Thursday, March 29, 2018 1:04 PM

Spectra

Def A SPECTRUM X is a collection of pointed spaces X_n for $n \geq 0$ and maps $\Sigma X_n \xrightarrow{\varepsilon_n^X} X_{n+1}$ ← STRUCTURE MAP

A map of spectra $f: X \to Y$ is a collection of pointed maps $f_n: X_n \to Y_n$ such that the following diagram commutes for $n \geq 0$.

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\varepsilon_n^X"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\varepsilon_n^Y"] & Y_{n+1} \end{tikzcd}$$

Here $\Sigma X_n = S^1 \wedge X_n$ (smash product) where for pointed space (A, a_0) and (B, b_0),

$$A \wedge B = A \times B / A \times b_0 \cup a_0 \times B$$

e.g. $S^m \wedge S^n \approx S^{m+n}$

Examples of spectra

① Let X be a pointed space and define its SUSPENSION SPECTRUM $\Sigma^\infty X$ by

$$(\Sigma^\infty X)_n = \Sigma^n X = S^n \wedge X$$

and

$$\Sigma(\Sigma^n X) \xrightarrow[\approx]{\varepsilon_n^{\Sigma^\infty X}} \Sigma^{n+1} X$$

e.g. $X = S^0$ and $(\Sigma^\infty S^0)_n = S^n$ is the SPHERE SPECTRUM

② Let A be an abelian group. The EILENBERG-MAC LANE spectrum for A, HA is defined by $(HA)_n = K(A, n)$. There is a

map $\Sigma K(A,n) \xrightarrow{\varepsilon_n^{HA}} K(A,n+1)$
given by the fact that

$$[X, K(A,n+1)] = H^{n+1}(X;A)$$

$$H^n(K(A,n);A) = \mathrm{Hom}(A,A)$$

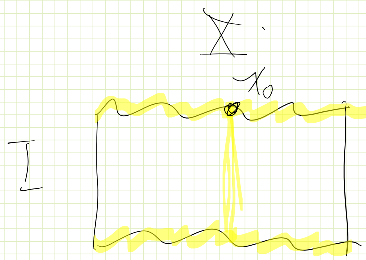
$$\cong$$

$$H^{n+1}(\Sigma K(A,n);A)$$

ε_n^{HA} corresponds to 1_A.

Prop There is a natural bijection between pointed maps $\Sigma X \xrightarrow{f} Y$ and map $X \xrightarrow{\hat{f}} \Omega Y$.

Proof Given a map $f: \Sigma X \longrightarrow Y$.
Consider $\Sigma X = I \times X$ / yellow



$\hat{f}(x)$ = closed path in Y
$\hat{f}(x)(t) = f(x,t) \in Y$ for $t \in I$ QED

We say $\hat{f}$ is the RIGHT adjoint of f,
f " LEFT " of $\hat{f}$.

In a spectrum $\underline{X}$ we have a structure maps $\varepsilon_n^X : \Sigma X_n \longrightarrow X_{n+1}$ $\forall n \geq 0$
Its right adjoint $X_n \xrightarrow{\eta_n^X} \Omega X_{n+1}$ is the nth COSTRUCTURE map of $\underline{X}$.
In the spectrum HA above, each costructure map is a htpy equivalence

Def A spectrum E is an Ω-SPECTRUM if each costructure map η^E_n is a htpy equivalence.

HA is an Ω-spectrum, $\Sigma^\infty X$ is not one.

Def Let E be a spectrum

$$H_k(E) = \varinjlim_n H_{n+k}\, E_n \qquad \text{and} \qquad \pi_k(E) = \varinjlim_n \pi_{n+k}\, E_n$$

where $H_i X = \pi_i X_n$ $= 0$ for $i < 0$

The map $\Sigma^X_n : \Sigma E_n \longrightarrow E_{n+1}$ induces

$_{n+k}E_n \longrightarrow$

and $\eta^X_n : E_n \longrightarrow \Omega E_{n+1}$

induces $\pi_{n+k} E_n \longrightarrow \pi_{n+k}\Omega E_{n+1} \cong \pi_{n+k+1} E_{n+1}$

Def A map of spectra $f : X \longrightarrow Y$ is a STABLE EQUIVALENCE if $\pi_*(f)$ is an isomorphism.

Remark In a spectrum X, X_n need NOT be $(n-1)$-connected

$\pi_k X$ is defined for all integers k. and $H_k X$

Example of an interesting map

Consider the Hopf map $S^3 \xrightarrow{\eta} S^2$

It is known that $S^{3+m} \xrightarrow{\Sigma^m \eta} S^{2+m}$ is essential for all $m \geq 0$.

