

$$\begin{array}{c}
 \dots \rightarrow H_{p+q-1}(K^{r-1}) \xrightarrow{d} H_{p+q-1}(K^r/K^{r-1}) \xrightarrow{d} H_{p+q-1}(K^{r+1}/K^{r-1}) \\
 \downarrow \scriptstyle{1} \\
 \dots \rightarrow H_{p+q-1}(K^{r+1}) \xrightarrow{d} H_{p+q-1}(K^{r+2}/K^{r-1}) \xrightarrow{d} H_{p+q-1}(K^{r+3}/K^{r-1}) \xrightarrow{d} \dots \\
 \downarrow \scriptstyle{\{d^{r-3}\}} \\
 \dots \rightarrow H_{p+q-1}(K^{r-2}) \xrightarrow{d} H_{p+q-1}(K^{r-1}/K^{r-2}) \xrightarrow{d} H_{p+q-1}(K^r/K^{r-2}) \xrightarrow{d} H_{p+q-1}(K^{r+1}/K^{r-2}) \\
 \downarrow \scriptstyle{1} \\
 \dots \rightarrow H_{p+q-1}(K^r/K^{r-1}) \xrightarrow{d} H_{p+q-1}(K^{r+1}/K^{r-1}) \xrightarrow{d} H_{p+q-1}(K^{r+2}/K^{r-1}) \xrightarrow{d} H_{p+q-1}(K^{r+3}/K^{r-1}) \xrightarrow{d} \dots \\
 \downarrow \scriptstyle{\{d^{r-1}\}} \\
 \dots \rightarrow H_{p+q-1}(K^r) \xrightarrow{d} H_{p+q-1}(K^{r+1}/K^r) \xrightarrow{d} H_{p+q-1}(K^{r+2}/K^r) \xrightarrow{d} H_{p+q-1}(K^{r+3}/K^r) \xrightarrow{d} \dots \\
 \downarrow \scriptstyle{\{d^{r-1}\}} \\
 \dots \rightarrow H_{p+q-1}(K^{r+1}/K^r) \xrightarrow{d} H_{p+q-1}(K^{r+2}/K^r) \xrightarrow{d} H_{p+q-1}(K^{r+3}/K^r) \xrightarrow{d} \dots \\
 \downarrow \scriptstyle{\{d^{r-1}\}} \\
 \dots \rightarrow H_{p+q-1}(K^{r+1}/K^r) \xrightarrow{d} H_{p+q-1}(K^{r+2}/K^r) \xrightarrow{d} H_{p+q-1}(K^{r+3}/K^r) \xrightarrow{d} \dots \\
 \downarrow \scriptstyle{\{d^{r-1}\}} \\
 \dots \rightarrow H_{p+q-1}(K^{r+1}/K^{r-1}) \xrightarrow{d} H_{p+q-1}(K^{r+2}/K^{r-1}) \xrightarrow{d} \dots
 \end{array}$$

Figure 1

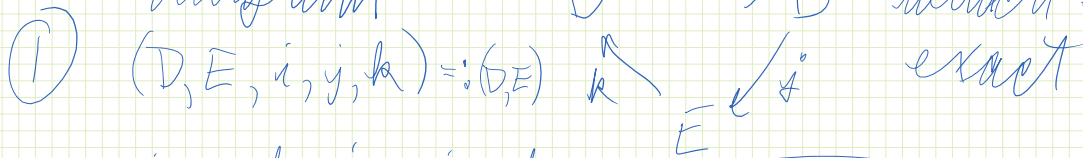
Sources: Mookherjee + Tangora Ch 7.
 Hilton + Stamback Ch. VIII
 Kreen book § 2.1, also gives other references.

Let \mathcal{C} be an ABELIAN CATEGORY, i.e. one in which exactness makes sense, e.g. R -modules, graded R -modules, chain complexes.

Def A DIFFERENTIAL OBJECT in \mathcal{C} is an object E with a morphism $E \xrightarrow{d} E$ s.t. $d \circ d = 0$. e.g. a chain complex where \mathcal{C} = category of graded abelian groups.

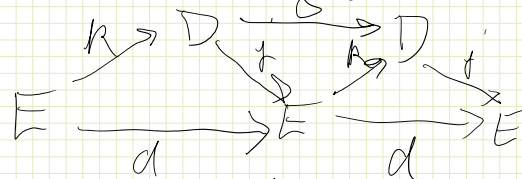
Then $Z(E) = \text{kernel of } d = \text{cycles}$
 $B(E) = \text{image of } d = \text{boundaries}$
 $H(E) = Z(E) / B(E) = \text{homology}$

Def An EXACT COUPLE in \mathcal{C} is a diagram



i.e. $\text{ker } i = \text{im } k$
 $\text{ker } j = \text{im } i$
 $\text{ker } k = \text{im } j$

Then we have



Let $d = j \circ k \circ i: E \rightarrow E$. Then $d^2 = 0$ so (E, d) is a differential object.

Def Given an exact couple as above,
the DERIVED COUPLE is a diagram

$$\begin{array}{ccc}
 D' & \xrightarrow{i'} & D' \\
 \uparrow k' & & \downarrow j' \\
 E & & E
 \end{array}
 \quad \text{where } D' = \text{im } i \subset D \\
 \quad \quad \quad E' = H(E) = \text{ker } d / \text{im } d \\
 \quad \quad \quad i' = i|_{D'} \\
 \text{For } x \in D, j' \text{ sends } \\
 i(x) \text{ to } [j(x)], \text{ the homology} \\
 \text{class of } j(x) \\
 k' \text{ sends } [y] \text{ (for } y \in \text{ker } d \subset E) \text{ to } k(y).$$

Theorem (2) is also an exact couple.

