Serre's program to calculate homotopy groups

HOLY GRAIL For your favourite space X, find $\pi_* X$, e.g. $X = S^n$ for $n > 1$.

VERY HARD PROBLEM

HUREWICZ Theorem There is a hom

$$\pi_n X \xrightarrow{\eta} H_n X$$ for all n as follows

Given a map $S^n \xrightarrow{f} X$, we get a map

$$1 \in \mathbb{Z} = H_n S^n \xrightarrow{f_*} H_n X \ni f_*(1)$$

η is a homomorphism. Suppose X is $(k-1)$-connected (i.e. $\pi_i X = 0$ for $i < k$). Then the HUREWICZ map in dimension k is an isomorphism when $k > 1$.

EILENBERG-MACLANE theorem. Hypotheses as above. There is a map $X \xrightarrow{p} K(\pi_k X, k)$ inducing an iso in π_k.

Serre's program: Suppose we know $H^* K(A, k)$ for all $k > 0$ and all abelian groups A.

FACT the source of p can be replaced by a homotopy equivalent space $\begin{tikzcd} \hat{X} \arrow[r, "\hat{p}"] & K(-) \\ X \arrow[u, "\simeq"] \arrow[ur, "p"'] & \end{tikzcd}$

such $\hat{p}$ is a Serre fibration. Let X_1 denote its fiber, so we have a fiber sequence

$$X_1 \longrightarrow X_0 \xrightarrow{p_0} K(\pi_k(X), k)$$
$$X_0 = \hat{X}, \quad K(\pi_k(X),k) = *$$

There is a long exact sequence

$$\cdots \longrightarrow \pi_i X_1 \longrightarrow \pi_i X_0 \xrightarrow{\pi_i(p_0)} \pi_i(K) \longrightarrow \pi_{i-1} X_1 \longrightarrow \cdots$$

$\pi_i(p_0)$ is $\begin{cases} \text{iso} & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases}$

It follows that $\pi_i X_1 = \begin{cases} 0 & \text{for } i \leq k \\ \pi_i X & \text{for } i > k \end{cases}$

We can use the Serre SS to compute $H^* X_1$. Using the Hurewicz theorem, we can find the first nontrivial htpy group of X_1, i.e. the second such gp for $X_0 = X$.

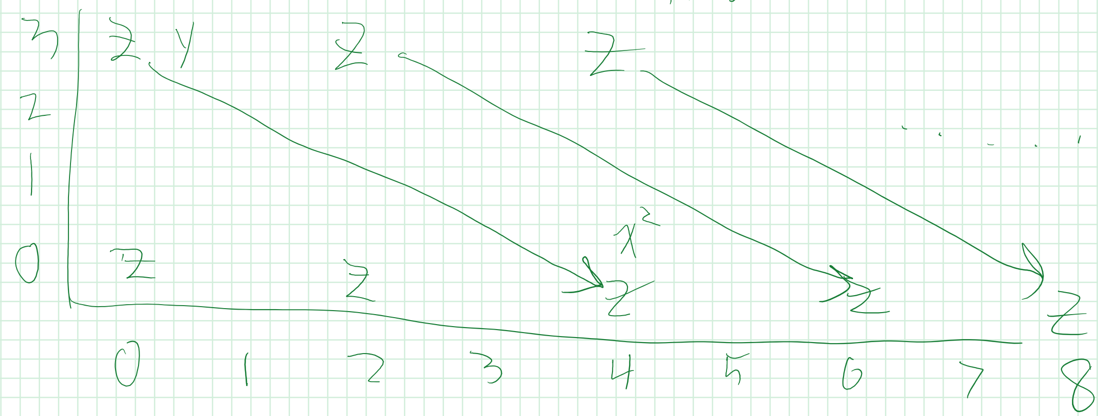
<u>An example</u> $X = S^2$. $\pi_2 S^2 = \mathbb{Z}$ and $\pi_1 S^2 = 0$

