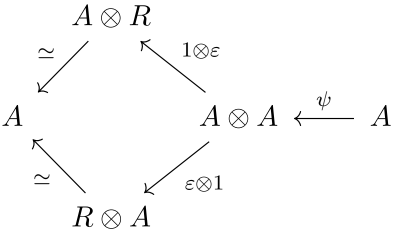
Let A be a connected graded R-module with a given R-homomorphism $\varepsilon : A \longrightarrow R$. Then $\varepsilon|_{A_0} : A_0 \longrightarrow R$ is an isomorphism.

Note that when we show the existence of unit (looks like 1), we consider the following diagram.

$$\begin{array}{ccccc} & & A\otimes R & & & & \\ & \overset{\simeq}{\nearrow} & & \overset{1\otimes\eta}{\searrow} & & & \\ A & & & & A\otimes A & \xrightarrow{\varphi} & A \\ & \underset{\simeq}{\searrow} & & \underset{\eta\otimes 1}{\nearrow} & & & \\ & & R\otimes A & & & & \end{array}$$

Both compositions are both the identity, where η is called **coagumentation**, is the inverse of the isomorphism $\varepsilon|_{A_0} : A_0 \longrightarrow R$.

A is a **coalgebra** (with co-unit) if there is an R-homomorphism $\psi : A \longrightarrow A \otimes A$ both compositions are both the identity in the following dual diagram.



i.e., For dim$a > 0$, the element $\psi(a)$ has the form

$$\psi(a) = a \otimes 1 + 1 \otimes a + \sum b_i \otimes c_i.$$

Definition. An element a in a coalgebra is called **primitive** if

$$\psi(a) = a \otimes 1 + 1 \otimes a.$$

Definition. Let A be an augmented graded algebra over a commutative ring R with a unit. We say A is a **Hopf algebra** if

1. A has a coalgebra structure with co-unit ε.
2. A has the comultiplication map $\psi : A \longrightarrow A \otimes A$.

with several commutative diagrams.

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Example. Let X be a connected topological group, with the group multiplication map $m : X \times X \longrightarrow X$ and the diagonal map $\Delta : X \longrightarrow X \times X$.

- $H_*(X; F)$ is a Hopf algebra with multiplication m_* and comultiplication map Δ_*.
- $H^*(X; F)$ is a Hopf algebra with multiplication Δ^* and comultiplication map m^*.

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- $H_*(X; F)$ is a Hopf algebra with multiplication m_* and comultiplication map Δ_*.
- $H^*(X; F)$ is a Hopf algebra with multiplication Δ^* and comultiplication map m^*.

Corollary. The Steenrod algebra $(\mathcal{A}, \phi, \psi)$ is a Hopf algebra.

This proof follows from the previous theorem that ψ is an algebra homomorphism.

Moreover, ψ has more good properties.

Recall that associativity and commutativity. By dualizing,

- ψ is **coassociative** if $(\psi \otimes 1) \circ \psi = (1 \otimes \psi) \circ \psi$. i.e., the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A \otimes A \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi \otimes 1} \\ A \otimes A & \xrightarrow[1 \otimes \psi]{} & A \otimes A \otimes A \end{array}$$

- ψ is **cocommutative** if $T \circ \psi = \psi$.

Note that the multiplication of the Steenrod algebra $\mathcal{A}$ is associative but not commutative. However,

Theorem. Comultiplication ψ of the Steenrod algebra $\mathcal{A}$ is coassociative and cocommutative.

Proof. Since ψ is an algebra homomorphism, it suffices to check on the generators. $\square$

Remark. In general, as for Hopf algebra, comultiplication need not be cocommutative. But always satisfy coassociative.

To sum up, the Steenrod algebra $\mathcal{A}$ is an

- φ associative,
- φ noncommutative,
- ψ coassociative,
- ψ cocommutative
- $(\mathcal{A}, \varphi, \psi)$ Hopf algebra.

