Recall we have a chain complex K with subcomplexes

$$0 = K^{-1} \subset K^0 \subset K^1 \subset \cdots \subset K^p \subset \cdots K$$

Assumptions

① $K^p = 0$ for $p < 0$

② $H_{p+q}(K^p/K^{p-1}) = 0$ for $q < 0$

③ $\bigcup K^p = K$

Def $F_{p,q} = \operatorname{im} H_{p+q}(K^p) \longrightarrow H_{p+q}(K)$

$E^\infty_{p,q} = F_{p,q} / F_{p-1,q+1}$

Note

$$F_{0,p+q} \subset F_{1,p+q-1} \subset F_{2,p+q-2} \cdots \cdots \subset F_{p+q,0} = H_{p+q}K$$

Knowing each $E^\infty_{p,q}$ does not determine $H_{p+q}K$ uniquely. e.g. $p+q=1$

$F_{0,1} \subset F_{1,0} = H_1 K$ so $F_{0,1} = E^\infty_{0,1}$ and $E^\infty_{1,0} = F_{1,0}/F_{0,1}$ and there is a SES

$$0 \longrightarrow E^\infty_{0,1} \longrightarrow H_1 K \longrightarrow E^\infty_{1,0} \longrightarrow 0$$

Finding the value of $H_1 K$ is called SOLVING THE EXTENSION PROBLEM.
It requires some extra information.

Suppose $H_* K$ has a multiplication

and that the subcomplexes K^p are chosen so that if

$x \in F_{p,q} = \operatorname{im} H_{p+q} K^p \subset H_{p+q} K$

$y \in F_{p',q'} = \operatorname{im} H_{p'+q'} K^{p'} \subset H_{p'+q'} K$

then $xy \in H_{p+p'+q+q'} K$ is in the image of $H_* K^{p+p'}$.

Suppose $x \notin F_{p-1,q+1}$ so it has nontrivial image $\bar{x}$ in $E^\infty_{p,q}$ and similarly for y, and that $xy \neq 0 \in H_* K$. There is a map $E^\infty_{p,q} \otimes E^\infty_{p',q'} \longrightarrow E^\infty_{p+p',q+q'}$ induced by multiplication, but it could happen that $\bar{x}\bar{y} = 0$, i.e. xy could be in the image of $H_* K^s$ for $s < p+p'$. This is the MULTIPLICATIVE EXTENSIO PROBLEM. Solving it may require additional information. Knowledge of E^∞ is only partial information.

Def $F_{p,q} = \operatorname{im} \; H_{p+q}(K^p) \longrightarrow H_{p+q}(K)$

$$E^\infty_{p,q} = F_{p,q} / F_{p-1,q+1}$$

Prop 4 (MT p 60 something)

$$E^\infty_{p,q} = \frac{\operatorname{im}\left(H_{p+q} K^p \longrightarrow H_{p+q}(K^p/K^{p-1})\right)}{\operatorname{im}\left(H_{p+q+1}(K/K^p) \xrightarrow{\partial} H_{p+q}(K^p/K^{p-1})\right)}$$

where ∂ is the connecting hom for

$$0 \longrightarrow K^p/K^{p-1} \longrightarrow K/K^{p-1} \longrightarrow K/K^p \longrightarrow 0$$

Recall our assumptions

① $K^p = 0$ for $p < 0$

② $H_{p+q}(K^p/K^{p-1}) = 0$ for $q < 0$

③ $\bigcup K^p = K$

They hold in the Serre SS.

Prop 5 If ① and ② hold then $E^r_{p,q} = E^{r+1}_{p,q}$ if $r > \max(p, q+1)$

Thm I (MT page 67) If ①, ② and ③ hold then $E^\infty_{p,q}$ (as defined above) is the same as $E^r_{p,q}$ (in the SS) for $r \gg 0$.

Some other spectral sequences.

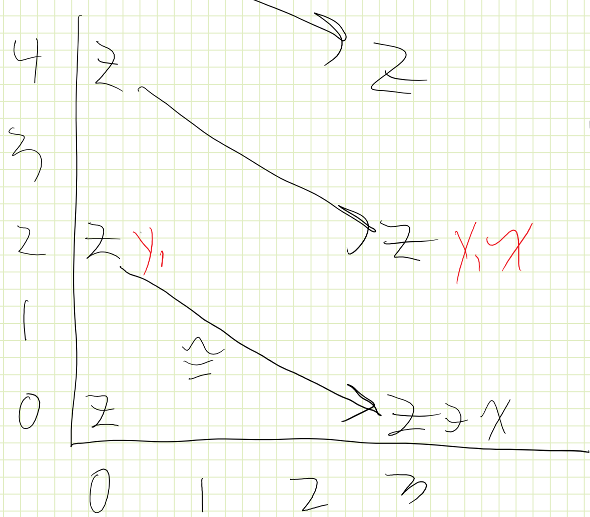
Consider $\Omega S^n \to PS^n \to S^n$

PS^n = path space

The Serre SS in H^* has

$$E^2_{p,q} = H^p(S^n; H^q \Omega S^n) = \begin{cases} H^q \Omega S^n & \text{for } p = 0 \text{ or } n \\ 0 & \text{else} \end{cases}$$

