Thursday, February 22, 2018 1:56 PM

Toward the Adams spectral sequence

GOAL: Find $\pi_{n+k} S^n$ (the 2-component) for $n >>$ $k < n$.

Our set up

$$\begin{tikzcd} S^n = X_0 \arrow[d, "f_0"] & X_1 \arrow[l] \arrow[d, "f_1"] & X_2 \arrow[l] \arrow[d, "f_2"] & \cdots \arrow[l] \\ K(\mathbb{Z}/2, n) = L_0 & L_1 & L_2 & \end{tikzcd}$$

where

① X_{s+1} is the fiber of f_s

② L_s is a product of $K(\mathbb{Z}/2, n+i)$'s

③ f_s is chosen so that $H^*(f_s; \mathbb{Z}/2)$ is onto

We have fiber sequences

$$L_s \leftarrow X_s \leftarrow X_{s+1}$$

We know $H^* X_s$ by induction

Choose a set of A-module generators for it and use them to define $f_s: X_s \to L_s$

Use the Serre SS to find $H^* X_{s+1}$.

Below dim $2n$, this Serre SS reduces to a short exact sequence

$$0 \leftarrow H^* X_s \xleftarrow{f_s^*} H^* L_s \leftarrow H^* X_{s+1} \leftarrow 0$$

QUESTION Why is L_0 not a product of $K(\mathbb{Z}/2, -)$'s like the L_s for $s > 0$?

ANSWER. It is possible to prove that X_s is roughly $(n+2s)$-connected, but if we replace $K(\mathbb{Z},n)$ by $K(\mathbb{Z}/2,n)$ this would not happen.

Suppose we replace S^n by $K(\mathbb{Z},n)$ and do a similar construction

$$\begin{array}{ccccc} K(\mathbb{Z},n) & \xleftarrow{2} & K(\mathbb{Z},n) & \xleftarrow{2} & K(\mathbb{Z},n) \leftarrow \cdots \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 \\ K(\mathbb{Z}/2,n) & & K(\mathbb{Z}/2,n) & & K(\mathbb{Z}/2,n) \end{array}$$

$H^t f_s$ is onto for each $s \geq 0$.

The bottom line is this.

There a spectral sequence with

$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}/2,\mathbb{Z}/2)$ computable by algebra

$E_r^{s,t} \xrightarrow{d_r} E_r^{s+r,\,t+r-1}$ ADAM INDEXING not SERRE INDEXING

and $E_\infty^{s,t}$ is a subquotient of

$\pi_{n+t-s}\, S^n$ for $t-s < n$

How to define $\mathrm{Ext}_A^{s,t}(\mathbb{Z}/2,\mathbb{Z}/2)$.

① Choose a free A-resolution of $\mathbb{Z}/2$

$$0 \leftarrow \mathbb{Z}/2 \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow \cdots$$

$$F_0 = A$$

free A-module

② apply the functor $\mathrm{Hom}_A(-, \mathbb{Z}/2)$

Let $\mathrm{Hom}_A(M, \mathbb{Z}/2) = M^*$

$$F_0^* \longrightarrow F_1^* \longrightarrow F_2^* \longrightarrow F_3^* \longrightarrow \cdots$$

It is a cochain complex of graded $\mathbb{Z}/2$-vector spaces

③ Take its cohomology. $H^{s,*} = \mathrm{Ext}_A^{s,*}(\mathbb{Z}/2, \mathbb{Z}/2)$

A TOY EXAMPLE

Replace A by the subalgebra E generated by Sq^1. The Adem relation tells us $Sq^1 Sq^1 = 0$. Our subalgebra has basis $\{1, Sq^1\}$ $\dim Sq^1 = 1$

Here is a free E-resolution of $\mathbb{Z}/2$

$$\mathbb{Z}/2 \leftarrow E \xleftarrow{Sq^1} \Sigma E \xleftarrow{Sq^1} \Sigma^2 E \xleftarrow{Sq^1} \Sigma^3 E \leftarrow \cdots$$

gen $\leftarrow\!\shortmid$ 1   1

0 $\leftarrow\!\shortmid$ Sq^1 $\leftarrow$ Sq^1

Applying $\mathrm{Hom}_E(-, \mathbb{Z}/2)$ gives us

$$\mathbb{Z}/2 \xrightarrow{0} \Sigma\mathbb{Z}/2 \xrightarrow{0} \Sigma^2\mathbb{Z}/2 \xrightarrow{0} \Sigma^3\mathbb{Z}/2 \to \cdots$$

