

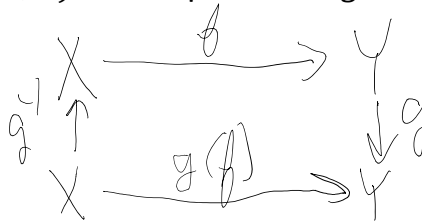
Some categorical notions

1. Enrichment, I. In a category C one has a set of morphisms for each pair of objects. This set may have some additional structure that is natural in C .

(i) $C = Ab$, the category of abelian groups or R -modules. Then $Ab(X, Y)$ is an abelian group in a natural way.

(ii) For $C = Top$, the category of (pointed) topological spaces, $Top(X, Y)$ has the compact open topology.

(iii) Let G be a group and let T^G be the category of pointed G -spaces. The space of (non-equivariant) continuous maps $T^G(X, Y)$ is a G -space. For $g \in G$ and $f: X \rightarrow Y$, we define $g(f): X \rightarrow Y$ by



T^G is enriched over itself. Note that the space of equivariant maps from X to Y is $(T^G(X, Y))^G$, a space without a G -action.

2. Adjoint functors. Let C and D be categories with functors $F: C \rightarrow D$ and $G: D \rightarrow C$. Let X and Y be objects of C and D respectively. Then if

$$D(F(X), Y) = C(X, G(Y)),$$

we say that F is the left adjoint of G and G is the right adjoint of F .

Example: (a) $C = Set, D = Ab$, and $G: D \rightarrow C$ is the forgetful functor. Then F is the free abelian group functor.

(b) Let $H \subset G$ be a subgroup. Let T^H and T^G be the categories as above. Let $i_H^*: T^G \rightarrow T^H$ be the forgetful or restriction functor. It has both a left adjoint L and a right adjoint R , where for an H -space Y ,

$$L(Y) = G_+ \wedge_H Y = (G/H)_+ \wedge Y$$

where G_+ is G with a disjoint base point and $G_+ \wedge_H Y$ denotes the orbit space of $G_+ \wedge Y$ and

$$R(Y) = T^H(G_+, Y) = \prod_W Y \text{ where } W = |G/H|, \text{ where } G \text{ permutes the factors, each of}$$

which is H -invariant.

3. A symmetric monoidal category (SMC) is a category C equipped with a map $C \times C \rightarrow C$ with natural associativity isomorphisms $(X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, natural symmetry isomorphisms $X \otimes Y \rightarrow Y \otimes X$ and a unit object 1 with unit isomorphisms $\iota_X: 1 \otimes X \rightarrow X$. The monoidal structure is closed if the functor $A \otimes (-)$ has a right adjoint $(-)^A$, the internal Hom with

$$C(1, X^A) = C(A, X).$$

4. A fancier definition of an enriched category. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, 1)$ be an SMC. A \mathcal{V} -category C has

a collection of objects $ob(C)$ and for each pair of objects X, Y an object $C(X, Y)$ in \mathcal{V} . For each object X in C we have a morphism $1 \rightarrow C(X, X)$ in \mathcal{V} . We have composition

$$C(Y, Z) \otimes C(X, Y) \rightarrow C(X, Z)$$

A functor $F: C \rightarrow D$ between \mathcal{V} -categories consists of a function $F: ob(C) \rightarrow ob(D)$ and each pair of objects X and Y in C , $C(X, Y) \rightarrow D(FX, FY)$ with naturality conditions.

See Appendix A of http://www.math.rochester.edu/people/faculty/doug/kervaire_061114.pdf.