Last time we saw that

$$H_* \Omega S^3 = \mathbb{Z}[x] \qquad x \in H_2$$

$$H^* \Omega S^3 = \Gamma[y] \qquad y \in H^2$$

Def An H-space (X, x_0) (H stands for Hopf) is one with a map $X \times X \xrightarrow{m} X$ satisfying

1)

$$\begin{array}{c} x \longmapsto (x_0, x) \\ X \rightrightarrows X \times X \xrightarrow{m} X \\ x \longmapsto (x, x_0) \end{array}$$

Both maps $X \longrightarrow X$ are homotopic to 1_X

If X were a topological group with $e = x_0$, then both maps would be 1_X.

2) Associativity

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times X} & X \times X \\ {\scriptstyle X \times m}\downarrow & & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow{m} & X \end{array}$$

commutes up to homotopy

If X is an H-space and k is a field we get

$$\begin{array}{ccc} H_*(X \times X; k) & \cong & H_*(X; k) \otimes H_*(X; k) \\ {\scriptstyle m_*}\downarrow & \swarrow & \\ H_*(X; k) & & \end{array}$$

PRODUCT

For any X the diagonal $X \xrightarrow{\Delta} X \times X$

induces a map $H_*(X;k) \xrightarrow{\Delta_*} H_*(X) \otimes H_*(X)$

COPRODUCT

In H^* we have the cup product

$$H^*(X) \otimes H^*(X) \xrightarrow{\Delta^*} H^*(X)$$

and (when X is an H-space)

$$H^*(X) \xrightarrow{m^*} H^*(X) \otimes H^*(X)$$

Note: m need not be commutative, even up to homotopy, i.e.

$$\begin{array}{ccc} (x_1, x_2) & \longmapsto & (x_2, x_1) \\ X \times X & \xrightarrow{t} & X \times X \\ & {\scriptstyle m} \searrow \quad \swarrow {\scriptstyle m} & \\ & X & \end{array}$$

need not commute, even up to homotopy

OTOH

$$\begin{array}{ccc} X \times X & \xrightarrow{t} & X \times X \\ & {\scriptstyle \Delta} \nwarrow \quad \nearrow {\scriptstyle \Delta} & \\ & X & \end{array}$$

always commutes

These two structures play nicely with each other i.e.

$X \times X \xrightarrow{m} X$ induces

$H^*(X) \otimes H^*(X) \xleftarrow{m^*} H^* X$ is a ring hom

i.e. an algebra map

Def A GRADED HOPF ALGEBRA over

a field k is a graded k-vector spaces with maps

$A \otimes A \longrightarrow A$ and $A \longrightarrow A \otimes A$

that play nicely with each other in the same way.

Thm S^2 is not an H-space, i.e. $\exists$ no map $S^2 \times S^2 \xrightarrow{m} S^2$ as above.

Proof Let $k = \mathbb{Q}$. Consider $H^*(m)$

Let x be a generator of $H^2(S^2)$

$m^*(x) \in H^2(S^2 \times S^2)$. It is $x \otimes 1 + 1 \otimes x$.

Note $m^*(x)^2 = (x \otimes 1 + 1 \otimes x)^2$

$$m^*(x)^2 = m^*(x^2) = 0$$

$$= x^2 \otimes 1 + 2x \otimes x + 1 \otimes x^2$$

$$= 2x \otimes x \in H^2(S^2) \otimes H^2(S^2)$$

$$\neq 0$$

CONTRADICTION derived from assuming m exists. QED

If we replace S^2 (or S^{2m}) by S^1 (or S^{2m-1}) the binomial expansions is different

$$(x\otimes 1 + 1\otimes x)^2 = (x\otimes 1)(x\otimes 1) + (x\otimes 1)(1\otimes x) + (1\otimes x)(x\otimes 1) + (1\otimes x)(1\otimes x)$$

$$= x^2\otimes 1 + \qquad 0 \qquad + 1\otimes x^2$$

$$= 0 \quad \text{NO CONTRADICTION.}$$

Recall $H_*(\Omega S^3; k) = k[x] \quad x \in H_2$

$$H^*(\Omega S^3; k) = \Gamma(x)$$

The diagonal map $\Omega S^3 \longrightarrow \Omega S^3 \times \Omega S^3$ induces $H_*(\Omega S^3) \longrightarrow H_*(\Omega S^3) \otimes H_*(\Omega S^3)$

$$H^*(\Omega S^3) \longleftarrow H^*\Omega S^3 \otimes H^*(\Omega S^3)$$

$$x \longmapsto x\otimes 1 + 1\otimes x$$

$$x^n \longmapsto (x\otimes 1 + 1\otimes x)^n$$

$$x^{i+j} \longmapsto \binom{i+j}{i} x^i \otimes x^j + \cdots$$

This means that if $y_i \in H^{2i}\Omega S^3$ is the generator, i.e dual to x^i, then

$$y_i y_j = \binom{i+j}{i} y_{i+j} \in H^*(\Omega S^3)$$

This is a divided power algebra.

If we consider $H^*(\Omega S^3 ; \mathbb{Z}/2)$.

