$$
\begin{tikzcd}[column sep=small]
 & & & \cdots \arrow[r] & H_{p+q-1}(K^{p-r}) \arrow[r, "j"] \arrow[d, "i"] & H_{p+q-1}(K^{p-r}/K^{p-r-1}) \\
 & & & \cdots \arrow[r] & H_{p+q-1}(K^{p-r+1}) \arrow[r, "j"] \arrow[dd, squiggly, "i^{(r-3)}"] & H_{p+q-1}(K^{p-r+1}/K^{p-r}) \\
 & & {} \arrow[d] & & & \cdots\cdots\cdots \\
 {} \arrow[r] & H_{p+q+1}(K^p/K^{p-1}) \arrow[r, "k_{p,q+1}"] & H_{p+q}(K^{p-1}) \arrow[r, "j"] \arrow[d, "i_{p-1,q+1}"] & H_{p+q}(K^{p-1}/K^{p-2}) \arrow[r, "k"] & H_{p+q-1}(K^{p-2}) \arrow[r, "j"] \arrow[d, "i"] & H_{p+q-1}(K^{p-2}/K^{p-3}) \\
 {} \arrow[r] & \begin{array}{c} H_{p+q+1}(K^{p+1}/K^p) \\ (= E_{p+1,q}) \end{array} \arrow[r, "k"] & \begin{array}{c} H_{p+q}(K^p) \\ (= D_{p,q}) \end{array} \arrow[r, "j_{p,q}"] \arrow[d, "i_{p,q}"] & \begin{array}{c} H_{p+q}(K^p/K^{p-1}) \\ (= E_{p,q}) \end{array} \arrow[r, "k_{p,q}"] & \begin{array}{c} H_{p+q-1}(K^{p-1}) \\ (= D_{p-1,q}) \end{array} \arrow[r, "j"] \arrow[d, "i_{p-1,q}"] & \begin{array}{c} H_{p+q-1}(K^{p-1}/K^{p-2}) \\ (= E_{p-1,q}) \end{array} \\
 {} \arrow[r] & H_{p+q+1}(K^{p+2}/K^{p+1}) \arrow[r, "k"] & H_{p+q}(K^{p+1}) \arrow[r, "j"] \arrow[dd, squiggly, "i^{(r-3)}"] & H_{p+q}(K^{p+1}/K^p) \arrow[r, "k"] & H_{p+q-1}(K^p) \arrow[r, "j"] \arrow[d, "i"] & H_{p+q-1}(K^p/K^{p-1}) \\
 & \cdots\cdots\cdots & & & {} & \\
 {} \arrow[r] & H_{p+q-1}(K^{p+r-1}/K^{p+r-2}) \arrow[r, "k"] & H_{p+q}(K^{p+r-2}) \arrow[r] & \cdots & &
\end{tikzcd}
$$

Figure 1

Exact couples and spectral sequences

Tuesday, February 27, 2018 10:30 AM

Sources: Mosher + Tangora Ch 7.
Hilton + Stamback Ch. $\overline{\underline{\text{VIII}}}$
Green book §2.1, also gives other references.

Let $\mathcal{C}$ be an ABELIAN CATEGORY, i.e. one in which exactness makes sense, e.g. R-modules, graded R-modules, chain complexes.

Def A DIFFERENTIAL OBJECT (E,d) in $\mathcal{C}$ is an object E with a morphism $E \xrightarrow{d} E$ s.t. $d \circ d = 0$. e.g. a chain complex where $\mathcal{C}$ = category of graded abelian groups.

Then $Z(E)$ = kernel of d = cycles
$B(E)$ = image of d = boundaries
$H(E) = Z(E)/B(E)$ = homology

Def An EXACT COUPLE in $\mathcal{C}$ is a diagram

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & E & \end{array}$$

which is exact

(1) $(D,E,i,j,k) =: (D,E)$

i.e. $\ker i = \operatorname{im} k$,
$\ker j = \operatorname{im} i$,
$\ker k = \operatorname{im} j$

Then we have

$$\begin{array}{ccccccc} & & D & \xrightarrow{i} & D & & \\ & {\scriptstyle k}\nearrow & & {\scriptstyle j}\searrow \quad {\scriptstyle k}\nearrow & & {\scriptstyle j}\searrow & \\ E & & \xrightarrow{d} & E & \xrightarrow{d} & & E \end{array}$$

Let $d = jk : E \to E$. Then $d^2 = 0$ so (E,d) is a differential object.

Def Given an exact couple as above, the DERIVED COUPLE is a diagram

(2)
$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

where $D' = \operatorname{im} i \subset D$

$E' = H(E) = \ker d / \operatorname{im} d$

$i' = i \,|\, D'$

For $x \in D$, j' sends $i(x)$ to $[j(x)]$, the homology class of $j(x)$

k' sends $[y]$ (for $y \in \ker d \subset E$) to $k(y)$.

Theorem (2) is also an exact couple.

Proof is diagram chase.

So we have a sequence of exact couples

$$(D, E) \rightsquigarrow (D', E') \rightsquigarrow (D'', E'') \rightsquigarrow \cdots\cdots$$

(E, d) is a differential object with $E' = H(E, d)$

(E', d') " $E'' = H(E', d')$

(E^{n+1}, d^{n+1}) " $E^{n+1} = H(E^n, d^n)$

so we have a SPECTRAL SEQUENCE.