Proof is diagram chase.

so we have a sequence of exact couples

$$\begin{array}{ccccccc}
 (D, E) & \rightsquigarrow & (D', E') & \rightsquigarrow & (D'', E'') & \rightsquigarrow & \dots \\
 (E, d) & & \text{"} & & \text{"} & & \\
 (E', d') & & \text{"} & & \text{"} & & \\
 (E^{m+1}, d^{m+1}) & & \text{"} & & \text{"} & & \\
 & & & & & & E^{m+1} = H(E^m, d^m)
 \end{array}$$

so we have a SPECTRAL SEQUENCE.

Examples

(1) Adams diagram of spaces

$$\begin{array}{ccccccc}
 X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \dots \\
 \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 & & \dots
 \end{array}$$

$K_0 \quad \downarrow \quad K_1 \quad K_2$ "fibers of f_s "
 where for each s , $X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s$
 is a fiber sequence, meaning there is
 a long exact sequence

$$\cdots \rightarrow \pi_n(X_{s+1}) \xrightarrow{(g_s)_*} \pi_n(X_s) \xrightarrow{(f_s)_*} \pi_n(K_s) \xrightarrow{h_s} \pi_n(X_{s+1}) \rightarrow$$

i.e. we have

$$\begin{array}{ccc}
 D = \pi_n X_s & \xrightarrow{g_s} & \pi_n X_s = D \\
 \uparrow h & & \downarrow f_s \\
 E = \pi_n K_s & &
 \end{array}$$

This is an exact couple, so
 it leads to a spectral sequence,
 the ADAMS SS.

Remark We have not said what this
 SS "means"

② Let X be a space and $C(X)$ its
 cellular or singular chain complex,
 i.e. a chain complex of free abelian
 with $H_*(C(X)) = H_*(X; \mathbb{Z})$. There is
 a SES for each prime p

$$0 \rightarrow C(X) \xrightarrow{p} C(X) \rightarrow C(X) \otimes \mathbb{Z}/p \rightarrow 0$$

This leads to a LES

$$\dots \rightarrow H_n(X; \mathbb{Z}) \xrightarrow{p} H_n(X; \mathbb{Z}) \xrightarrow{j} H_n(X; \mathbb{Z}/p) \xrightarrow{k} H_{n-1}(X; \mathbb{Z})$$

Hence there is an exact couple

$$\begin{array}{ccc} D = H_n(X) & \xrightarrow{p} & H_n(X) = D \\ \uparrow k & & \swarrow j \\ E = H_n(X; \mathbb{Z}/p) & & \end{array}$$

This leads to the BOCKSTEIN SS.

③ FILTERED CHAIN COMPLEX

$$0 = K^{-1} \rightarrow K^0 \hookrightarrow K^1 \hookrightarrow K^2 \hookrightarrow K^3 \hookrightarrow \dots \subset K$$

each K^i is a chain complex. We have short exact sequences for each $p \geq 0$.

$$0 \rightarrow K^{p-1} \rightarrow K^p \rightarrow K^p / K^{p-1} \rightarrow 0$$

leading to a LES

$$\begin{array}{ccccccc} \dots \rightarrow H_{p+q}(K^{p-1}) & \xrightarrow{i} & H_{p+q}(K^p) & \xrightarrow{j} & H_{p+q}(K^p/K^{p-1}) & \xrightarrow{k} & H_{p+q-1}(K^{p-1}) \\ \parallel & & \parallel & & \parallel & & \parallel \\ D_{p-1, q+1} & & D_{p, q} & & E_{p, q} & & D_{p-1, q} \end{array}$$

This leads to an exact couple (of bigraded abelian groups)

$$\begin{array}{ccc} D & \xrightarrow{j} & D \\ \uparrow k & & \swarrow j \\ E & & \end{array}$$

This leads to the Serre SS

when the chain complex is \dots .

See the diagram Figure 1 (page 63) of MT

$$H_{p+q}(K^p/K^{p-1}) = E_{p,q} \xrightarrow{d_1 = j^k} E_{p-1,q} = H_{p+q-1}(K^{p-1}/K^{p-2})$$

This is the first differential

Prop 3 of MT page 64

$$E_{p,q}^m = \frac{\text{im} \left(H_{p+q}(K^p/K^{p-m}) \longrightarrow H_{p+q}(K^p/K^{p-1}) \right)}{\text{im} \left(H_{p+q+1}(K^{p+m-1}/K^p) \longrightarrow H_{p+q}(K^p/K^{p-1}) \right)}$$

Proof takes a full page of MT

Questions about convergence:

① Can we say $E_{p,q}^m$ is independent of m for $m \gg 0$?

② What does $E_{p,q}^\infty$ have to do with anything??

Consider the Serre SS for

consider the Serre SS for

$$F \rightarrow E \rightarrow B$$

$$E_2^{p,q} = H^p(B; H^q(F)) \quad \text{and}$$

nontrivial
 ONLY
 for $p, q \geq 0$

$$E_m^{p-m, q+m-1} \xrightarrow{d_m} E_m^{p, q} \xrightarrow{d_m} E_m^{p+m, q+1-m}$$

$$E_{m+1}^{p, q} = \ker d_m^{p, q} / \text{im } d_m$$

For $m \gg 0$, $E_{m+1}^{p, q} = E_m^{p, q}$ so $E_\infty^{p, q}$ is defined.