We would like to have a map of spectra $\Sigma^\infty S^1 \xrightarrow{\eta} \Sigma^\infty S^0$ whose nth component is

$$S^{n+1} \xrightarrow{\Sigma^{n-2}\eta} S^n$$

but what about $n = 0$ and 1?

WHAT TO DO ?

Replace $\Sigma^\infty S^0$ by a spectrum Y with

$$Y_n = \begin{cases} S^n & \text{for } n \geq 2 \\ * & \text{for } n = 0, 1 \end{cases}$$

$$X_n = \begin{cases} S^{n+1} & \text{for } n \geq 2 \\ * & \text{for } n = 0, 1 \end{cases}$$

There is a map $X \longrightarrow Y$ whose n th component is $\Sigma^{n-2}\eta$ for $n \geq 2$. Note

$$\begin{array}{ccccccc} * & = & \Sigma X_1 & \xrightarrow{\varepsilon_1^X} & X_2 & = & S^3 \\ & & {\scriptstyle \Sigma f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle \eta} \\ * & = & \Sigma Y_1 & \xrightarrow{\varepsilon_1^Y} & Y_2 & = & S^2 \end{array}$$

Note there are maps $X \xrightarrow{\alpha} \Sigma^\infty S^1$ and $Y \xrightarrow{\beta} \Sigma^\infty S^0$

where
$$\alpha_n = \begin{cases} * \longrightarrow S^{n+1} & \text{for } n = 0, 1 \\ S^{n+1} \longrightarrow S^{n+1} & \text{for } n \geq 2 \end{cases}$$

$$\beta_n = \begin{cases} * \longrightarrow S^n & \text{for } n = 0, 1 \\ S^n \longrightarrow S^n & n \geq 2 \end{cases}$$

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \Sigma^\infty S^1 \\ {\scriptstyle f}\big\downarrow & \nexists & \vdots \end{array}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \Sigma S^{\cdot} \arrow[d, dashed] \\ Y \arrow[r, "\beta"] & \Sigma^\infty S^0 \end{tikzcd}$$

$\nexists$

CLAIM α and β are both stable equivalences.

Let X be a spectrum. Consider the maps

$$X_n \xrightarrow{\eta_n^X} \Omega X_{n+1} \xrightarrow{\Omega \eta_{n+1}^X} \Omega^2 X_{n+2} \longrightarrow \cdots\cdots$$

Let $\tilde{X}_n$ be the MAPPING TELESCOPE of the above sequence of maps

Def The MAPPING TELESCOPE of a sequence of pointed maps

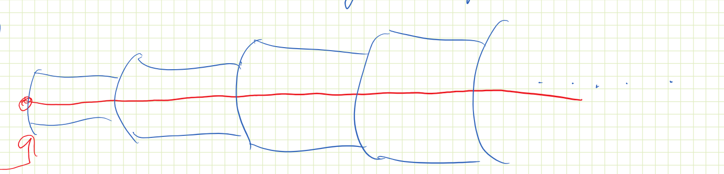
$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \longrightarrow \cdots$$

$$TA = \coprod_{n \geq 0} A_n \times I \Big/ (x_n, 1) \sim (a_n(x_n), 0)$$

for $x_n \in A_n$

also shrink basepoint line to a single point.

$\tilde{X}_n$



X_n

Then $\tilde{X}_n$ is the nth component of a spectrum $\tilde{X}$ with

1) There is a structure map $\Sigma \tilde{X}_n \xrightarrow{\Sigma_n^{\tilde{X}}} \tilde{X}_{n+1}$ (exercise)

2) The structure map $\tilde{X}_n \xrightarrow{\eta_n^{\tilde{X}}} \Omega \tilde{X}_{n+1}$

is a htpy equivalence, so

$\tilde{X}$ is an Ω-spectrum

3) There is a map $X \longrightarrow \tilde{X}$ which is a stable equivalence.

Hence every spectrum X is stably equivalent to an Ω-spectrum $\tilde{X}$.

Thm A stable equivalence $X \xrightarrow{f} Y$ induces a map

$$\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y}$$

such that $\tilde{f}_n : \tilde{X}_n \longrightarrow \tilde{Y}_n$ is an equivalence of pointed spaces for each $n \geq 0$.

Example Let X be $\Sigma^\infty S^0$, so $X_n = S^n$. Then

$$\tilde{X}_n = T(S^n \longrightarrow \Omega S^{n+1} \longrightarrow \Omega^2 S^{n+2} \longrightarrow \cdots)$$