

Let $\text{Top} =$ category of topological spaces
 $\mathcal{J} =$ pointed

In Top let $i_n: S^{n-1} \hookrightarrow D^n$ this is a cofibration

e.g. $i_0: \emptyset \hookrightarrow *$

$$j_n: I^n \longrightarrow I^n \times I$$

$$\begin{matrix} \downarrow & \downarrow \\ X & \longrightarrow (X, 0) \\ \downarrow & \downarrow \\ X & \longrightarrow I \end{matrix}$$

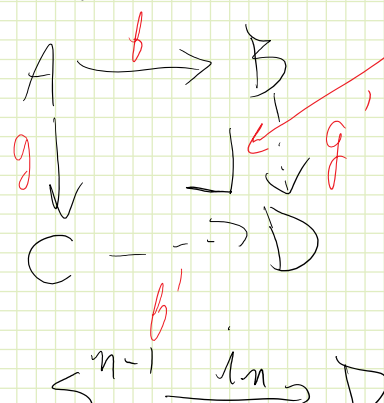
this is a trivial cofibration

Let $\mathcal{I} = \{i_n: n \geq 0\}$ and $\mathcal{J} = \{j_n: n \geq 0\}$

In \mathcal{J} , define i_{n+} and j_{n+} by adding a disjoint base point to source + target and get sets \mathcal{I}_+ and \mathcal{J}_+

Consider the set of morphisms that can be derived from those in \mathcal{I} (or \dots) by the following operations

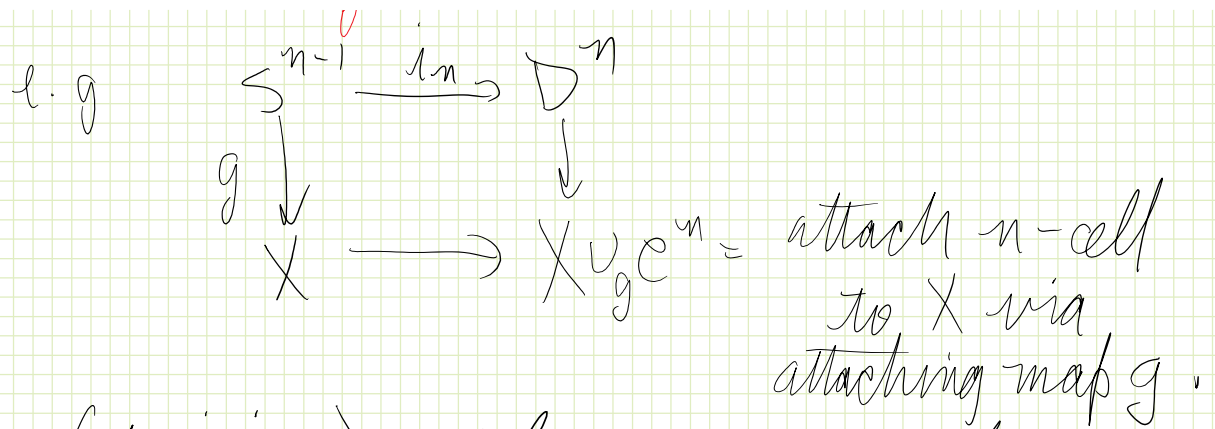
- ① composition (possibly transfinite)
- ② disjoint union (or one pt union)
- ③ pushouts



