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Recall $\underline{\mathcal{T}_G}$ denotes the TGC of pointed G-spaces and nonequivariant maps.
$\mathcal{J}_G$ denotes the TGC whose obects are finite dimensional orthogonal representations of G and
$\mathcal{J}_G(V,W) = Thom(O(V,W); W-V)$ where $O(V,W)$ is the Stiefel manifold of orthogonal embeddings $V \to W$. For each such embedding, $W-V$ denotes the orthogonal complement of V in W.
If $\dim(W) < \dim(V)$, there are no embeddings and $\mathcal{J}_G(V,W) = *$.
If $\dim(W) = \dim(V)$, $\mathcal{J}_G(V,W) = O(V,W)_+$.
If $\dim(W) > \dim(V)$ then the space is connected.

One has composition maps
$\mathcal{J}_G(V,W) \wedge \mathcal{J}_G(U,V) \to \mathcal{J}_G(U,W)$

A G-spectrum is a functor $X: \mathcal{J}_G \to \underline{\mathcal{T}_G}$. This means we have pointed G-spaces X_V and structure maps $\mathcal{J}_G(V,W) \wedge X_V \to X_W$.

$\mathcal{S}^G$ denotes the category of G-spectra and nonequivariant maps. $\mathcal{S}_G$ denotes the category of G-spectra and and G-maps.
Since a spectrum is a functor, a map between spectra is a natural transformation. These categories are complete and cocomplete, meaning they are have all small limits and colimits.

For a spectrum X and space A we define a spectrum $X \wedge A$ by $(X \wedge A)_V = X_V \wedge A$.

The spectrum S^{-V} is defined by $(S^{-V})_W = \mathcal{J}_G(V,W)$. With structure maps being the composition maps above.

Yoneda Lemma: Let C be a category with an object A. Then $h^A := C(A,-)$ is a covariant set valued functor on C. Let F be another such functor. Then the set of natural tarsnformations from h^A to F is $F(A)$.

If C is enriched in some way, then the functors h^A and F will take values in the new category, eg $\underline{\mathcal{T}_G}$.

Application: $C = \mathcal{J}_G$, $A = V$ and $F = X$. The "set" of natural transformations is the space of spectrum maps $S^{-V} \to X$ is X_V.

Reflexive coequalizers: Consider the category W with two objects A and B and three morphisms $f, g: A \to B$ and $s: B \to A$ with $fs = gs = 1_B$. A functor $W \to C$ is a diagram in C which may have a colimit called the reflexive coequalizer.

Example: $C = Ab$ and $A_n = Z$ for $n \geq 0$. Let there be a map $A_n \to A_{n+1}$ that is multiplication by 2. The colimit of

$\mathfrak{Cat}_G$

symmetric monoidal category

$\underline{\mathcal{T}_G}$

$(\mathcal{T}^G, \wedge, S^0)$

$(\mathcal{T}, \wedge, S^0)$

$\mathcal{T}^G$

$\mathcal{J}_G$

$\mathcal{S}^G$

$\mathcal{S}_G$

$A_0 \to A_1 \to A_2 \to \cdots$ is $Z[\frac{1}{2}]$. It can also be described as a reflexive coequalizer as follows.

$A = \bigoplus_{m,n\geq 0} A_{m,n}$ where $A_{m,n} = Z$
$B = \bigoplus_{m\geq 0} A_m$ where $A_m = Z$
$f: A_{m,n} \to A_{m+n}$ via multiplication by 2^n
$g: A_{m,n} \to A_m$ via multiplication by 1.
$s: A_m \to A_{m,0}$ via multiplication by 1.
Then the reflexive coequalizer is $Z[\frac{1}{2}]$.

Here is a diagram whose reflexive coequalizer is a spectrum X.

$$A = \bigvee_{V,W} S^{-W} \wedge \mathcal{J}_G(V,W) \wedge X_V$$

$$B = \bigvee_{V} S^{-V} \wedge X_V$$

$f = i \wedge X_V$ where $i: S^{-W} \wedge \mathcal{J}_G(V,W) \to S^{-V}$ is the structure map. On U its is
$\mathcal{J}_G(W,U) \wedge \mathcal{J}_G(V,W) \to \mathcal{J}_G(V,U)$, the composition map.

$g = S^{-w} \wedge j$ where $j: \mathcal{J}_G(V,W) \wedge X_V \to X_W$ is the structure map for X.
$g: S^{-W} \wedge \mathcal{J}_G(V,W) \wedge X_V \; to \; S^{-W} \wedge X_W$.

$$A_U = \bigvee_{V,W} \mathcal{J}_G(W,U) \wedge \mathcal{J}_G(V,W) \wedge X_V$$

$$B_U = \bigvee_{V} \mathcal{J}_G(V,U) \wedge X_V$$

The coequalizer is X_U. Hence the spectrum coequalizer is X.

This can be written as
$X = \mathrm{colim}_V\, S^{-V} \wedge X_V$, a colimit of "desuspension" spectra. The is the tautological presentation of X.