MOVING TOWARD THE ADAMS SS.

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If we have a fiber sequence

$$F \xrightarrow{i} E \xrightarrow{p} B$$

where all spaces are (roughly) n-connected, then dimension $\sim 2n$ there is a LES

$$\cdots \longrightarrow H^i B \xrightarrow{p^*} H^i E \xrightarrow{i^*} H^i F \xrightarrow{\delta} H^{i+1} B \longrightarrow \cdots$$

<u>Suppose p^* is onto in our range of dimensions</u>

Hence i^* is 0 and δ is one-to-one. Hence we have a short exact sequence

(1) $$0 \longrightarrow H^{i-1} F \xrightarrow{\delta} H^i B \xrightarrow{p^*} H^i E \longrightarrow 0$$

<u>Example</u> The first step in Serre's program to find $\pi_{n+i} S^n$ for small i is to study the fiber sequence

$$F \xrightarrow{i} S^n \xrightarrow{p} K(\mathbb{Z}, n)$$

$$0 \longleftarrow H^n S^n \xleftarrow[\cong]{p^*} H^n K(\mathbb{Z}, n) \longleftarrow H^{n-1} F \longleftarrow \cdots$$

we get a short exact sequence

$$0 \longrightarrow H^{n+i-1} F \xrightarrow{\delta} H^{n+i} K(\mathbb{Z}, n) \xrightarrow{p^*} H^{n+i} S^n \longrightarrow 0$$

for $0 \le i < n-1$

We learn that $\pi_{n+1} S^n = \pi_{n+1} F = \mathbb{Z}/2$ (for $n > 2$)

Serre would look at a map

$$F' \xrightarrow{i} F \xrightarrow{p} K(\mathbb{Z}/2, n+1)$$

However p^* is not onto, so we will not get the SES of (1).

Adams: Replace $K(\mathbb{Z}/2, n+1)$ by another space L with

i) L is n-connected and $\pi_{n+2} L = \mathbb{Z}/2$

ii) $H^* L$ is known

iii) p^* is onto in our range

Lemma For any $(m-1)$-connected space there is a map $X \xrightarrow{p} L$ with L is also $(m-1)$-connected, $H^* L$ is known and p^* is onto below dim $2m$.

Consider the map $S^n \to K(\mathbb{Z}, n)$

$$H^* K(\mathbb{Z}, n) = \mathbb{Z}/2 \left[Sq^I x_n : \text{for certain } I \right]$$

Hence in our range, below dim $2n$ it is

$$\mathbb{Z}/2 \left\{ Sq^I x_n : \sim\!\sim \right\}$$

The $I = (i_1, i_2 \cdots i_\ell)$ must satisfy $i_\ell > 1$, $e(I) < n$. Each Sq^I with dimension $< n$ has excess $< n$.

(EASY EXERCIZE).

$H^* K(\mathbb{Z},n)$ is a certain cyclic A-module

The Steenrod algebra A is a VERY complicated noncommutative $\mathbb{Z}/2$-algebra.

Recall it is generated by Sq^i for $i>0$ subject to the ADEM RELATIONS.

$$Sq^a Sq^b = \sum_{0 \le i \le a/2} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i \quad \text{for } a<2b$$

$$Sq^1 Sq^{2n} = Sq^{2n+1} \quad \text{for } n \geq 0$$

$$Sq^2 Sq^{4n} = \sum \binom{4n-1-i}{2-2i} Sq^{4n+2-i} Sq^i$$

$$= \binom{4n-1}{2} Sq^{4n+2} + \binom{4n-2}{0} Sq^{4n+1} Sq^1$$

LUCAS ~~Wilson's~~ Theorem $\binom{n}{k}$

Let $n = \sum c_i 2^i$ for $c_i = 0,1$

$k = \sum b_i 2^i$ $\quad b_i = 0,1$

Then $\binom{n}{k} \equiv \prod_i \binom{c_i}{b_i} \mod 2$

$$Sq^2 Sq^{4n} = Sq^{4n+2} + Sq^{4n+1} Sq^1$$

$$= Sq^{4n+2} + Sq^1 Sq^{4n} Sq^1$$

$$Sq^{4n+2} = Sq^2 Sq^{4n} + Sq^1 Sq^{4n} Sq^1 \quad \text{for } n \geq 0$$

We learn that A is generated as $\mathbb{Z}/2$-algebra by $\{Sq^{2^j} : j \geq 0\}$

Why is Serre's map $F \xrightarrow{p} K(\mathbb{Z}/2, n+1)$ NOT onto in H^*?

We have a SES

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{n+i-1} F & \longrightarrow & H^{n+i} K(\mathbb{Z}, n) & \longrightarrow & H^{n+i} S^n & \longrightarrow & 0 \\
 & & & & x_n & \longrightarrow & x_n & & \\
 & & y = y_{n+1} = \text{bottom class} & \longrightarrow & Sq^2 x_n & \longrightarrow & 0 & & \\
 & & Sq^1 y_{n+1} & \longrightarrow & Sq^3 x_n & \longrightarrow & 0 & & \\
 & & y_{n+3} & \longrightarrow & Sq^4 x_n & \longrightarrow & 0 & & \\
 & & Sq^1 y_{n+3} & \longrightarrow & Sq^5 x_n & \longrightarrow & 0 & & \\
 & & & \longrightarrow & Sq^6 x_n,\ Sq^4 Sq^2 x_n & \longrightarrow & 0 & &
\end{array}$$

$\{y_{n+2^j-1} : j \geq 0\}$

y_{n+3} is not in the image of p^* for Serre's map $F \longrightarrow K(\mathbb{Z}/2, n+1)$.

How to fix this (ADAMS). Choose a set $\{z_1, z_2, \dots\}$ in H^*F that generate set $\{z_1, z_2, \dots\}$ in H^*F that generate it as an A-module. Each determines a map to ~~some~~ $K(\mathbb{Z}/2, ?)$, so collectively they give a map p to $\prod K(\mathbb{Z}/2, ?)$. Since the set $\{z_\alpha\}$ generates H^*F as an A-module, p^* is onto.

<u>Lemma</u> For any n-connected space X there is a map $p: X \longrightarrow L$ where L is a product of $K(\mathbb{Z}/2, i)$s and p^* is onto.

Proof as above.

$$\begin{tikzcd}
X_0 = S^n \arrow[r, "p_0"] & K(\mathbb{Z}, n) = L_0 \\
X_1 = \text{fiber} \arrow[u] \arrow[r, "p_1"] & L_1 = \text{product of } K(\mathbb{Z}/2, -) \\
X_2 \arrow[u] \arrow[r, "p_2"] & L_2 = \text{"} \\
X_3 \arrow[u] & \cdots
\end{tikzcd}$$

Each p_i induces surjection in H^*

This is an ADAMS RESOLUTION

We get a collection of SES of A-modules

DIMENSION SHIFT

$$0 \leftarrow H^*X_0 \leftarrow H^*L_0 \xleftarrow{\circ} H^*X_1 \leftarrow 0$$

$$0 \leftarrow H^*X_1 \leftarrow H^*L_1 \xleftarrow{\circ} H^*X_2 \leftarrow 0$$

$$0 \leftarrow H^*X_s \leftarrow H^*L_s \xleftarrow{\circ} H^*X_{s+1} \leftarrow 0$$

to be continued.