Let Top = category of topological spaces

$\mathcal{T}$ = " pointed "

In Top let $i_n : S^{n-1} \hookrightarrow D^n$ this is a cofibration

e.g. $i_0 : \emptyset \hookrightarrow *$

$j_n : I^n \longrightarrow I^n \times I$ this is a trivial cofibration

$x \longmapsto (x, 0)$

$j_0 : * \longrightarrow I$

Let $\mathcal{I} = \{ i_n : n \geq 0 \}$ and $\mathcal{J} = \{ j_n : n \geq 0 \}$

In $\mathcal{T}$, define i_{n+} and j_{n+} by adding a disjoint base point to source + target. and get sets $\mathcal{I}_+$ and $\mathcal{J}_+$.

Consider the set of morphisms that can be derived from those in $\mathcal{I}$ (or ...) by the following operations

1. composition (possibly transfinite)
2. disjoint union (or one pt union)
3. pushouts

pushout symbol

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, dashed, "g'"] \\ C \arrow[r, dashed, "f'"'] \arrow[ur, phantom, "\lrcorner"] & D \end{tikzcd}$$

If f is a cofibration, so is f'.

e.g. $S^{n-1} \xrightarrow{i_n} D^n$

e.g.

$$\begin{tikzcd} S^{n-1} \arrow[r, "in"] \arrow[d, "g"'] & D^n \arrow[d] \\ X \arrow[r] & X \cup_g e^n \end{tikzcd}$$

$X \cup_g e^n$ = attach n-cell to X via attaching map g.

Any (trivial) cofibration can be derived from those in I (J) by these operations. Similarly in the pointed case.

<u>Def</u> A map $X \xrightarrow{p} Y$ is a SERRE FIBRATION if for each diagram

$$\forall n \geq 0 \qquad \begin{tikzcd} I^n \arrow[r, "\alpha"] \arrow[d, "j_n"'] & X \arrow[d, "p"] \\ I^{n+1} \arrow[r, "\beta"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

$\exists$ lifting h with $\alpha = h j_n$ and $\beta = p h$.

It is a TRIVIAL SERRE FIBN if

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d, "in"'] & X \arrow[d, "p"] \\ D^n \arrow[r, "\beta"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

We say that I and J are COFIBRANT

We say that I and J are COFIBRANT GENERATING SETS for the model structure on Top.
This is a convenient way to describe a model category.

Def An object X is COFIBRANT if the unique map $\phi \to X$ is a cofibration. It is FIBRANT if the unique $X \to *$ is a fibration.

Example In Top every object is fibrant, and cofibrant objects are CW-complexes.
In Ch_R (chain cxs of R-modules) every object is fibrant, and cofibrant objects are chain complexes of projective R-modules.

Morphisms from cofibrant to fibrant objects have the nicest properties.

properties.

Consider the unique map

$$\begin{tikzcd} & QY \arrow[dr, "\text{trivial fibr, so a weq}"] & \\ \phi \arrow[ur, "\text{cofib}"] \arrow[rr] & & Y \end{tikzcd}$$

Y = any object Y

QY is the COFIBRANT REPLACEMT for Y

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & * \\ & RX \arrow[ur, "\text{fibration}"'] & \end{tikzcd}$$

RX = FIBRANT REPLACEMT for X.

In Ch_R let M be an R-module

and Y is $M \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots$

Its cofibrant replacement is

$$\begin{tikzcd} P \arrow[d] & P_0 \arrow[d, "f_0"'] & P_1 \arrow[l] \arrow[d, "f_1"'] & P_2 \arrow[l] \arrow[d, "f_2"'] & P_3 \arrow[l] \arrow[d, "f_3"'] & \cdots \arrow[l] \\ Y & M & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

where the map f is a trivial fibr,

i.e. f_0 is onto and $H_* P \xrightarrow[\cong]{f} H_* Y$, i.e.

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

is exact, so it is a projective resolution of M.

New topic. Let $\mathcal{M}$ be a cofibrantly generated model category (e.g. Top or $\mathcal{S}$) and let J be a small category.

Let $\mathcal{M}^J$ be the category of functors $J \to \mathcal{M}$ (J-shaped diagrams in $\mathcal{M}$)

e.g. G = finite gp

J_G = one object category with a morphism $\forall\ \gamma \in G$.

A functor $J_G \longrightarrow \mathrm{Top}$ is a G-space X

Notation: Given a functor $J \xrightarrow{X} \mathcal{M}$

1. $X \in \mathcal{M}^J$

$$\begin{array}{ccc} j & \longmapsto & X_j \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle X_f} \\ k & \longmapsto & X_k \end{array}$$

$k \longmapsto X_k$

We want a model structure on the category $\mathcal{M}^J$. What is a morphism in $\mathcal{M}^J$? It is a NATURAL TRANSFORMATION $X \xrightarrow{\theta} Y$

i.e. for each $j \in J$ we have a morphism

$$X_j \xrightarrow{\theta_j} Y_j$$

and for each morphism $j \xrightarrow{b} k$ in J, the following diagram commutes

$$\begin{tikzcd} X_j \arrow[r, "\theta_j"] \arrow[d, "X_b"'] & Y_j \arrow[d, "Y_b"] \\ X_k \arrow[r, "\theta_k"] & Y_k \end{tikzcd}$$

Def In the PROJECTIVE MODEL structure on $\mathcal{M}^J$, a morphism $X \xrightarrow{\theta} Y$ is a weq or a fibration if $X_j \xrightarrow{\theta_j} Y_j$ is one for each $j \in J$.

A cofibration is defined in terms of lifting properties;

i.e. a map $A \xrightarrow{\psi} B$ in M^J is a cofibration if for any trivial fibration $X \xrightarrow{p} Y$ and any diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\psi"'] & X \arrow[d, "p"] \\ B \arrow[r, "\sigma"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

in M^J

ψ is a cofibration if $\exists h$ in all cases.

In the INJECTIVE model structure ......

We can rephrase ① by saying there are STRUCTURE MAPS

$$J(j,k) \times X_j \xrightarrow{\varepsilon^X_{j,k}} X_k$$

set of morphisms $j \to k$ in J

. It should play nicely with

composition of morphisms in

$$\begin{tikzcd} j \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & k \arrow[r, "g"] & l \end{tikzcd} \quad \text{in } J$$

$$J(k,l) \times J(j,k) \xrightarrow{C_{j,k,l}} J(j,l)$$

The following diagram should commute

$$\begin{tikzcd} J(k,l) \times J(j,k) \times X_j \arrow[r, "J(k,l)\times \varepsilon^X_{j,k}"] \arrow[d, "C_{j,k,l}\times X_j"'] & J(k,l)\times X_k \arrow[d, "\varepsilon^X_{k,l}"] \\ J(j,l) \times X_j \arrow[r, "\varepsilon^X_{j,l}"] & X_l \end{tikzcd}$$