pushout symbol

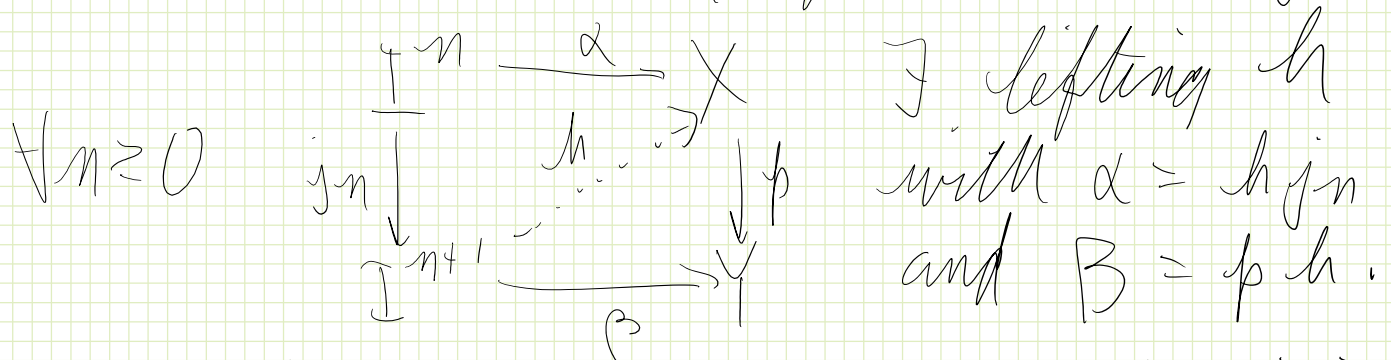
If f is a cofibration, so is f'

e.g. $S^{n-1} \xrightarrow{i_n} D^n$

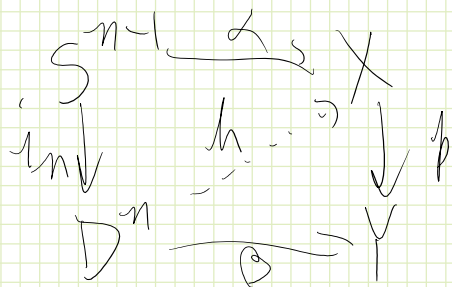


Any (trivial) cofibration can be derived from those in $\downarrow (f)$ by these operations. Similarly in the pointed case.

Def A map $X \xrightarrow{p} Y$ is a SERRE FIBRATION if for each diagram



\mathcal{A} is a TRIVIAL SERRE FIBN if



We say that \mathcal{A} and \mathcal{B} are COFIBRANT

We say that \mathcal{C} and \mathcal{F} are COFIBRANT
GENERATING SETS for the
model structure on Top .

This is a convenient way to
describe a model category.

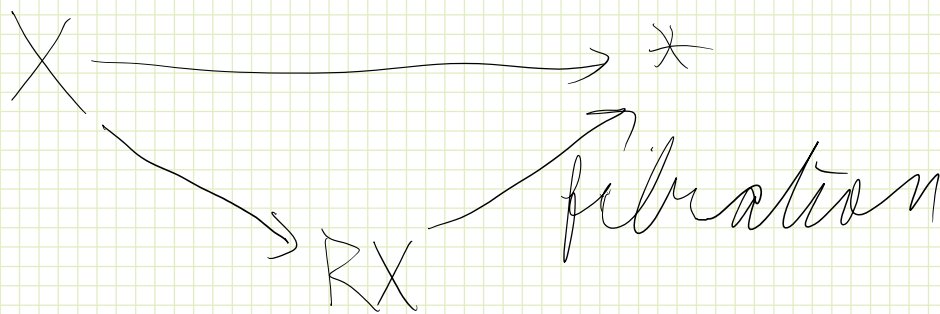
Def An object X is COFIBRANT if
the unique map $\phi \rightarrow \mathbb{I}$ is a
cofibration. It is FIBRANT if the
unique $X \rightarrow *$ is a fibration.

Example In Top every object is fibrant,
and cofibrant objects are CW-complexes.

In Ch_R (chain cxs of R -modules)
every object is fibrant, and
cofibrant objects are chain complexes
of projective R -modules.

Morphisms from cofibrant to
fibrant objects have the nicest
properties.