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To every connected Hopf algebra (A, φ, ψ), there is associated the **daul Hopf algebra** (A^*, ψ^*, φ^*), where the homomorphisms

$$A^* \xrightarrow{\varphi^*} A^* \otimes A^* \xrightarrow{\psi^*} A^*$$

are the duals in the sense explained below:

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are the duals in the sense explained below: Let R be a field.

- $(A^*) = (A_i)^*$. i.e., dual vector over R.
- The mulitpication φ of A gives the diagonal map φ^* of A^*.
- The comulitpication map ψ of A gives the multiplication map ψ^* of A^*.

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Remark. The daul Hopf algebra is Hopf algebra.

Question. Why Dual?

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It is natural to study the dual Steenrod algebra.

	$\mathcal{A}$ the Steenrod Algebra	$\mathcal{A}^*$ the Dual Steenrod Algebra
Multiplication	φ Associative	ψ^* Coassociative
	φ Noncommutative	ψ^* **Commutative!!**
Comultiplication	ψ Coassociative	φ^* Coassociative
	ψ Cocommutaive	φ^* Noncocomutative
Hopf algebra	O	O

Table: The comparison the Steenrod algebra $\mathcal{A}$ with its dual $\mathcal{A}^*$

From now on, let us study a beautiful description of the dual Steenrod algebra $\mathcal{A}^*$.

Denote
$\mathcal{R} := \{(i_1, i_2, \cdots) \mid i_k \in \mathbb{Z}_{\geq 0},$ finitely many i_k are non-zero$\}$.

Definition. A sequence $I \in \mathcal{R}$ is called **admissible** if there exists $r \geq 0$ such that

$$\begin{cases} i_r > 0, i_q \geq 2i_{q+1} & \text{for } 1 \leq q < r \\ i_s = 0 & \text{for } s > r. \end{cases}$$

Denote $\mathcal{J} \subset \mathcal{R}$ be the set of all admissible sequenceses.

Example. Let $I^k := (2^{k-1}, \cdots, 2, 1, 0, 0, \cdots)$. Then I^k are admissible.

Let us do some combinatorics to obtain our main theorem.

Definition. Let ξ_i be the element of $\mathcal{A}^*_{2^i-1}$ such that

$$\langle \xi_k, Sq^I \rangle = \begin{cases} 1 & \text{for } I = I^k \\ 0 & \text{Otherwise} \end{cases}$$

where I be admissible and $k \geq 1$.
Furthemore, for arbitrary I, $\langle \xi_k, Sq^I \rangle = 0$ unless I is obtained from I^k by interspersion of zeros.

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Question. $\{\xi_k\}$ form a basis of $\mathcal{A}^*$?

Let us do some combinatorics to obtain our main theorem.

Definition. Let ξ_i be the element of $\mathcal{A}^*_{2^i-1}$ such that

$$\langle \xi_k, Sq^I \rangle = \begin{cases} 1 & \text{for } I = I^k \\ 0 & \text{Otherwise} \end{cases}$$

where I be admissible and $k \geq 1$.

Furthemore, for arbitrary I, $\langle \xi_k, Sq^I \rangle = 0$ unless I is obtained from I^k by interspersion of zeros.

Question. $\{\xi_k\}$ form a basis of $\mathcal{A}^*$?

Answer. No, remember $\{Sq^I \mid I \text{ adimissible}\}$ form a basis of $\mathcal{A}$. Then who can be a basis of $\mathcal{A}^*$? Also, I am going to show it's true they generate $\mathcal{A}^*$ as an algebra.

Define

- For each $R = (r_1, r_2, \cdots) \in \mathcal{R}$,

$$\xi^R := (\xi_1)^{r_1} (\xi_2)^{r_2} \cdots \in \mathcal{A}^*.$$

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- a set bijection $\gamma : \mathcal{J} \longrightarrow \mathcal{R}$ by

$$\gamma((a_1, \cdots, a_k, 0, 0, \cdots)) := (a_1 - 2a_2, a_2 - 2a_3, \cdots, a_k, 0, 0, \cdots).$$

Note that for $I \in \mathcal{J}$, degSq^I =deg$\xi^{\gamma(I)}$.

