The Serre spectral sequence:

Let $F \to E \to B$ be a fiber bundle of CW-complexes. Suppose we know the homologies or cohomologies of F and B, and we want to find that of E.

Consider the bigraded group $E_2^{i,j} = H^i(B; H^j F)$. This group vanishes if $i < 0$ or $j < 0$.

It turns out that for each bidegree there is homomorphism

$d_2: E_2^{i,j} \to E_2^{i+2,j-1}$ with $d_2 d_2 = 0$,

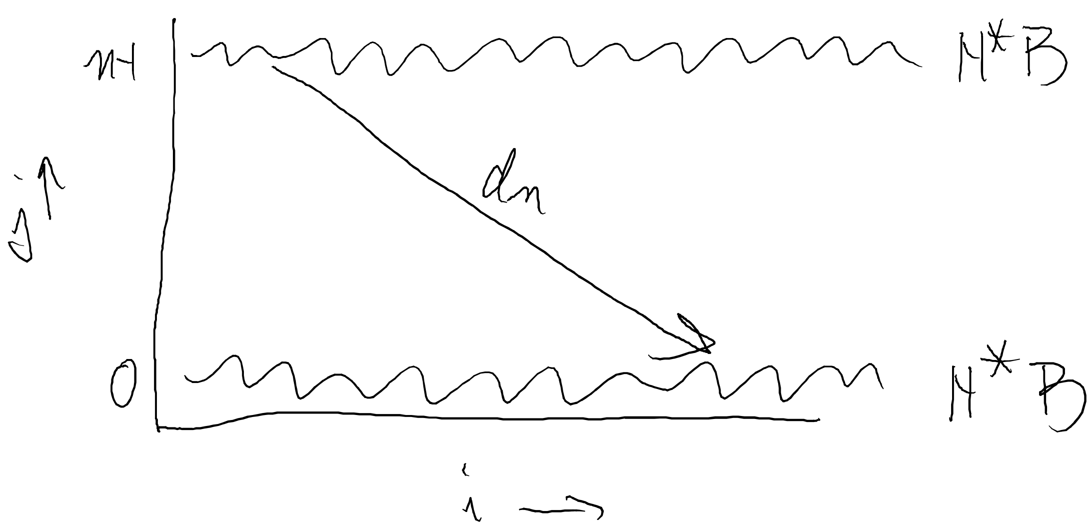
so we have a bigraded cochain complex. We denote its cohomology by $E_3^{i,j}$. Inductively we have maps

$d_r: E_r^{i,j} \to E_r^{i+r,j-r+1}$ with $d_r d_r = 0$,

so we have a bigraded cochain complex. We denote its cohomology by $E_{r+1}^{i,j}$. Note that $E_2^{i,j} = 0$ if either i or j is negative. This means that for fixed i, j, for $r \gg 0$, the incoming and outgoing d_r are both trivial, so $E_r^{i,j} = E_\infty^{i,j}$.

EXAMPLE

Suppose $F = S^{n-1}$. This means that $E_2^{i,0} = E_2^{i,n-1} = H^i B$ and $E_2^{i,j} = 0$ for other values of j. The only possible nontrivial differential is d_n.

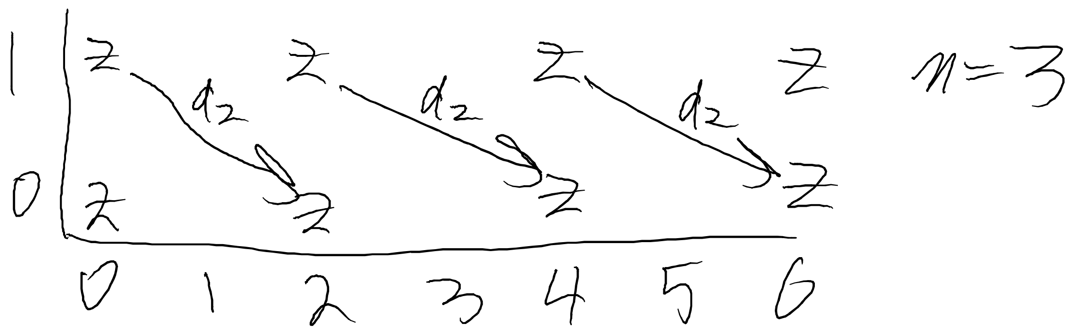


$$H^iB = E_n^{i,n-1} \xrightarrow{d_n} E_n^{i+n,0} = H^{i+n}B$$

Suppose our sphere bundle is the unit sphere bundle for an n-plane bundle ξ over B. Then it tutrn out that d_n is multiplication (via cup product) by a class $e(\xi) \in H^n B$ called the Euler class of \xi. The kernel of d_n is $E_{n+1}^{i,n-1}$ and the cokernel is $E_{n+1}^{i+n,0}$. Furthermore, there are no more differentials, so $E_{n+1} = E_\infty$. We get a long exact sequence