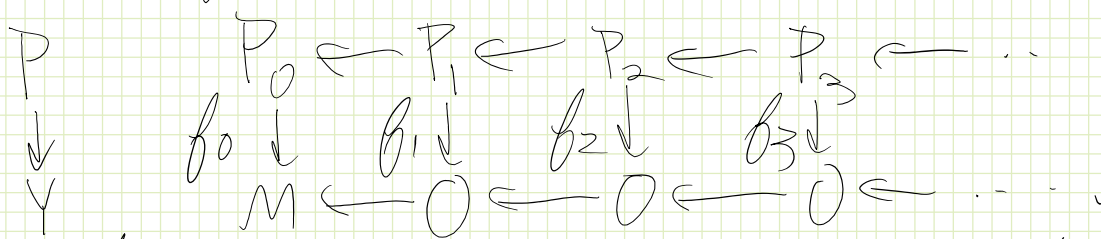
properties
 Consider the unique map
 ϕ $\begin{array}{ccc} \text{cofibr} & \rightarrow & QY \\ & \searrow & \text{trivial fib} \\ & & Y \end{array}$ as a way
 $Y = \text{any object } Y$
 QY is the COFIBRANT REPLACEMENT
 for Y



$RX = \text{FIBRANT REPLACEMENT}$
 for X .

In Ch_R let M be an R -module
 and Y is $M \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \dots$

Its cofibrant replacement is



where the map ϕ is a trivial fib.

i.e. f_0 is onto and $H_x P \xrightarrow[\cong]{} H_x Y$, i.e.

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$$

is exact, so it is a projective resolution of M .

New topic. Let \mathcal{M} be a cofibrantly generated model category (e.g. Top on \mathcal{J}) and let \mathcal{J} be a small category.

Let $\mathcal{M}^{\mathcal{J}}$ be the category of functors $\mathcal{J} \rightarrow \mathcal{M}$ (\mathcal{J} -shaped diagrams in \mathcal{M})

e.g. $G = \text{finite gp}$

$\mathcal{J}_G = \text{one object category with a morphism } \forall \gamma \in G.$

A functor $\mathcal{J}_G \rightarrow \text{Top}$ is a G -space X .

Notation: Given a functor $\mathcal{J} \xrightarrow{X} \mathcal{M}$

① $X \in \mathcal{M}^{\mathcal{J}}$

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{X} & \mathcal{M} \\ \downarrow j & \longrightarrow & \downarrow j \\ \mathcal{B} & \longrightarrow & X \end{array}$$

$$k \longrightarrow X_k$$

We want a model structure on the category M^J . What is a morphism in M^J ? \mathcal{A} is a NATURAL

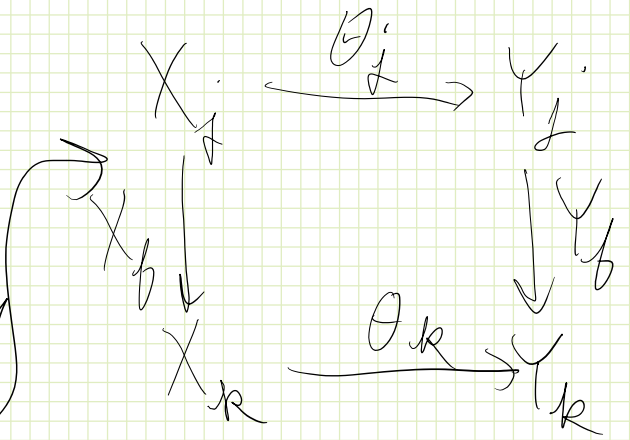
TRANSFORMATION $X \xrightarrow{\theta} Y$