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Note that for $I \in \mathcal{J}$, degSq^I =deg$\xi^{\gamma(I)}$.

Let us give an order to the sequences of $\mathcal{J}$ lexicographically from the right.

Example.

$$\{7, 3, 2, 0, 0, \cdots\} > \{8, 3, 1, 0, 0, \cdots\} > \{8, 3, 0, 0, \cdots\} > \{10, 2, 0, 0, \cdots\}$$

Theorem. For $I, J \in \mathcal{J}$,

$$\left\langle \xi^{\gamma(J)}, Sq^I \right\rangle = \begin{cases} 0 & \text{for } I < J \\ 1 & \text{for } I = J \end{cases}$$

In particular, $\left\{\xi^{\gamma(J)}\right\}$ form a vector space basis for $\mathcal{A}^*$.

Sketch of Proof. Proof by a downward induction.

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Sketch of Proof. Proof by a downward induction.

Step 1. For $J = (a_1, \cdots, a_k, 0, 0, \cdots), I = (b_1, \cdots, b_k, 0, 0, \cdots)$, $J \geq I$, define

$$J' := (a_1 - 2^{k-1}, a_2 - 2^{k-2}, \cdots, a_k - 1, 0, 0, \cdots).$$

Then $\gamma(J) = \gamma(J')$ except for k component.

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Then $\gamma(J) = \gamma(J')$ except for k component.

Step 2. Show that

$$\left\langle \xi^{\gamma(J)}, Sq^I \right\rangle = \left\langle \xi^{\gamma(J')}, Sq^{I - I^k} \right\rangle$$

Descent on b_k and k completes the proof.

Corollary. As an algebra,

$$\mathcal{A}^* \simeq \mathbb{Z}_2[\xi_1, \xi_2, \cdots].$$

Proof.

- Note that $\{Sq^I\}$ is a basis for $\mathcal{A}$, where I is admissible.
- If J runs through $\mathcal{J}$, then $\xi^{\gamma(J)}$ runs through all the monomials in the ξ_i.
- $\{\xi^{\gamma(J)}\}$ form a vector space basis for $\mathcal{A}^*$ by theorem.
- Notice that a polynomial ring is characterized by the fact that the monomials in the generators form a vector space basis. □

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The Steenrod Algebra $\mathcal{A}$ with

- Multiplication map :

$$\varphi = \circ$$

- Comultiplication map :

$$\psi(Sq^i) = \sum_j Sq^j \otimes Sq^{i-j}$$

The dual Steenrod Algebra $\mathcal{A}^*$ with

- Multiplication map :

$$\psi^*(\xi_i \otimes \xi_j) = \xi_i \xi_j$$

- Comultiplication map :

$$\varphi^* = \ ?$$

Definition. Set $H_* := H_*(X;\mathbb{Z}_2), H^* := H^*(X;\mathbb{Z}_2)$.

Given the trivial action $\mu : \mathcal{A} \otimes H^* \longrightarrow H^*$, by $\mu(\theta, y) = \theta(y)$,

Definition. Set $H_* := H_*(X;\mathbb{Z}_2), H^* := H^*(X;\mathbb{Z}_2)$.

Given the trivial action $\mu : \mathcal{A} \otimes H^* \longrightarrow H^*$, by $\mu(\theta, y) = \theta(y)$,

- Define $\lambda : H_* \otimes \mathcal{A} \longrightarrow H_*$ by

$$\langle \lambda(x,\theta), y\rangle = \langle x, \mu(\theta, y)\rangle ,$$

  where $y \in H^*, x \in H_*, \theta \in \mathcal{A}$.