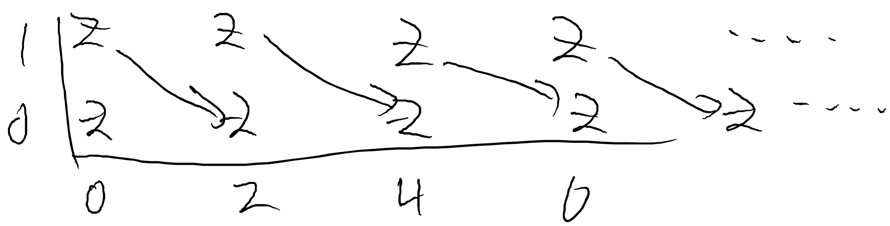
$$\ldots \to H^{k-n}B \xrightarrow{d_n} H^k B \to H^k E \to H^{k-n-1}B \to \cdots$$

Specific example. $S^1 \to S^{2n+1} \to CP^n$. Think of S^{2n+1} as the unit sphere in C^{n+1}. The unit circle acts on it by scalar multiplication. The orbit is CP^n, the space of complex lines thru the origin.

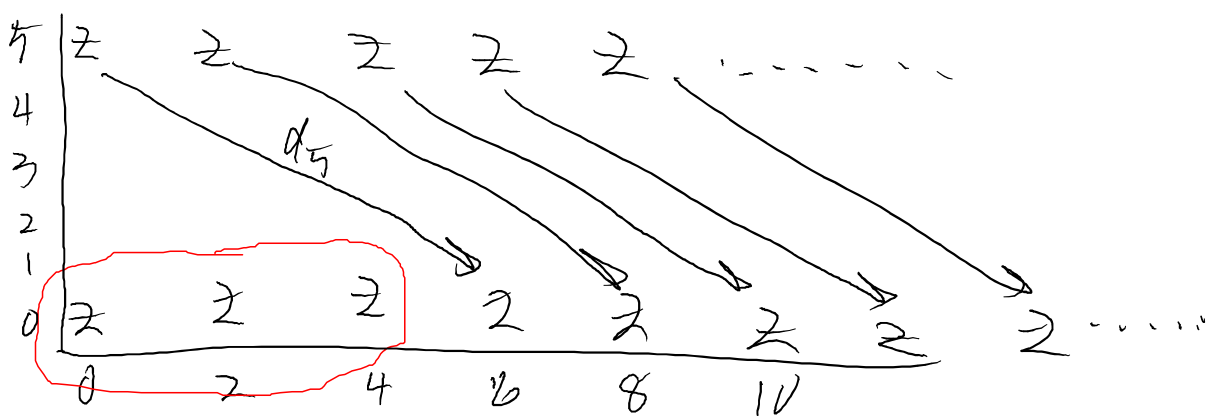


This leads to $H^*E = H^*S^7$.

Consider the canonical complex line bundle λ over CP^∞. $H^*CP^\infty = Z[x]$ where $x \in H^2$. The Euler class is x. The fiber sequence is $S^1 \to S^\infty \to CP^\infty$.