i.e. for each $j \in J$ we have a morphism in

and for each morphism

$$j \xrightarrow{b} k \text{ in } J,$$

the following diagram commutes

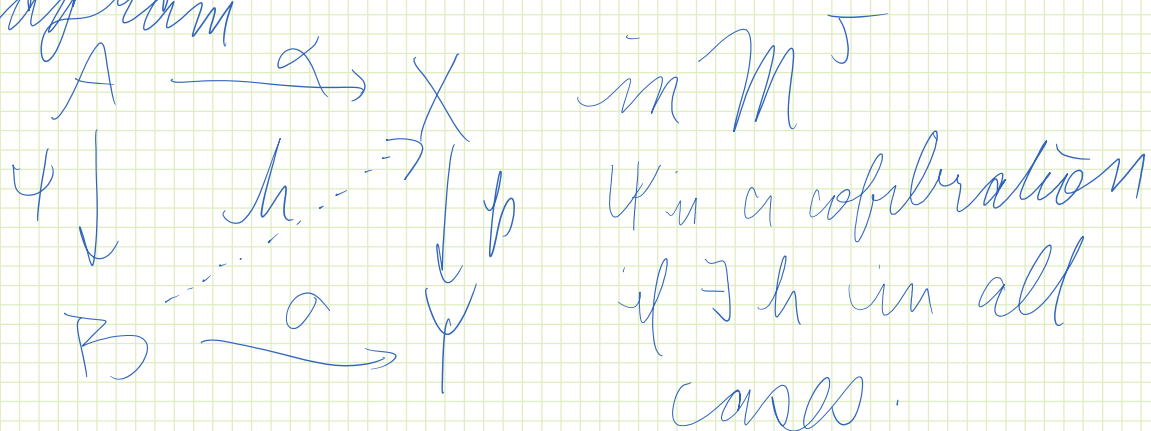


Def In the PROJECTIVE MODEL structure on M^J , a morphism $X \xrightarrow{\theta} Y$ is a weak fibration if

$X_j \xrightarrow{\theta_j} Y_j$ is one for each $j \in J$.

A cofibration is defined in terms of lifting properties;

i. e. a map $A \xrightarrow{\psi} B$ in \mathcal{M} is a cofibration if for any trivial fibration $X \xrightarrow{p} Y$ and any diagram



In the INJECTIVE model structure

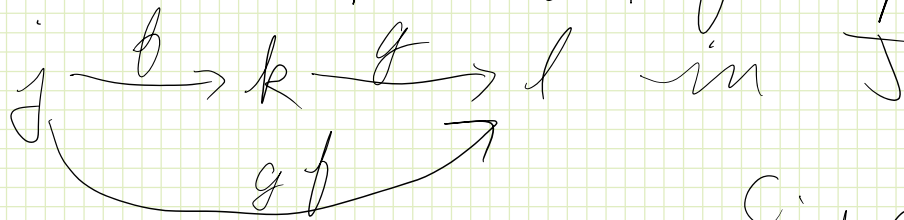
We can rephrase ① by saying there are STRUCTURE MAPS

$$J(j, k) \times \prod_j \xrightarrow{\sum_{j \rightarrow k} \epsilon_{j \rightarrow k}} X_k$$

↑
 set of morphisms
 $j \rightarrow k$ in J

It should play nicely with

composition of morphisms in \mathcal{J}



$$\mathcal{J}(k, l) \times \mathcal{J}(j, k) \xrightarrow{C_{j, k, l}} \mathcal{J}(j, l)$$

The following diagram should commute

$$\begin{array}{ccc}
 \mathcal{J}(k, l) \times \mathcal{J}(j, k) \times \prod_j \mathcal{I}_j & \xrightarrow{\mathcal{J}(k, l) \times \mathcal{E}_{j, k}^{\mathcal{I}}} & \mathcal{J}(k, l) \times \mathcal{I}_k \\
 \downarrow C_{j, k, l} \times \mathcal{I}_j & & \downarrow \mathcal{E}_{k, l}^{\mathcal{I}} \\
 \mathcal{J}(j, l) \times \prod_j \mathcal{I}_j & \xrightarrow{\mathcal{E}_{j, l}^{\mathcal{I}}} & \mathcal{I}_l
 \end{array}$$