Some categorical notions

1. Enrichment, I. In a category C one has a set of morphisms for each pair of objects. This set may have some additional structure that is natural in C.
   (i) $C = Ab$, the category of abelian groups or R-modules. Then $Ab(X, Y)$ is an abelian group in a natural way.
   (ii) For $C = Top$, the category of (pointed) topological spaces, $Top(X, Y)$ has the compact open topology.
   (iiI) Let G be a group and let T^G be the category of pointed G-spaces. The space of (non-equivariant) continuous maps $T^G(X, Y)$ is a G-space. For $g \in G$ and $f: X \to Y$, we define $g(f): X \to Y$ by

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "g"] \\ X \arrow[u, "g^{-1}"] \arrow[r, "g(f)"] & Y \end{tikzcd}$$

T^G is enriched over itself. Note that the space of equivariant maps from X to Y is $(T^G(X, Y))^\wedge G$, a space without a G-action.

2. Adjoint functors. Let C and D be categories with functors $F: C \to D$ and $G: D \to C$. Let X and Y be objects of C and D respectively. Then if
   $$D(F(X), Y) = C(X, G(Y)),$$
   we say that F is the left adjoint of G and G is the right adjoint of F.
   Example: (a) $C = Set, D = Ab$, and $G: D \to C$ is the forgetful functor. Then F is the free abelian group functor.
   (b) Let $H \subset G$ be a subgroup. Let T^H and T^G be the categories as above. Let $i_H^*: T^G \to T^H$ be the forgetful or restriction functor. It has both a left adjoint L and a right adjoint R, where for an H-space Y,
$$L(Y) = G_+ \wedge_H Y = (G/H)_+ \wedge Y$$
where G_+ is G with a disjoint base point and $G_+ \wedge_H Y$ denotes the orbit space of $G_+ \wedge_{\square} Y$ and
$$R(Y) = T^H(G_+, Y) = \prod_W Y \text{ where } W = \left|\frac{G}{H}\right|$$, where G permutes the factors, each of which is H-invariant.

3. A <u>symmetric monoidal category</u> (SMC)is a category C equipped with a map $C \times C \to C$ with natural associativity isomorphisms $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, natural symmetry isomorphisms $X \otimes Y \to Y \otimes X$ and a unit object 1 with unit isomorphisms $\iota_X: 1 \otimes X \to X$. The monoidal structure is <u>closed</u> if the functor $A \otimes (-)$ has a right adjoint $(-)^A$, the internal Hom with
   $$C(1, X^A) = C(A, X).$$

4. A fancier definition of an enriched category. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, 1)$ be an SMC. A $\mathcal{V}$-category C has

a collection of objects $ob(C)$ and for each pair of objects X, Y an object $C(X,Y)$ in $\mathcal{V}$. For each object X in C we have a morphism $1 \to C(X,X)$ in $\mathcal{V}$. We have composition

$$C(Y,Z) \otimes C(X,Y) \to C(X,Z)$$

A functor $F: C \to D$ between $\mathcal{V}$-categories consists of a function $F: ob(C) \to ob(D)$ and each pair of obects X and Y in C, $C(X,Y) \to D(FX,FY)$ with naturality conditions.

See Appendix A of http://www.math.rochester.edu/people/faculty/doug/kervaire_061114.pdf.