$n = 3$



Conclusion:

$$H^i \Omega S^3 = \begin{cases} \mathbb{Z} & \text{for } i \text{ even} \\ 0 & \text{for } i \text{ odd} \end{cases}$$

Denote a generator of $H^{2i} \Omega S^3$ by y_i

What about cup products in $H^* \Omega S^3$?

$d_3 y_1 = x$ so $d_3 y_1^2 = 2 y_1 x$

but $d_3 y_2 = y_1 x$ so $y_1^2 = 2 y_2$

$d_3 y_1^3 = 3 y_1^2 x = 6 y_2 x$, but $d_3(y_3) = y_2 x$

so $y_1^3 = 6 y_3$

Similarly $y_1^n = n! \, y_n$ and $y_m y_n = \binom{n+m}{n} y_{m+n}$.

$$H^* \Omega S^3 = \mathbb{Z}[y_1, y_2, y_3 \dots] \Big/ \left(y_m y_n - \binom{m+n}{n} y_{m+n} \right)$$

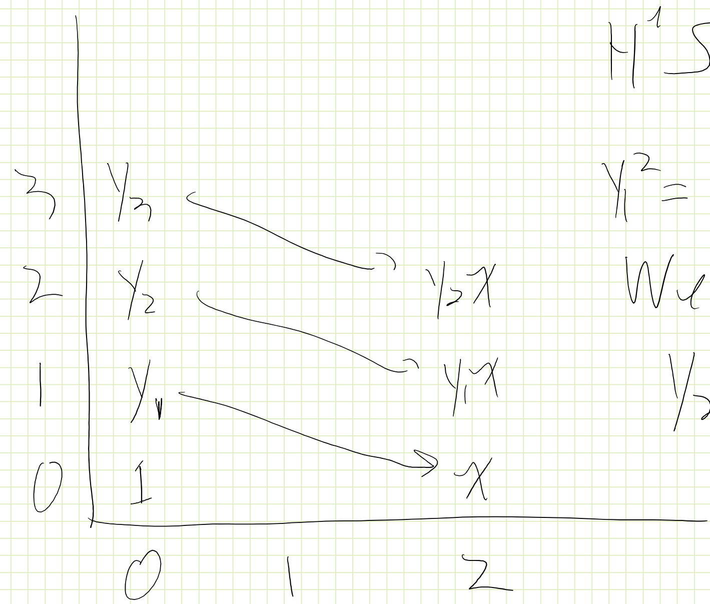
$$=: \Gamma(y_1) = \text{DIVIDED POWER ALGEBRA on } y_1$$

$$y_i \in H^{2i}(\Omega S^3)$$

$$H^*(\Omega S^{2m+1}) = \Gamma(y) \quad \text{with } y \in H^{2m}$$

$$\Omega S^n \longrightarrow * \longrightarrow S^n$$

$n = 2$



$H^i \Omega S^2 = \mathbb{Z}$ for each $i \geq 0$

$y_1^2 = -y_1^2$, so $y_1^2 = 0$

We find

$$y_{2i+1} = y_1 \, y_{2i}$$

$$y_{2i} \, y_{2j} = \binom{i+j}{j} y_{2(i+j)}$$

$$y_2^n = n! \; y_{2n}$$

<u>Thm</u> $H^* \Omega S^{2m} = E(y_1) \otimes \Gamma(y_2)$ where $y_1 \in H^{2m-1}$ and $y_2 \in H^{4m-2}$

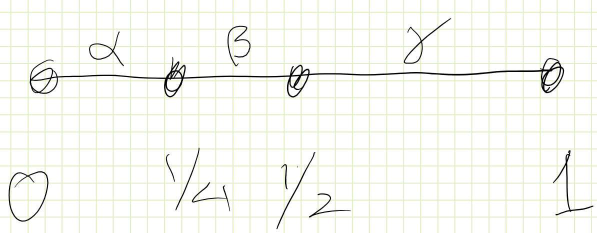
$\Lambda(-) = E(-)$ stands for EXTERIOR ALGEBRA

