Definition: A *spectrum* E is a sequence of spaces {E_n} with maps 1960s

$$\Sigma E_n \xrightarrow{f_n} E_{n+1}.$$

for $n >> 0$
The spaces are pointed

or equivalently maps

$$E_n \xrightarrow{g_n} \Omega E_{n+1}$$

Σ means reduced suspension

In a *suspension spectrum*, each f_n is an equivalence. In an *Ω-spectrum*, each g_n is an equivalence.

For a pointed space X there is a suspension spectrum where the nth space is the *n*th suspension of X.

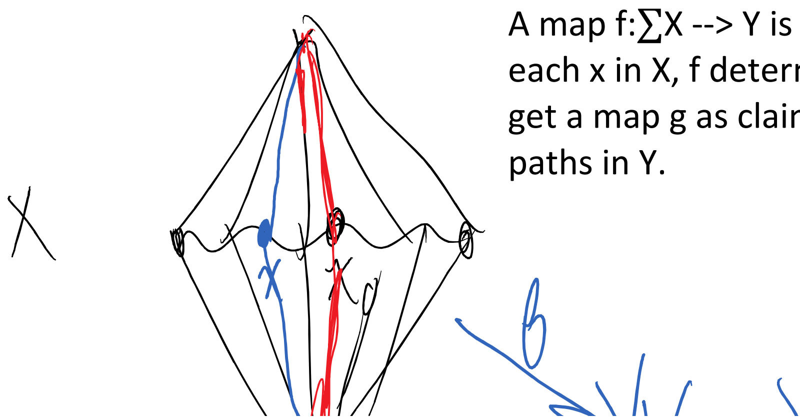
Any spectrum can be converted to an Ω-spectrum by replacing E_n with the space

$$\tilde{E}_n = \lim_{\overrightarrow{k}} \Omega^k E_{n+k}.$$

The sphere spectrum S^0 is the suspension spectrum for X=S^0.

The original example of an Ω-spectrum is the Eilenberg-Mac Lane spectrum for an abelian group G in which E_n=K(G, n).

The *reduced suspension* of a pointed space X is the double cone on X with the line through the base point x_0 collapsed to a point, which is the new base point.



A map f:∑X --> Y is equivalent to a map g:X --> ΩY. For each x in X, f determines a closed path in Y. Hence we get a map g as claimed since ΩY is the space of closed paths in Y.

In a spectrum we have maps g_: E_n --> ΩE_{n+1}

$$E_n \xrightarrow{g_n} \Omega E_{n+1} \xrightarrow{\Omega g_{n+1}} \Omega^2 E_{n+2} \xrightarrow{\Omega^2 g_{n+2}} \Omega^3 E_{n+3} \to \cdots$$

Call the limit $\hat{E}_n$. It follows that

$$\hat{E}_n \simeq \Omega \hat{E}_{n+1}.$$

A map between spectra E--->F could be a sequence of maps E_n --> F_n compatible with the structure maps. This is too restrictive!

EXAMPLE. Consider the suspension spectra S^1 and S^0. In the former, the nth space is S^{n+1}, and the latter it is S^n. We have the Hopf map of spaces \eta:S^3 --> S^2. However \eta is NOT the suspension of any map S^2 --> S^1.

Classifying spaces

Let G be a topological group. Let E_n G bed the (n+1)-fold join of G with itself. Given spaces X and Y, their join X*Y is the quotient of the product X x I x Y, where I=[0,1], where

$(x',0,y) \sim (x'',0,y)$ for any $x',x'' \in X$, $y \in Y$.

and $(x,1,y') \sim (x,1,y'')$ " $x \in X$, $y',y'' \in Y$

Exercises: S^m * S^n = S^{n+m+1}. For G=C_2, E_n G = S^n with antipodal action. In general E_n G is (n-1)-connected.

Definition G acts freely on E_n G by left multiplication n each coordinate in G. Let B_n G be its orbit space. Eg $B_1 G$ unreduced suspension of G. Let the *classifying space BG* be the limit of the $B_n G$ and EG the limit of the $E_n G$. EG is contractible and has a free G action. This construction is functorial in G, i.e. a homomorphism $G \to H$ induced a map $BG \to BH$.

When G is discrete and acts freely on a space X, we get an equivariant map $X \to EF$ and a map from X/G to BG. The homotopy class of the latter determines the group action.

Consider the cases $G=O(n)$ or $U(n)$, the nth orthogonal or unitary group.
We have spaces $BO(n)$ and $BU(n)$. They come equipped with n-dimensional real or complex vector bundles γ_n and γ^C_n. Each has an associated unit disk bundle D and unit sphere bundle S. Consider the space D/S. This is called a Thom space. Call them $MO(n)$ and $MU(n)$. We can use them to construct two spectra MO and MU.

In the orthogonal case, $MO_n = MO(n)$. We need a map $\Sigma MO(n) \to MO(n+1)$. Cosnder the direct sum γ'_n of the vector bundle γ_n with a trivial line bundle. It is an $(n+1)$-plane bundle induced by the map $BO(n) \to BO(n+1)$ incued by the inclusion of $O(n)$ into $O(n+1)$. The Thom space for γ'_n is $\Sigma MO(n)$, so we have our map from it to $MO(n+1)$.

In the unitary case, a similar construction gives a map $\Sigma^2 MU(n) \to MU(n+1)$. We define the spectrum MU by $MU_{2n}=MU(n)$ and $MU_{2n+1}= \Sigma MU(n)$.

The spectra MO and MU are very important.

Remark: In each example spectrum E so far, the space E_n is $(n-1)$-connected. This definition does NOT require this. Such a spectrum is said to be *connective*.

Example of a nonconnective spectrum:

Bott Periodicity Theorem: $\Omega^2 BU$ is equivalent to $Z \times BU$. It implies that
$\pi_k BU =\pi_{k+2} BU$. The orthogonal analog is $\Omega^8 BO$ is equivalent to $Z \times BO$, so
$\pi_k BO =\pi_{k+8} BO$.

We can use this to construct spectra K and KO. $K_{2n} =Z \times BU$, $K_{2n+1} = U$. For any G, $\Omega BG = G$, so $\Omega K_{2n+2}=K_{2n+1}$, and $\Omega^2 K_{2n+2}=K_{2n}$. This leads to an Ω-spectrum K. It is NOT connective.

To continued Monday 3:25 in 1101. Read Intro and Chapter 1 of Lewis-May-Steinberger.