- Denote λ^* be the dual of λ. i.e.,

$$\lambda^* : H^* \longrightarrow (H_* \otimes \mathcal{A})^* = H^* \otimes \mathcal{A}^*.$$

Proposition 1. λ is a module operation and λ^* is an comodule operation. i.e., The following diagrams commute.

$$\begin{array}{ccc} H_* \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\lambda \otimes 1} & H_* \otimes \mathcal{A} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \lambda} \\ H_* \otimes \mathcal{A} & \xrightarrow{\lambda} & H_* \end{array} \qquad \begin{array}{ccc} H^* \otimes \mathcal{A}^* \otimes \mathcal{A}^* & \xleftarrow{\lambda^* \otimes 1} & H^* \otimes \mathcal{A}^* \\ {\scriptstyle 1\otimes\varphi^*}\uparrow & & \uparrow{\scriptstyle \lambda^*} \\ H^* \otimes \mathcal{A}^* & \xleftarrow{\lambda^*} & H^* \end{array}$$

Proposition 2. λ is a coalgebra homomorphism and λ^* is an algebra homomorphism. i.e., The following diagrams commute.

$$\begin{array}{ccccc} H_* \otimes H_* \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{1\otimes T\otimes 1} & H_* \otimes \mathcal{A} \otimes H_* \otimes \mathcal{A} & \xrightarrow{\lambda\otimes\lambda} & H_* \otimes H_* \\ {\scriptstyle \Delta_*\otimes\psi}\uparrow & & & & \uparrow{\scriptstyle \Delta_*} \\ H_* \otimes \mathcal{A} & & \xrightarrow{\qquad\lambda\qquad} & & H_* \end{array}$$

Theorem. The comultiplication map φ^* of $\mathcal{A}^*$ is given by

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Sketch of Proof.

Step 1. Prove the following are equivalent for $y \in H^*$:

1. $\lambda^*(y) = \sum y_i \otimes w_i$
2. $\mu(\theta, y) = \sum \langle \theta, w_i \rangle \, y_i$ for all $\theta \in \mathcal{A}$.

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Step 2. Let x generate $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. Show that

$$\lambda^*(x) = \sum_{i \geq 0} x^{2^i} \otimes \xi_i.$$

i.e.,show $\mu(Sq^I, x) = \sum \langle Sq^I, \xi_i \rangle x^{2^i}$ and enough to check I is admissible.

Step 3. Show that

$$\lambda^*(x^{2^i}) = \sum_{j\geq 0} x^{2^{i+j}} \otimes (\xi_j)^{2^i}.$$

Proof. $\lambda^*(x^{2^i}) \stackrel{(2)}{=} (\lambda^* x)^{2^i} = \sum_j (x^{2^j} \otimes \xi_j)^{2^i} = \sum_j x^{2^{i+j}} \otimes (\xi_j)^{2^i}$ □

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Step 4. Use the commuting diagram in proposition 1.

$$(1 \otimes \varphi^*)\lambda^*(x) = (1 \otimes \varphi^*)(\sum_k x^{2^k} \otimes \xi_k) = \sum_k x^{2^k} \otimes \varphi^*(\xi_k)$$

$$(\lambda^* \otimes 1)\lambda^*(x) = (\lambda^* \otimes 1)(\sum_i x^{2^i} \otimes \xi_i) = \sum_i \lambda^*(x^{2^i}) \otimes \xi_i$$

$$= \sum_{i,j} x^{2^{i+j}} \otimes (\xi_j)^{2^i} \otimes \xi_i.$$

By comparing them, we get $\varphi^*(\xi_k) = \sum_i (\xi_{k-i})^{2^i} \otimes \xi_i$. □

Summary.

Algebra	$\mathcal{A}$ the Steenrod Algebra	$\mathcal{A}^*$ the Dual Steenrod Algebra
Structure	a graded noncommutative, cocommutaive Hopf algebra	a graded commutative, non-cocommutative Hopf algebra
Basis	$\{Sq^I\}$, where I : admissible	$\{\xi^R\}$, where R : any sequence
As an algebra	$\left\{Sq^{2^k}\right\}$ generate $\mathcal{A}$ and subject to Adem's realtions	$\{\xi_k\}$ freely generate $\mathcal{A}^*$
Comultiplication	$\psi(Sq^k) = \sum_j Sq^j \otimes Sq^{k-j}$	$\varphi^*(\xi_k) = \sum_{i=0}^k (\xi_{k-i})^{2^i} \otimes \xi_i$

Table: The comparison the Steenrod algebra $\mathcal{A}$ with its dual $\mathcal{A}^*$

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Remember finding primitive elements is difficult. But there is a nice 1-1 correspondence primitive elements in $\mathcal{A}$ and indecomposables in $\mathcal{A}^*$.