(or GRASSMANN ALGEBRA)
on a set of generators $\{x_\alpha\}$
with $x_\alpha x_\beta = - x_\beta x_\alpha$, e.g. $x_\alpha^2 = 0$

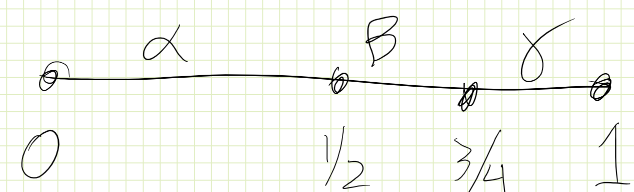
We also have a map $\Omega X \times \Omega X \longrightarrow \Omega X$ induced by CONCATENATION of closed paths in X.

Recall if α, β, γ are closed paths in X, i.e. maps $[0,1] \longrightarrow X$ sending $0, 1$ to $x_0 \in X$ (basepoint). Then

$(\alpha * \beta) * \gamma$ is



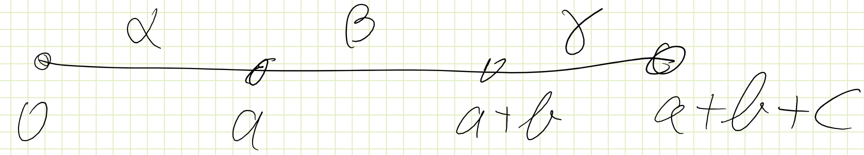
$\alpha * (\beta * \gamma)$ is



New definition: Consider closed path of all positive finite lengths

Suppose α, β and γ have length a, b and c. Then

$(\alpha * \beta) * \gamma$



$\alpha * (\beta * \gamma)$ same

We get an associative multiplication on the space ΩX. The map $\Omega X \times \Omega X \longrightarrow \Omega X$ induces a multiplication in H_*.

Consider the Serre SS IN HOMOLOGY for $\Omega S^n \longrightarrow * \longrightarrow S^n$

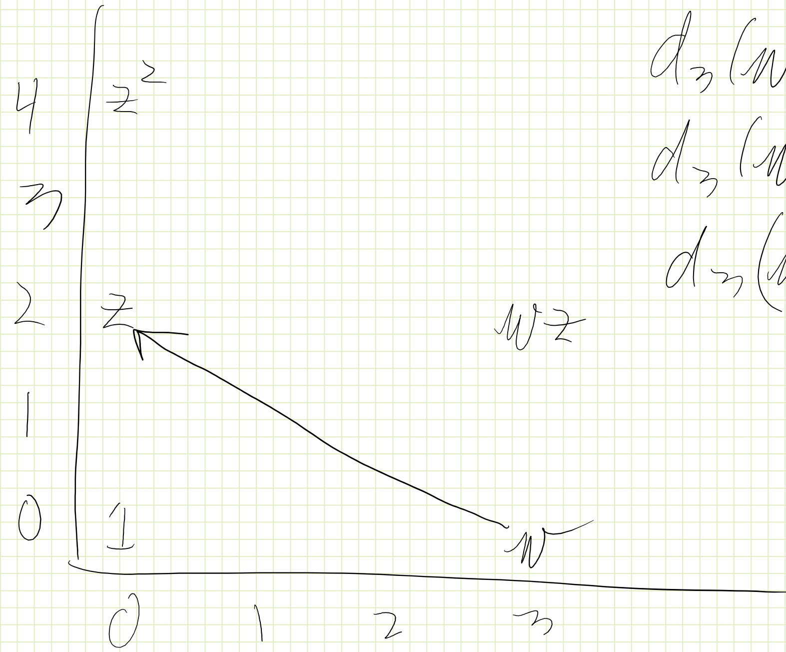
In general for $F \longrightarrow E \longrightarrow B$,

$E^2_{p,q} = H_p(B; H_q F)$ with differentials

$$E^n_{p,q} \longrightarrow E^n_{p-n,\, q+n-1}$$

$$\Omega S^n \longrightarrow * \longrightarrow S^n$$

$n = 3$



$d_3(w) = z$

$d_3(wz) = z^2$

$d_3(wz^k) = z^{k+1}$

$H_* \Omega S^3 = \mathbb{Z}[y] \qquad y \in H_2 \Omega S^3$

$H_* \Omega S^{2m+1} = \mathbb{Z}[y] \quad \text{for } y \in H_{2m} \Omega S^{2m+1}$