Let $K_n = K(\mathbb{Z}/2, n)$

= EM space with $\pi_i \begin{cases} \mathbb{Z}/2 & \text{for } i = n \\ 0 & \text{for } i \neq n \end{cases}$

$H^*(\ )$ always means $H^*(\ ; \mathbb{Z}/2)$

Recall $H^*(K_1) = \mathbb{Z}/2[x_1]$ with $x_1 \in H^1$

$\cong H^*(RP^\infty)$

$H^*(K_2) = \mathbb{Z}/2[x_2, x_3, x_5 \cdots x_{1+2^i}, \cdots]$

On $H^*(K_1)$, let $x = x_1$ and consider the classes

$S_1 = \{x, x^2, x^4, x^8, x^{16} \cdots\} = \{x^{2^i} : i \geq 0\}$

This is a SIMPLE SYSTEM of generators

i.e a basis for H^*K_1 consists of all possible finite products (without repetition) of elements of S.

$\{\text{Powers of } x\} \longleftrightarrow \{\text{Finite subsets of } S\}$

A simple system of generators for $H^*(K_2)$ is $\{x_{1+2^i}^{2^j} : i, j \geq 0\} = \{x_2, x_3, x_2^2, x_5, x_3^2, x_2^4 \ldots\}$

Thm (A Borel) Suppose we have a fiber sequence

$F \longrightarrow E \longrightarrow B$ where

1) H^*F has a simple system of generators $S = \{x_1, x_2, x_3 \cdots\}$

2) $H^*E \cong H^*(pt)$

3) Each x_i is TRANSGRESSIVE

Then $H^*B = \mathbb{Z}/2[y_1, y_2, y_3, \cdots]$ where

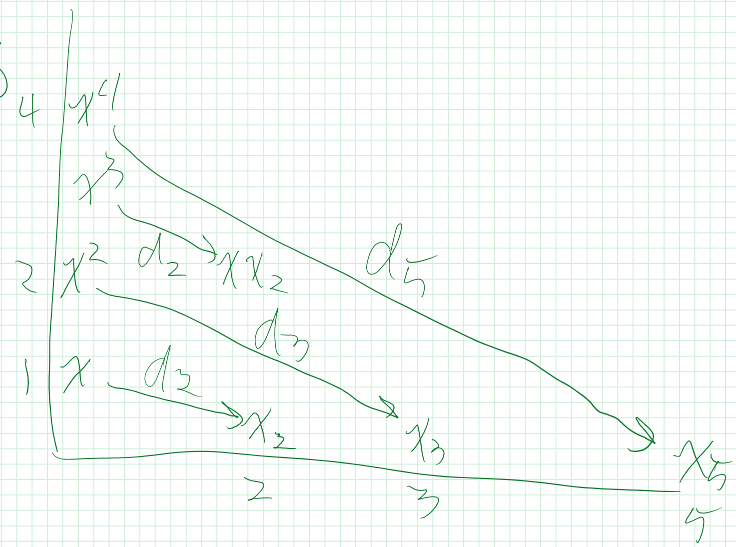
$|y_i| = 1 + |x_i|$ and in the Serre SS

$$d_{1+|x_i|}\, x_i = y_i$$

<u>Example</u> (from 1/23) $B = K_2$, $F = K_1$

$S = \{x, x^2, x^4, \dots\}$

x^3 is not transgressive but each element of S is transgressive.



<u>Prop</u> If $R = \mathbb{Z}/2[a_1, a_2, a_3, \dots]$ then the set $\{a_i^{2^j} : i > 0, j \geq 0\}$ is a simple system of generators for it.

Recall $K_n := K(\mathbb{Z}/2, n)$ = the nth mod 2 Eilenberg-Mac Lane space.

We can compute $H^* K_n$ by induction on n using the Serre SS and the Borel theorem.

$H^* K_1 = \mathbb{Z}/2[x_1]$

$H^* K_2 = \mathbb{Z}/2[x_{1+2^i} : i \geq 0]$

$H^* K_3 = \mathbb{Z}/2[x_{2+2^i+2^j} : 0 \leq i < j]$?

