Recall that $\mathcal{S}_G$ denotes the topological G-category of orthogonal G-spectra and nonequivariant maps, that is functors $\mathcal{J}_G \to \mathcal{T}_G$, while $\mathcal{S}^G$ denotes the topological category of orthogonal G-spectra and equivariant maps. We use the symbols $\boldsymbol{Comm}_G$, $\boldsymbol{Alg}_G$, $\boldsymbol{Comm}^G$ and $\boldsymbol{Alg}^G$ to denote the categories of commutative and associative algebras (under smash product) within these two categories. They are bicomplete, and the evident forgetful functors have left adjoints of the form

$$X \mapsto Sym(X) = \bigvee_{k\geq 0} Sym^k(X) \quad \text{and} \quad X \mapsto T(X) = \bigvee_{k\geq 0} X^{(k)}.$$

One can define left and right modules M over such algebras A. The left modules form a category $\mathcal{M}_A$. When A is commutative, the category $\mathcal{M}_A$ inherits a symmetric monoidal product $M \wedge_A N$ defined by the reflexive coequalizer diagram

$$M \wedge A \wedge N \rightrightarrows M \wedge N \to M \wedge_A N.$$

Let $H \subset G$ be a subgroup of index m. We have seen (3/31) that there is a norm functor $N_H^G\colon \mathcal{S}^H \to \mathcal{S}^G$ that is right adjoint to the forgetful functor. We also have $N_H^G\colon \boldsymbol{Comm}^H \to \boldsymbol{Comm}^G$ that is *left* adjoint to the forgetful functor.

We know what a weak equivalence of G-spectra is, namely a map that induces an isomorphisms of homotopy groups on all of the fixed point sets. It is less clear what the fibrations and cofibrations should be. There is a theory that helps here, namely that of **homotopical categories**, not to be confused with homotopy categories. Roughly speaking they are categories with weak equivalences waiting for model structures. They were introduced in 2004 by Kan and 3 of his former students in

Homotopy Limit Functors on Model Categories and Homotopical Categories,

and are treated in Chapter 2 of Riehl's book, Categorical homotopy theory

Def. A homotopical category $\mathcal{M}$ is a category with a wide (includes all object and all isomorphisms) subcategory $\mathcal{W}$ satisfying the 2-of-6 condition: Given a diagram

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

In $\mathcal{M}$, with gf and hg are in $\mathcal{W}$, then so are f, g, h, and hgf.

Its homotopy category $Ho(\mathcal{M})$ is obtained by formally inverting all weak equivalences. There is a functor $\gamma\colon \mathcal{M} \to Ho(\mathcal{M})$. A functor $F\colon \mathcal{M} \to \mathcal{N}$ between homotopical categories may or may not preserve weak equivalences.

Def. (2.2.1 of Riehl) A <u>left deformation</u> on a homotopical category $\mathcal{M}$ is a endofunctor

$Q: \mathcal{M} \to \mathcal{M}$ with a natural transformation $q: Q \Rightarrow 1$. Let $\mathcal{M}_Q$ be the subcategory of objects defined by Q, the "cofibrant" objects.

Example. Let $\mathcal{M}$ be a model category with functorial cofibrant replacement Q.

Def. (Riehl 2.2.4) A <u>left deformation for a functor $F: \mathcal{M} \to \mathcal{N}$</u> between homotopical categories is a left deformation Q such that F is homotopical on the subcategory $\mathcal{M}_Q$. If such exists, we say F is <u>left deformable</u>.

Def (2.2.1 of Hovey) Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. A functor $F: \mathcal{C} \to \mathcal{D}$ is <u>left Quillen</u> if it is a left adjoint and preserves cofibrations and acyclic cofibrations. A right Quillen functor is …. A pair of adjoint functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

Is a <u>Quillen adjunction</u> if F is left Quillen. (This makes G a right Quillen functor.)

Example:

$$|-| : sSet \rightleftarrows Top : Sing$$

Ken Brown's Lemma. A left (right)Quillen functor preserves weak equivalences of cofibrant (fibrant) objects.

When a functor F is left deformable, there is a maximal subcategory on which it is homotopical.