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Observation.

- Let $I = (10, 4, 2, 1), I^4 = (8, 4, 2, 1)$. Then we get
$$I - I^4 = (2, 0, 0, 0) = 2I^1.$$
So $I = I^4 + 2I^1$.
- Let $I = (27, 13, 6, 2), 2I^4 = (16, 8, 4, 2), 2I^3 = (8, 4, 2)$. Then we get
$$I - 2I^4 - 2I^3 = (3, 1, 0, 0) = I^2 + I^1.$$
So $I = 2I^4 + 2I^3 + I^2 + I$.

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So $I = 2I^4 + 2I^3 + I^2 + I$.

Fact. Any admissible I can be written uniquely as a linear combination of I^ks.

Note that $I^k \longleftrightarrow \xi_k$ by $\langle \xi_k, Sq^{I_k} \rangle = 1$.

Observation.

$$I = 2I^4 + 2I^3 + I^2 + I \longleftrightarrow \xi_4^2 \xi_3^2 \xi_2 \xi_1.$$

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Observation.

$$I = 2I^4 + 2I^3 + I^2 + I \longleftrightarrow \xi_4^2 \xi_3^2 \xi_2 \xi_1.$$

There is a bijection between admissible sequences and monomials in the ξ_k in a such way. (Here, $\xi_0 = 1$.)

$$\{\text{Primitives in } \mathcal{A}\} \longleftrightarrow \{\text{Indecomposables in } \mathcal{A}^*\}$$

$$Q_1 := Sq^1 \longleftrightarrow \xi_1$$

$$Q_2 := [Sq^2, Sq^1] \longleftrightarrow \xi_2$$

$$= Sq^2 Sq^1 + Sq^1 Sq^2$$

$$= Sq^2 Sq^1 + Sq^3$$

$$= [Sq^2, Q_1]$$

$$Q_3 := [Sq^4, Q_2] \longleftrightarrow \xi_3$$

$$Q_{n+1} := [Sq^{2^n}, Q_n] \longleftrightarrow \xi_{n+1}$$

Moreover, we have the following bijection.

$$\begin{array}{ccc} \{\text{Indecomposables in } \mathcal{A}\} & \longleftrightarrow & \{\text{Primitives in } \mathcal{A}^*\} \\ Sq^{2^k} & \longleftrightarrow & \xi_1^{2^k} \end{array}$$

Remark. The only primitive elements in $\mathcal{A}^*$ are $\xi_1^{2^k}$. It's more simpler than primitives in $\mathcal{A}$.

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One might wonder if we can use the dual basis of $\{\xi^R\}$ to study the Steenrod algebra instead of Cartan-Serre basis.

It is called the **Milnor basis**.

Recall. $\{\xi^R\}, R \in \mathcal{R}$ forms a basis for $\mathcal{A}^*$. Now we can dualize back!

Definition. The dual basis of $\{\xi^R\}, R = (r_1, r_2, \cdots, r_k, 0, 0, \cdots) \in \mathcal{R}$, whose elements are denoted $\{Sq^R\}$ or $Sq(r_1, \cdots, r_k)$, is called the **Milnor basis** for the Steenrod algebra $\mathcal{A}$.

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Remark. 1) By difinition, $\left\langle \xi^R, Sq^{R'} \right\rangle = \begin{cases} 1 & \text{for } R = R' \\ 0 & \text{Otherwise} \end{cases}$.

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2) This is different from the Serre-Cartan basis. i.e., not the same as the composite $Sq^{r_1} Sq^{r_2} \cdots Sq^{r_k}$.

But, in some case, they are same.

Proposition. $Sq(i, 0, 0, \cdots) = Sq^i$.