We find $\mathrm{Ext}_E^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{for } t = s \\ 0 & \text{for } t \neq s \end{cases}$

General properties of Ext. (Pure algebra)

Suppose we have a short exact sequence of A-modules $\boxed{\mathrm{Hom} = \mathrm{Ext}^0}$

$$0 \leftarrow M_3 \leftarrow M_2 \leftarrow M_1 \leftarrow 0$$

Apply our functor $\mathrm{Hom}(-, \mathbb{Z}/2)$ and get a long exact sequence

$$0 \to \mathrm{Hom}_A(M_3, \mathbb{Z}/2) \to \mathrm{Hom}_A(M_2, \mathbb{Z}/2) \to \mathrm{Hom}_A(M_1, \mathbb{Z}/2)$$
$$\to \mathrm{Ext}^1(M_3, \mathbb{Z}/2) \to \mathrm{Ext}^1_A(M_2, \mathbb{Z}/2) \to \mathrm{Ext}(M_1, \mathbb{Z}/2)$$
$$\to \mathrm{Ext}(M_3, \mathbb{Z}/2) \to \cdots$$

Suppose $M_3 = \mathbb{Z}/2$ and $M_1 = \Sigma^{?}\mathbb{Z}/2$

We get a map $\mathrm{Hom}_A(\Sigma^{?}\mathbb{Z}/2, \mathbb{Z}/2) \to \mathrm{Ext}^1(\mathbb{Z}/2, \mathbb{Z}/2)$

$$0 \leftarrow \mathbb{Z}/2 \leftarrow \{1, Sq^1\} \leftarrow \Sigma\mathbb{Z}/2 \leftarrow 0$$

E identity $\longmapsto h_0 \in \mathrm{Ext}^{1,1}$

$$\mathrm{Hom}_A(\Sigma \mathbb{Z}/2, \mathbb{Z}/2) \to \mathrm{Ext}^1(\mathbb{Z}/2, \mathbb{Z}/2)$$

$$\overset{\shortparallel}{\mathrm{Ext}^0_A}(\Sigma \mathbb{Z}/2, \mathbb{Z}/2)$$

$$0 \leftarrow \mathbb{Z}/2 \leftarrow \{1, Sq^{2^j}\} \leftarrow \Sigma^{2^j} \mathbb{Z}/2 \leftarrow 0$$

$$\rightsquigarrow h_j \in \mathrm{Ext}_A^{1, 2^j}(\mathbb{Z}/2, \mathbb{Z}/2)$$

for any $j \geq 0$

Yoneda correspondance:

Let R be a (graded) $\mathbb{Z}/2$-algebra

There is relation between $\mathrm{Ext}^s_R(\mathbb{Z}/2, \mathbb{Z}/2)$ and exact sequences of R-modules of the form

(1) $0 \to \mathbb{Z}/2 \to M_1 \to M_2 \to \cdots M_s \to \mathbb{Z}/2 \to 0$

Suppose we have another exact sequence

(2) $0 \to \mathbb{Z}/2 \to M_1' \to M_2' \cdots \to M_s' \to \mathbb{Z}/2 \to 0$

Let $\alpha \in \mathrm{Ext}^s_R(\mathbb{Z}/2, \mathbb{Z}/2)$ corresponding to (1)

$\beta \in \mathrm{Ext}^{s'}_R(\mathbb{Z}/2, \mathbb{Z}/2)$ " (2)

Splicing (1) and (2) gives us

$$0 \to \mathbb{Z}/2 \to M_1 \to \cdots M_s \to M_1' \to \cdots M_{s'}' \to \mathbb{Z}/2 \to 0$$

which represents an element

$$\alpha\beta \in \mathrm{Ext}_R^{s+s'}(\mathbb{Z}/2, \mathbb{Z}/2)$$

Hence $\mathrm{Ext}_R(\mathbb{Z}/2, \mathbb{Z}/2)$ is a graded ring. If R is graded, then Ext is bigraded.

(A) We have seen two examples of spectral sequences due Serre and Adams. There are others besides these in algebraic topology. All are constructed in similar ways. There are 2 ways to do it

① EXACT COUPLE

② FILTERED CHAIN COMPLEX.

(B) How to study n-connected space for $n \to \infty$

STABLE HOMOTOPY THEORY

Spaces are replaced by SPECTRA

COMPUTATIONAL PRECEDES THEORY

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Make friends with the Steenrod algebra.

Spectral sequences via filtered chain complexes.