Binomial coefficient exercise:

For $i>0$ $\quad y_1^2 = \binom{2i}{i} y_{2i} \equiv 0$

$$y_{2i}\, y_1 = \binom{2i+1}{1} y_{2i+1} \neq 0$$

$$y_{4i}\, y_2 = \binom{4i+2}{2} y_{4i+2} \neq 0$$

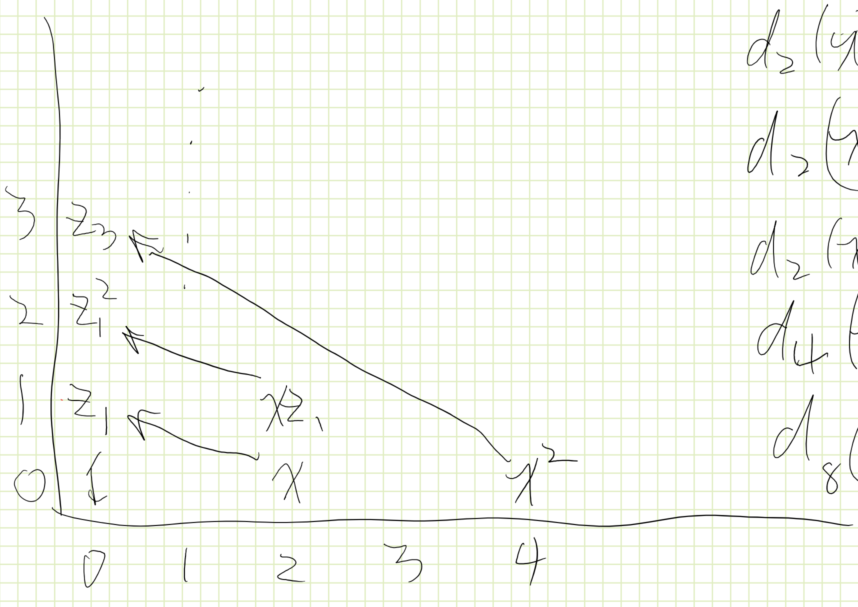
etc

$$\rightsquigarrow \Gamma(y_1) = E(y_1, y_2, y_4, y_8, \dots)$$

Consider the path fibration

$$\Omega^2 S^3 \longrightarrow * \longrightarrow \Omega S^3$$

Will use the Serre SS in $H_*(\ \ ; \mathbb{Z}/2)$



$d_2(x) = z_1$

$d_2(x z_1^n) = z_1^{n+1}$

$d_2(x^2) = 2 x z_1 = 0$

$d_4(x^2) = z_3$

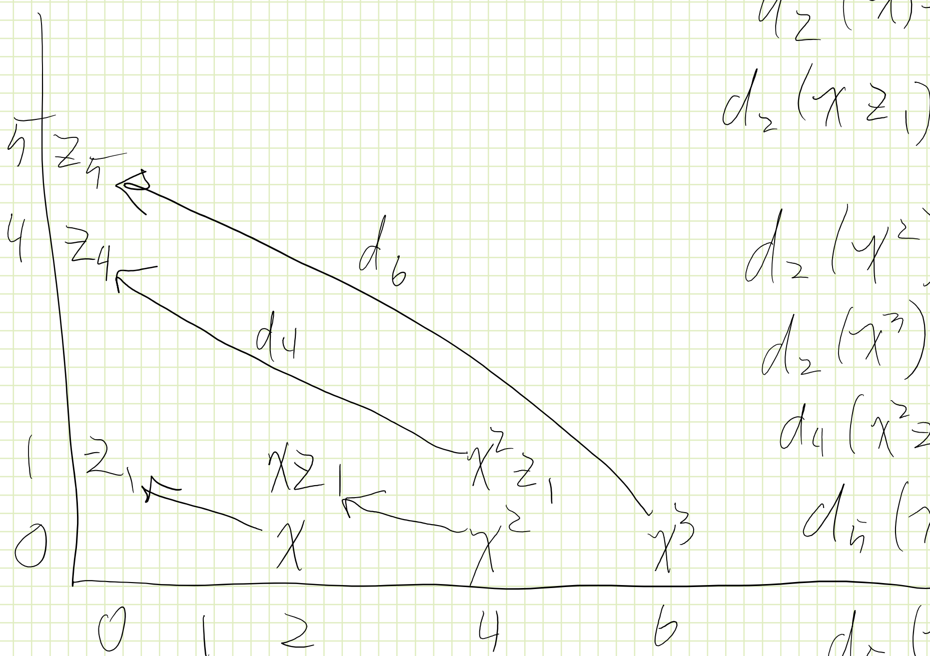
$d_8(x^4) = z_7$

⋮

$H_*(\Omega^2 S^3) = \mathbb{Z}/2[z_1, z_3, z_7, z_{15}, \dots]$ $\quad z_i \in H_i$

There is a Borel-like theorem here.

Now do this mod 3.



$d_2(x) = z_1$

$d_2(x z_1) = z_1^2 = -z_1^2 = 0$

$d_2(x^2) = 2 x z_1 \neq 0$

$d_2(x^3) = 3x^2 z_1 = 0$

$d_4(x^2 z_1) = z_4$

$d_6(x^3) = z_5$

$d_6(x^9) = 3x^6 z_5 = 0$

so $d_{18}(x^9) = z_{17}$

and $d_{12}(x^6 z_5) = z_{16}$

$\rightsquigarrow H_*(\Omega^2 S^3; \mathbb{Z}/3)$

$= E(z_1, z_5, z_{17}, \dots z_{2\cdot 3^i - 1} \dots)$

$\otimes \mathbb{Z}/3[z_4, z_{16}, \dots z_{2\cdot 3^i - 2} \dots]$