COHOMOLOGY OPERATIONS

Recall $H^n(X) = [X, K_n] :=$ the set of homotopy classes of natural map $X \longrightarrow K_n$

This set has a $\mathbb{Z}/2$-vector space structure, i.e. given a map $X \xrightarrow{f} Y$

we get $[X; K_n] \xleftarrow[\mathbb{Z}/2\text{-linear}]{f^*} [Y; K_n]$

$$\begin{array}{c} Y \\ \downarrow \\ K_n \end{array}$$

Suppose we have $\theta \in H^i(K_n) = [K_n, K_i]$

$$X \xrightarrow{x} K_n \xrightarrow{\theta} K_i$$

$x \in H^n(X)$ $\quad \theta x : X \longrightarrow K_i \quad \theta x \in H^i(X)$

θ determines a map $H^n X \xrightarrow{\theta} H^i X$.

It need not be a homomorphism, but it is natural in X

$$X \xrightarrow{f} Y \xrightarrow{y} K_n \xrightarrow{\theta} K_i$$

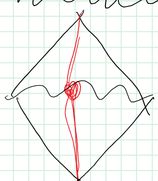
$y \in H^n Y \xrightarrow{\theta} H^i Y$

$$\begin{array}{ccc} y \in H^n Y & \xrightarrow{\theta} & H^i Y \\ \downarrow f^* & & \downarrow f^* \\ H^n X & \xrightarrow{\theta} & H^i X \end{array}$$

Diagram commutes

In principle we know $H^* K_n$ for all n
so we can find lots of θ's.

Consider the following



$$\Sigma K_n \xrightarrow{\sigma_n} K_{n+1} \qquad \pi_{n+1}(\text{both}) = \mathbb{Z}/2$$

adjoint to $K_n \longrightarrow \Omega K_{n+1} \simeq K_n$

σ_n is adjoint to the identity on K_n.

Suppose $K_n \xrightarrow{Sq^i} K_{n+i}$ is a map

$$\begin{array}{ccc} \Sigma K_n & \xrightarrow{\Sigma Sq^i} & \Sigma K_{n+i} \\ \sigma_n \downarrow & & \downarrow \sigma_{n+i} \\ K_{n+1} & \xrightarrow{Sq^i} & K_{n+i+1} \end{array} \qquad (1)$$

If $\exists\ \theta'$ we say θ is STABLE (?)

Thm (page 22 of MT) For each $i \geq 0$ there is a family of maps $K_n \xrightarrow{Sq^i} K_{n+i}$ —

[Sq^i = STEENROD SQUARE]

with certain properties:

1) Diagram ① can always be filled in

2) The map $H^n(X) \xrightarrow{Sq^i} H^{n+i}(X)$ is a natural homomorphism

3) If $i > n$ then $Sq^i = 0$ and when $i = n$, $Sq^n x = x^2$,

4) Sq^0 is the identity map

5) CARTAN FORMULA

$$Sq^n(xy) = \sum_{0 \le i \le n} Sq^i(x) Sq^{n-i}(y)$$

6) ADEM RELATIONS. For $a < 2b$

$$Sq^a Sq^b = \sum_{i \ge 0} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i$$

$Sq^1 Sq^1 = 0$ $\qquad$ $Sq^1 Sq^n = (n+1) Sq^{n+1}$

$Sq^1 Sq^2 = Sq^3$ $\qquad$ for $n > 0$

Back to EM spaces

$K_1 = K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty$

$H^* X = \mathbb{Z}/2[x] \qquad x \in H^1$

$Sq^0 x = x \qquad Sq^1 x = x^2$ and $Sq^i x = 0$ for $i > 1$

<u>Prop</u> $Sq^i(x^n) = \binom{n}{i} x^{n+i}$

<u>Proof</u> Consider the "total Steenrod square"

$$Sq = \sum_{i \geq 0} Sq^i$$

Claim The Cartan formula can be rewritten as $Sq(xy) = Sq(x)\, Sq(y)$

$$Sq^n(xy) = \sum_{0 \leq i \leq n} Sq^i(x)\, Sq^{n-i}(y)$$

$$\sum_{n \geq 0} Sq^n(xy) = \sum_{n \geq 0} \sum_{0 \leq i \leq n} Sq^i(x)\, Sq^{n-i}(y)$$

$$= \sum_{i, j \geq 0} Sq^i(x)\, Sq^j(y)$$

$$Sq(xy) = Sq(x)\, Sq(y).$$

For $x \in H^1 \mathbb{R}P^\infty = H^1 K_1$

$$Sq(x) = x + x^2 = x(1+x)$$

so $A_\varphi(x^n) = A_\varphi(x)^n = x^n(1+x)^n$

$$= x^n \sum_{0 \le i \le n} \binom{n}{i} x^i$$

$$= \sum_{0 \le i \le n} \binom{n}{i} x^{n+i}$$

QED