Formula.[6]

$$Sq(r_1, r_2, \cdots)Sq(s_1, s_2, \cdots) = \sum_X Sq(t_1, t_2, \cdots)$$

where the sum is taken over all matrices $X = \langle x_{ij} \rangle$ satisfying:

$$\sum_i x_{ij} = s_j, \quad \sum_j 2^j x_{ij} = r_i, \quad \prod_h (x_{h0}, x_{h-1,1}, \cdots, x_{0h}) \equiv 1 \ (\text{mod} 2)$$

where $(n_1, \cdots, n_m)$ is the multinomial coefficient $(n_1 + \cdots + n_m)!/(n_1! \cdots n_m!)$. (The value of x_{00} is never used and may be taken to be 0.) Each such allowable matrix produces a summand $Sq(t_1, t_2, \cdots)$ given by

$$t_h = \sum_{i+j=h} x_{ij}.$$

Example. How to express $Sq(4,2)Sq(2,1)$ using the Milnor basis?

Let $R=(4,2), S=(2,1)$. Then we get

$$\begin{aligned} x_{10}+2x_{11}+4x_{12}+\cdots &= 4 = r_1 \\ x_{20}+2x_{21}+4x_{22}+\cdots &= 2 = r_2 \\ x_{01}+x_{11}+x_{21}+\cdots &= 2 = s_1 \\ x_{02}+x_{12}+x_{22}+\cdots &= 1 = s_2 \end{aligned}$$

For row 1,

$$(4,0,0)<(2,1,0)<(0,2,0)<(0,0,1).$$

For row 2,

$$(2,0,0)<(0,1,0).$$

$$\begin{pmatrix} * & 2 & 1 \\ 4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} (4,2)(2,0,1)Sq(6,3) = Sq(6,3)$$

$$\begin{pmatrix} * & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} (2,1)(2,1,1)Sq(3,4) = 0$$

$$\begin{pmatrix} * & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} (0,0)(2,2,1)Sq(0,5) = 0$$

$$\begin{pmatrix} * & 2 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} (0,2)(2,0,0)(0,1)Sq(2,2,1) = Sq(2,2,1)$$

$$\begin{pmatrix} * & 1 & 1 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} (4,1)(0,0,1)(1,0)Sq(5,1,1) = Sq(5,1,1)$$

$$\begin{pmatrix} * & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} (2,0)(0,1,1)(1,0)Sq(2,2,1) = 0$$

$$\begin{pmatrix} * & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} (0,1)(0,0,0)(1,1)Sq(1,0,2) = 0$$

Therefore, we find that

$$Sq(4,2)Sq(2,1) = Sq(6,3) + Sq(2,2,1) + Sq(5,1,1).$$

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Further comments for the Steenrod algebra $\mathcal{A}$.

- Every element of $\mathcal{A}$ is nilpotent.
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- Milnor's clear description of the rich structure of the Steenrod algebra played a key role in the development of the Adams spectral sequence (Adams [1958, 1960]).
- The Adams spectral sequence and its generalizations by Novikov [1967] are the tools of choice in the study of stable homotopy theory.
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Not the end. It is only the beginning.

- R.R.Bruner, *An Admas Spectral Sequence Primer*, (2009).
- A. Gurjale, *Personal Discussion with him*, (2018).
- B. Guillou, *Class Notes*, (2017).
- J. Milnor, *The Steenrod Algebra and Its Dual*, Annals of Mathematics, (1958).
- J. Milnor, J. McCleary, *Homotopy, Homology, and Manifolds*, American Mathematical Society, (2009).
- K.G. Monks, *Change of Basis, Monomial Realtions, and P Bases for the Steenrod Algebra*, Journal of Pure and Applied Algebra, (1998), **125**, 235-260 .
- R. Mosher, M. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, American Mathematical Society, (2009).

- D. Ravenel, *Personal Discussion with him*, (2018).
- Wikipedia, The Free Encyclopedia.
- C. Woo, *Steenrod algebra*, A personal paper.

Thank you for your attention!