There is a map $S^2 \to K(\mathbb{Z},2) = \mathbb{C}P^\infty$

Consider the fiber sequence

$$S^1 \simeq \Omega K(\mathbb{Z},2) \to X_1 \to S^2 \to K(\mathbb{Z},2) \qquad x \in H^2$$

$$E_2^{p,q} = H^p(K(\mathbb{Z},2),\ H^q X_1) \Rightarrow H^* S^2 \qquad H^* K(\mathbb{Z},2) = \mathbb{Z}[x]$$



$d_4(y) = x^2$ so $d_4(x^n y) = x^{n+2}$

Conclusion $H^i(X_1) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 3 \\ 0 & \text{else} \end{cases}$

In fact $X_1 \simeq S^3$. Recall the Hopf fibration

$$\begin{array}{c} S^1 \longrightarrow S^3 \longrightarrow S^2 \\ \shortparallel \qquad\qquad\qquad\quad \\ K(\mathbb{Z},1) \qquad\qquad\quad \end{array}$$

The LES in π_* shows

$$\pi_i S^3 = \begin{cases} 0 & \text{for } i < 3 \\ \pi_i S^2 & \text{for } i \geq 3 \end{cases}$$

$$\pi_3 S^2 = \mathbb{Z}$$

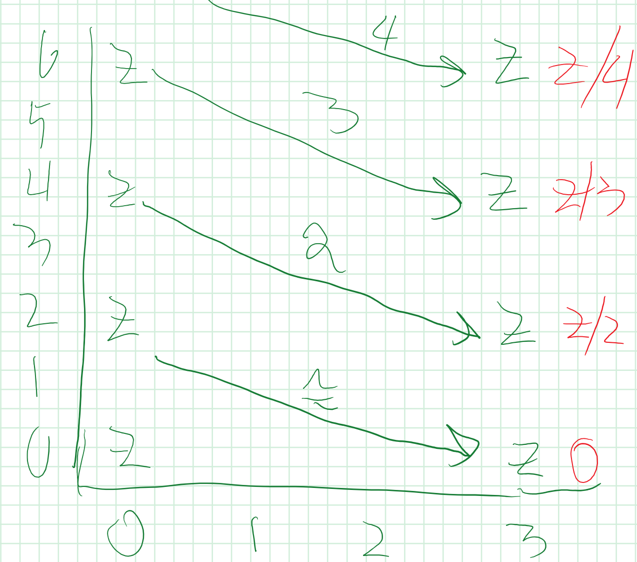
<u>Thm</u> If $F \to E \to B$ is a fiber sequence, so is $\Omega B \to F \to E$ where ΩB, the loop space of B, has $\pi_i \Omega B = \pi_{i+1} B$.

Consider $X \longrightarrow S^3 \longrightarrow K(\mathbb{Z},3)$

We also have $K(\mathbb{Z},2) \longrightarrow X \longrightarrow S^3$

$E_2^{p,q} = H^p(S^3; H^q K(\mathbb{Z},2)) \Longrightarrow H^* X$ and

we know $\pi_i X \cong \pi_i S^3 \cong \pi_i S^2$ for $i \geq 4$



Let $x \in H^2 \mathbb{CP}^\infty$

$y \in H^3 S^3$

$d_3 x = y$

$d_3(x^n) = n x^{n-1} y$

$\boxed{X \text{ is the 3-connected cover of } S^3 \text{ and of } S^2}$

$$H^i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/n & i = 2n+1 \\ 0 & \text{else} \end{cases}$$

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/n & i = 2n \end{cases}$$

$$H_i(X) = \begin{cases} \mathbb{Z}/n & i = 2n \\ 0 & \text{else} \end{cases}$$

e.g. $H_4(X) = \mathbb{Z}/2$ so $\pi_4 X = \mathbb{Z}/2$

$$\pi_4 S^3 = \pi_4 S^2$$

Back to $K(\mathbb{Z}/2, n)$ for various n.

