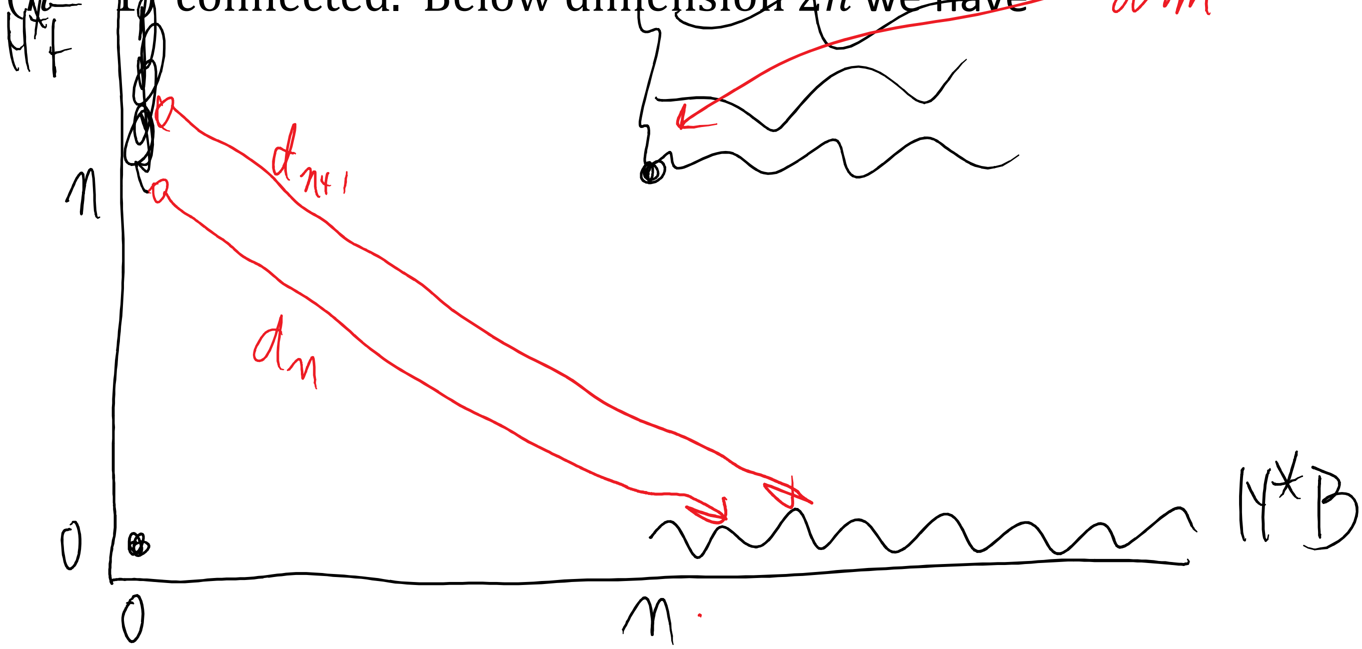
Let $F \to E \to B$ be a fiber sequence where F and B are $(n-1)$-connected. Below dimension $2n$ we have



The only differential in this range go from the x-axis to the y-axis. Such differentials are called *transgressions*. We get a long exact sequence

$$0 \to H^nB \to H^nE \to H^nF \to H^{n+1}B \to H^{n+1}E \to H^{n+1}F \to H^{n+2}B \ldots$$

This LES looks like the one for the cofiber sequence $F \to E \to E/F$. There is a map $\frac{E}{F} \to B$. This leads to a map between long exact sequences below dimension 2n and we see that in this range, B and E/F have the same cohomology.

In the world of connective spectra, we have this in all nonnegative dimensions. In this world, fiber sequences and cofiber sequences coincide. This is a good thing!

Back to mod 2 Eilenberg-Mac Lane spaces. Let $K_n = K(Z/2\,, n)$. We want to know $H^*(K_n,\ Z/2\ )$ for all n. For $n = 1$ we know $H^*\ K_1 = P(x)$ where $x \in H^1$ and P denotes polynomial algebra. $K_1 = RP^\infty$. In the Serre spectral sequence for $K_1 \to * \to K_2$ we see that $H^* K_2 = P(x_2, x_3, x_5, \dots .)$.

Definition. A *simple system of generators* for a graded $Z/2$-algebra R is a collection $\{x_1, x_2, \dots\}$ such that the set of all products of the form $x_{i_1} x_{i_2} \dots .$ where $i_1 < i_2 < i_3 \dots$ is a basis for R.

Example. A simple system of generators for $P(x)$ is $\{x^{2^i} : i \geq 0\}$

Theorem (Borel 1953) Let $F \to E \to B$ be a fiber sequence where

(i) $H^*E = H^*\ (point)$.

(ii) H^*F has a simple system of generators $\{x_1, x_2, \dots\}$ with $|x_i| =: n_i$.

Then $H^*B = P(y_1, y_2, \dots)$ where $|y_i| = 1 + n_i$.

This was proved using the Serre spectral sequence.

We saw this in the relations between H^*K_1 and H^*K_2.

Steenrod operations: For each $i \geq 0$ we have homomorphisms

$Sq^i \colon H^n(X, Y) \to\ H^{n+i}(X, Y)$ with the following porperties:

(i) Naturality.

(ii) Commutes with connecting homomorphism δ.

(iii) (Cartan formula) $Sq^i(ab) = \sum_{0 \leq k \leq i} Sq^k(a) Sq^{i-k}(b)$

(iv) If $i = |x|$, then $Sq^i\ x = x^2$ and if $i > |x|$, then $Sq^i\ x = 0$.

(v) $Sq^0 x = x$.

(vi) Sq^1 is the Bockstein map for the sequence

$$0 \to Z/2 \to Z/4 \to Z/2 \to 0\,.$$

These can be iterated. For a sequence of nonnegative integers
$I = (i_1, i_2, \dots, i_r)$, let $Sq^I = Sq^{i_1}\, Sq^{i_2} \dots$.
We will say that I is admissible if $i_k \geq 2i_{k+1}$ for each k. Define
$n(I) = i_1 + i_2 + \cdots$ and the excess

$$e(I) = 2i_1 - n(I) = (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + i_r.$$

Suppose that $H^*F = P(z_1, z_2, \dots)$ with $n_i = |z_i|$. Let
$L(a, r) = (2^{r-1}a \dots, 2a, a)$. Then $Sq^{L(n_i,r)}(z_i) = z_i^{2^r}$.
A simple system of generators for H^*F is

$$\{Sq^{L(n_i,r)}(z_i) \colon i = 1,2, \dots; r \geq 0\}.$$