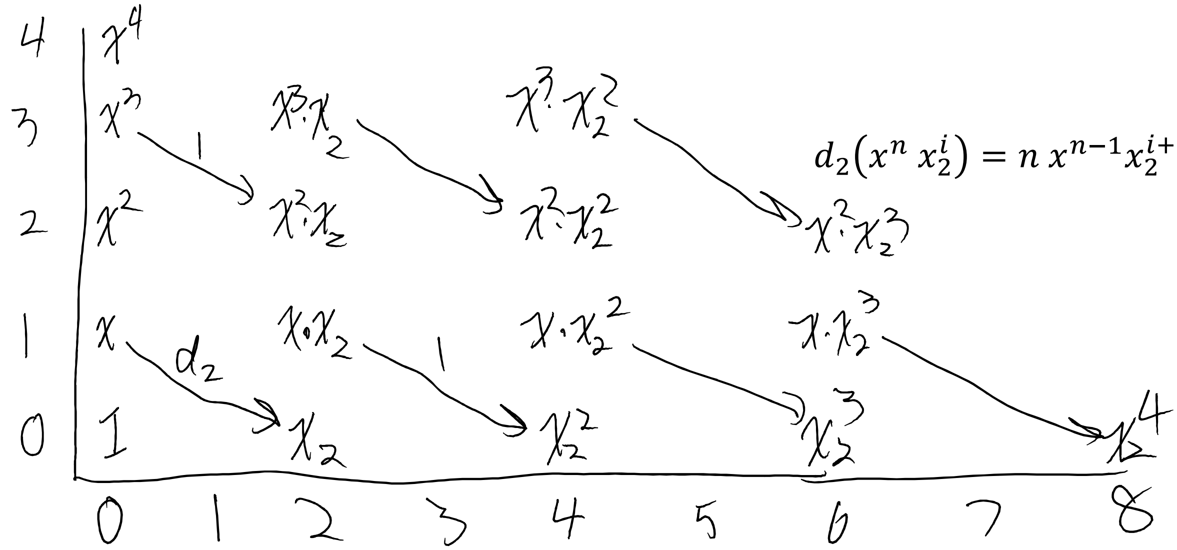


Conclusion is that $H^*S^\infty = H^*(point)$. Consideer the n-fold direct sum of λ with itself. Then $e(n\lambda) = e(\lambda)^n = x^n$. For $n = 3$ we get $S^5 \to E \to CP^\infty$.



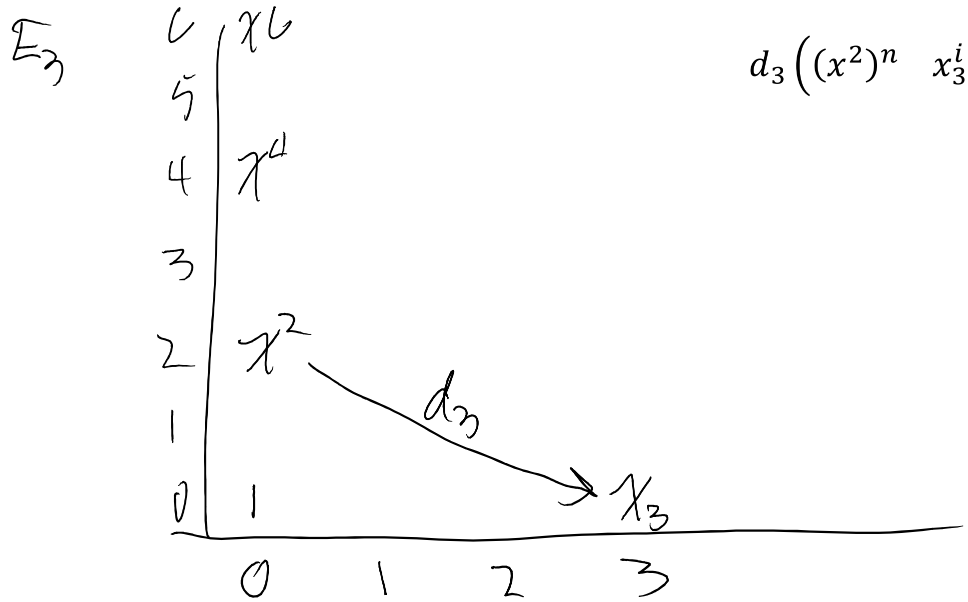
Conclusion : $H^*E = H^*CP^2$. Exercise: show that E is homotopy equivalent to CP^2.

EXAMPLE 2. $K(Z/2,1) = F \to E \to K(Z/2,2) = B$ where E is the path pace of B and $F = \Omega B$. We know that $K(Z/2,1) = RP^\infty$ so its mod 2 cohomology is $Z/2[x]$ where $x \in H^1$ and E has the cohomology of a point.



$d_2(x^n\, x_2^i) = n\, x^{n-1} x_2^{i+1}$.

d_2 is a derivation with respect to cup product, i.e., it behaves like the product rule in calculus. $d_2(x^n) = n\, x^{n-1} d_2(x) = n\, x^{n-1}\, x_2$. This vansihses for even n but not for odd n.



$d_3\left((x^2)^n \quad x_3^i\right) = n\,(x^2)^{n-1} x_3^{i+1}$.

Conclusion: $H^*(K\left(\frac{Z}{2},2\right) = F_2[x_2, x_3, x_5, x_9, \ldots\ldots]$ where $x_{1+2^i} \in H^{1+2^i}$.