$K_n = K(\mathbb{Z}/2, n)$ $\qquad K_1 = \mathbb{R}P^\infty$

Will study $H^*(K_n; \mathbb{Z}/2)$

Recall $H^* K_1 = H^* \mathbb{R}P^\infty = \mathbb{Z}/2[x_1]$ $\qquad x_1 \in H^1$

$\rightsquigarrow H^* K_2 = \mathbb{Z}/2[x_2, x_3, x_5, \ldots, x_{1+2^i}, \ldots]$

$x_i \in H^i$

A similar calculation shows

$$H^*(K(\mathbb{Z},3); \mathbb{Z}/2) = \mathbb{Z}/2[x_3, x_5, x_9, \ldots]$$

There is a fiber sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

Can show there is a (LONG!) fiber sequence

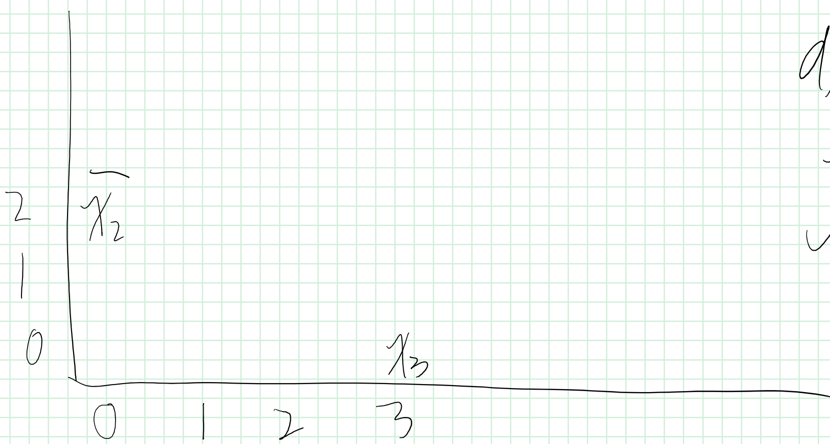
$$K(\mathbb{Z},1) \xrightarrow{2} K(\mathbb{Z},1) \to K(\mathbb{Z}/2,1) \to K(\mathbb{Z},2) \xrightarrow{2} K(\mathbb{Z},2) \to K(\mathbb{Z}/2,2) \to K(\mathbb{Z},3) \to \cdots$$

$\hookrightarrow K(\mathbb{Z},3) \to \cdots$

Consider the fiber sequence

$$F = K(\mathbb{Z},2) \longrightarrow K(\mathbb{Z}/2,2) \longrightarrow K(\mathbb{Z},3)$$
$$\quad\; \| \qquad\qquad\qquad \| \qquad\qquad\quad \|$$
$$\quad CP^\infty \qquad\qquad\quad E \qquad\qquad\quad B$$

$$\mathbb{Z}/2[\bar{x}_2] \longleftarrow \mathbb{Z}/2[x_2, x_3, x_5, x_9 \cdots]$$



$d_n(\bar{x}_2) = 0$ because $\bar{x}_2$ is in the image of H^*E

$E_2 = H^*F \otimes H^*B$ with no differentials, so H^*B is as claimed

$$H^*K_2 = \mathbb{Z}/2[x_2, x_3, x_5, x_9 \cdots]$$
$$Sq^1 \quad Sq^2 \quad Sq^4 \quad \cdots$$

$$= \mathbb{Z}/2[x_2, Sq^1 x_2, Sq^2 Sq^1 x_2, Sq^4 Sq^2 Sq^1 x_2, \cdots]$$

Recall some properties of Sq^i

6) Adem relation: For $a < 2b$,

$$Sq^a Sq^b = \sum \binom{b-1-i}{\;\cdot\;} Sq^{a+b-i} Sq^i$$

$$Sq^a Sq^b = \sum_{i \ge 0} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i$$

Claim that each term on the right $Sq^k Sq^l$ has $k \ge 2l$

We know that $\binom{b-1-i}{a-2i}$ is defined only for $2i \le a < 2b$; so $i < b$

Claim these imply $a+b-i \ge 2i$

$$a+b \ge 3i$$

but $a \ge 2i$

and $b > i$.

so $a+b > 3i$