Examples

(1) Adams diagram of spaces

$$\begin{tikzcd} X_0 & X_1 \arrow[l, "g_0"'] \arrow[d, "h_1"] & X_2 \arrow[l, "g_1"'] \arrow[d, "h_2"] & \cdots \arrow[l, "g_2"'] \\ {} \arrow[from=u, "h_0"] & {} & {} & \end{tikzcd}$$

$\cdots\cdots$

$$K_0 \qquad K_1 \qquad K_2 \quad \text{,, fiber of } f_s$$

where for each s, $X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s$
is a fiber sequence, meaning there is a long exact sequence

$$\cdots \to \pi_n(X_{s+1}) \xrightarrow{(g_s)_*} \pi_n(X_s) \xrightarrow{(f_s)_*} \pi_n(K_s) \xrightarrow{h_s} \pi_{n-1}(X_{s+1}) \to$$

i.e. we have

$$\begin{tikzcd} D = \pi_* X_* \arrow[rr, "g_*"] & & \pi_* X_* = D \arrow[dl, "f_*"] \\ & E = \pi_* K_* \arrow[ul, "h"] & \end{tikzcd}$$

This is an exact couple, so it leads to a spectral sequence, the ADAMS SS.

<u>Remark</u> We have not said what this SS "means"

(2) Let X be a space and $C(X)$ its cellular or singular chain complex, i.e. a chain complex of free abelian with $H_*(C(X)) = H_*(X; \mathbb{Z})$. There is a SES for each prime p

$$0 \to C(X) \xrightarrow{p} C(X) \to C(X) \otimes \mathbb{Z}/p \to 0$$

This leads to a LES

$$\cdots \to H_n(X;\mathbb{Z}) \xrightarrow{p} H_n(X;\mathbb{Z}) \xrightarrow{j} H_n(X;\mathbb{Z}/p) \xrightarrow{k} H_{n-1}(X;\mathbb{Z})$$

Hence there is an exact couple

$$\begin{tikzcd} D = H_*(X) \arrow[rr, "p"] & & H_*(X) = D \arrow[dl, "j"] \\ & E = H_*(X;\mathbb{Z}/p) \arrow[ul, "k"] & \end{tikzcd}$$

This leads to the BOCKSTEIN SS.

(3) FILTERED CHAIN COMPEX

$$0 = K^{-1} \to K^0 \hookrightarrow K^1 \hookrightarrow K^2 \hookrightarrow K^3 \hookrightarrow \cdots \subset K$$

each K^i is a chain complex. We have short exact sequences for each $p \geq 0$.

$$0 \to K^{p-1} \to K^p \to K^p/K^{p-1} \to 0$$

leading to a LES

$$\begin{array}{ccccccc} \cdots \to H_{p+q}K^{p-1} & \xrightarrow{i} & H_{p+q}K^p & \xrightarrow{j} & H_{p+q}(K^p/K^{p-1}) & \xrightarrow{\partial = k} & H_{p+q-1}K^{p-1} \\ \| & & \| & & \| & & \| \\ D_{p-1,q+1} & & D_{p,q} & & E_{p,q} & & D_{p-1,q} \end{array}$$

This leads to an exact couple (of bigraded abelian groups)

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "\partial"] & \end{tikzcd}$$

This leads to the Serre SS

when the chain complex is ~~~.

See the diagram Figure I′ (page 63) of MT

$$H_{p+q}(K^p/K^{p-1}) = E_{p,q} \xrightarrow{d_1 = j k} E_{p-1,q} = H_{p+q-1}(K^{p-1}/K^{p-2})$$

This is the first differential

Prop 3 of MT page 64

$$E^r_{p,q} = \frac{\operatorname{im}\left(H_{p+q}(K^p/K^{p-r}) \longrightarrow H_{p+q}(K^p/K^{p-1})\right)}{\operatorname{im}\left(H_{p+q+1}(K^{p+r-1}/K^p) \xrightarrow{\partial} H_{p+q}(K^p/K^{p-1})\right)}$$

Proof takes a full page of MT

Questions about convergence:

1. Can we say $E^r_{p,q}$ is independent of r for $r \gg 0$?
2. What does $E^\infty_{p,q}$ have to do with anything??

Consider the Serre SS for

Consider the Serre SS for

$$F \longrightarrow E \longrightarrow B$$

$E_2^{p,q} = H^p(B; H^q(F))$ and $\boxed{\begin{array}{l}\text{nontrivial}\\ \text{ONLY}\\ \text{for } p,q \geq 0\end{array}}$

$$E_r^{p-r,\,q+r-1} \xrightarrow{\tilde{d_r}} E_r^{p,q} \xrightarrow{d_r^{p,q}} E_r^{p+r,\,q+1-r}$$

$$E_{r+1}^{p,q} = \ker d_r^{p,q} / \operatorname{im} \tilde{d_r}$$

For $r >> 0$, $E_{r+1}^{p,q} = E_r^{p,q}$ so $E_\infty^{p,q}$ is defined.