Let C be a chain complex whose homology you desperately want to know.

An INCREASING FILTRATION ON C is a sequence of sub-chain complexes $F_0C \subset F_1C \subset F_2C \to F_3C \subset \cdots$ with C = union of all of them.

We have short exact sequences

$0 \to F_{n-1}C \to F_nC \to F_n/F_{n-1}C \to 0$

and we know $H_*(F_n/F_{n-1}C)$. How can we use it to find H_*C?

Remarks

1) We discuss cochain complexes

2) One could also have a DECREASING filtration

$C = F^0C \supset F^1C \supset F^2C \supset F^3C \supset \cdots$

with $\bigcap_n F^nC = 0$

we have short exact sequence

$0 \to F^{n+1}C \to F^nC \to F^n/F^{n+1}C \to 0$

Suppose we know $H_*(F^n/F^{n+1}C)$ for all n.

<u>Example</u> Suppose we have a fiber sequence $F \to E \xrightarrow{p} B$ = simply connected

Suppose these are CW-complexes with cellular chain complexes and they "play nicely" with each other. B has skeleta

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{p} & B \\ \| & & \uparrow & & \uparrow \\ F & \longrightarrow & p^{-1}(B^n) & \longrightarrow & B^n = n\text{-skeleton} \end{array}$$

Let C be the cellular chain cx of E and let $F_n C =$ cellular " for $p^{-1}(B^n) \subset E$

What $F_n C / F_{n-1} C$?

$$0 \to C(B^{n-1}) \longrightarrow C(B^n) \longrightarrow \underset{\underset{C(\vee S^n)}{\|}}{C(B^n/B^{n-1})} \to 0$$

$$0 \to C\,p^{-1}(B^{n-1}) \longrightarrow C(p^{-1}(B^n)) \longrightarrow C(F) \otimes C(\vee S^n) \to 0$$

$H_*(C(F) \otimes C(\vee S^n)) = \bigoplus \Sigma^n H_* F =$ known quantity.

Back to our filtered complex story

Consider the diagram

$$0 \to F_{n-2} C \longrightarrow F_{n-1} C \longrightarrow F_{n-1}/F_{n-2} C$$

$$\begin{array}{ccccccc}
0 \to & F_{n-2}C & \longrightarrow & F_{n-1}C & \longrightarrow & F_{n-1}/F_{n-2}C & \\
& \| & & \downarrow & & \downarrow & \\
0 \to & F_{n-2}C & \longrightarrow & F_nC & \longrightarrow & F_nC/F_{n-2}C & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & \longrightarrow & F_n/F_{n-1}C & = & F_nC/F_{n-1}C & \to 0 \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 &
\end{array}$$

It has exact rows and columns.
Consider the LES in H_* for the right column

$$\xrightarrow{d_{n,i+1}} H_i F_{n-1}/F_{n-2} \longrightarrow H_i F_n/F_{n-2} \longrightarrow H_i F_n/F_{n-1} \xrightarrow{d_{n,i}} H_{i-1} F_{n-1}/F_{n-2} \longrightarrow \cdots$$

($H_{i+1} F_n/F_{n-1} \xrightarrow{d_{n,i+1}} H_i F_{n-1}/F_{n-2}$)

$$E_1^{i,n} \xrightarrow{d_{n,i}} E_1^{i-1,n-1}$$

It follows there a SES

$$0 \to \operatorname{coker} d_{n,i+1} \longrightarrow H_i F_n/F_{n-2} \longrightarrow \ker d_{n,i} \to 0$$

Suppose we can find the middle group.

$$\begin{array}{ccccc}
F_{n-3}C & \longrightarrow & F_{n-2}C & \longrightarrow & F_{n-2}/F_{n-3}C \\
\| & & \downarrow & & \downarrow \\
F_{n-3} & \longrightarrow & F_nC & \longrightarrow & F_n/F_{n-3}C
\end{array}$$

$$\begin{tikzcd} F_{n-3} \arrow[r] & F_n C \arrow[r] \arrow[d] & F_n/F_{n-3}C \arrow[d] \\ & F_n/F_{n-2}C \arrow[r, equal] & F_n/F_{n-2}C \end{tikzcd}$$

As before we get a LES with connecting homomorphisms and we deduce a SES with $H_* F_n/F_{n-3}$ in the middle.

. . . .

Eventually we learn $H_*(F_n C)$ and then $H_* C$ itself.

All of this can be encoded as a spectral sequence.

Let $E_1^{i,n} = H_i F_n/F_{n-1}$

to be continued