MATH 549 4-5-10

Note Title 4/5/2010

$\pi_*^u MU^{(4)} = \mathbb{Z}[m_i(j) : i > 0, \ 1 \le j \le 4]$ $\quad m_1(j) \in \pi_{2,1}$

The gen $\gamma \in C_8$ sends $\quad m_i(j) \mapsto \begin{cases} m_i(j+1) & \text{for } j \ne 4 \\ (-1)^i m_i(1) & \text{for } j = 4 \end{cases}$

$\gamma^4 m_i(j) = (-1)^i m_i(j)$

Recall in $\pi_4^u(MU^{(4)})$ we have 4 orbits of the C_8-action

$\hat{S}(p_4)$ $\quad \{m_1(1)\, m_1(3),\ m_1(2)\, m_1(4)\}$

$\hat{S}(2p_2)$ $\left\{\begin{array}{l} \{m_2(j)\} \quad \{m_1(j)^2\} \\ \{m_1(1)\, m_1(2),\ m_1(2) m_1(3),\ m_1(3) m_1(4),\ -m_1(4)\, m_1(1)\} \end{array}\right.$

where $\tilde{S}(m\rho_h) = G_+ \wedge_H S^{m\rho_h}$ $h = 1, 2, 4, 8$

ρ_h = reg rep of C_h

Recall our general strategy

invert something in $\pi_\star^{G} MU^{(4)}$ $G = C_8$

Call the new spectrum $\tilde{\Omega}$

Prove 4 thms

1) $\tilde{\Omega}^{hG}$ has periodicity

2) $\tilde{\Omega}^{G}$ has gap

3) $\tilde{\Omega}^{hG} \simeq \Omega^{G}$ fixed pt thm

4) $\tilde{\Omega}^{hG}$ detects all θ_j's

DETECTION

$\pi_*(\tilde{\Omega}^{hG})$ can be computed by homotopy fixed pt SS (~1995)

$\pi_*(\tilde{\Omega}^{G})$ can be computed by slice SS (2009)

In $MU^{(4)}$ the degree 4 slice is

$$\left(\hat{S}(\rho_4) \vee \hat{S}(2\rho_2) \vee \hat{S}(2\rho_2) \vee \hat{S}(2\rho_2)\right) \wedge H\underline{\mathbb{Z}}$$

You can work out the other slices by similar calculations.

The degree 8 slice has a summand of the $S^{\rho_8} \wedge H\mathbb{Z}$

We never get an orbit with 8 elements, so $\tilde{S}(m\rho_1) = G_+ \wedge S^m$ never appears in our calculation.

We need to calculate

$$\pi_*^G\left(\tilde{S}(m\rho_h) \wedge H\mathbb{Z}\right) \text{ for } h = 2, 4, 8$$

Recall $\hat{S}(m\rho_h) = G_+ \wedge_H S^{m\rho_h}$ so

$$\pi_*^G(\hat{S}(m\rho_h) \wedge H\underline{\mathbb{Z}}) = \pi_*^H(S^{m\rho_h} \wedge H\underline{\mathbb{Z}})$$

$$\hat{S}(m\rho_h)^G = (S^{m\rho_h})^H$$

Will compute $\pi_*(S^{\rho_8} \wedge H\underline{\mathbb{Z}})$. Note

$$(S^{\rho_8})^H = \begin{cases} S^1 & H = C_8 \\ S^2 & H = C_4 \\ S^4 & H = C_2 \\ S^8 & H = \{e\} \end{cases}$$

We have a chain cx of $\mathbb{Z}[C_8]$-modules

$$\begin{array}{ccccccccccccccc} 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \\ \mathbb{Z} & \xleftarrow{\varepsilon} & \mathbb{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbb{Z}[C_4] & \xleftarrow[1+\gamma]{} & \mathbb{Z}[C_4] & \xleftarrow[(1-\gamma)(1+\gamma^2)]{} & \mathbb{Z}[C_8] & \xleftarrow{1+\gamma} & \mathbb{Z}[C_8] & \xleftarrow[(1-\gamma)(1+\gamma^2)(1+\gamma^4)]{} & \mathbb{Z}[C_8] & \xleftarrow{1+\gamma} & \mathbb{Z}[C_8] \end{array}$$

C:

The boundary is uniquely determined by

$$H_*(C) = \overline{H}_*(S^8)$$

Passing to fixed pts gives

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z}$$

This determines $\pi_*^G(S^{\rho_8}, H\mathbb{Z})$

$$(S^{2\rho_8})^H = \begin{cases} S^2 & H=C_8 \quad \mathbb{Z} \\ S^4 & H=C_4 \quad \mathbb{Z}[C_8/C_4] = \mathbb{Z}^2 \\ S^8 & H=C_2 \quad \mathbb{Z}[C_8/C_2] = \mathbb{Z}^4 \\ S^{16} & H=\{e\} \quad \mathbb{Z}[C_8] = \mathbb{Z}^8 \end{cases}$$

$$\begin{array}{cccccccccccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$

$$\mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}^2 \xleftarrow{1-\gamma} \mathbb{Z}^2 \xleftarrow{1+\gamma} \mathbb{Z}^4 \xleftarrow{1-\gamma} \mathbb{Z}^4 \leftarrow \mathbb{Z}^4 \xleftarrow{1-\gamma} \mathbb{Z}^4 \xleftarrow[T_4]{} \mathbb{Z}^8 \xleftarrow{1-\gamma} \mathbb{Z}^8 \leftarrow \mathbb{Z}^8 \xleftarrow{1-\gamma} \mathbb{Z}^8 \xleftarrow[T_8]{} \mathbb{Z}^8 \leftarrow \cdots$$

$$T_4 = (1+\gamma)(1+\gamma^2) \qquad T_8 = (1+\gamma)(1+\gamma^2)(1+\gamma^4)$$

Fixed pts

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{8} \mathbb{Z} \leftarrow \cdots$$

Sketch the proof of the periodicity thm for larger (than C_2) cyclic gps.

Facts we used in C_2 case

$$\Phi^{C_2} MU_{\mathbb{R}} = MO \Rightarrow d_n\, u^{2^k} = \sim$$

$$\Phi^{C_4} MU^{(2)} = MO \qquad \Phi^{C_2} MU^{(2)} = MO^{(2)}$$

$$\Phi^{C_8} MU^{(4)} = MO, \quad \Phi^{C_4} MU^{(4)} = MO^{(2)}, \quad \Phi^{C_2} MU^{(4)} = MO^{(4)}$$

This also forces certain differentials

in the slice SS

Slice Differentials Thm. In the slice SS for $MU^{(4)}$ (as a C_8-spectrum) we have

$$d_{1+8(2^k-1)}\, u_{2^k\sigma} = a_\sigma^{2^k} f_{2^k-1} \quad \text{for } k \geq 1$$

σ = sign rep of C_8 $\qquad u_{2^k\sigma} = u_{2\sigma}^{2^{k-1}}$

$f_i \in \pi_i^{G} MU^{(4)}$

$$S^i \xrightarrow{a_{i\rho_8}} S^{i\rho_8} \xrightarrow{m_i(1)\cdots m_i(4)} MU^{(4)}$$

Applying the functor Φ^G to this map gives

$$S^i \longrightarrow S^i \xrightarrow{y_i} MO$$

where $\pi_* MO = \mathbb{Z}/2[y_2, y_4, y_5 \cdots]$

y_j for $j \neq 2^k - 1$

$$y_{2^k-1} = 0$$

Hence f_1 must be killed by some power of a_6

$$\pi^G_\star MU^{(4)} \wedge \tilde{E}C_2 = a_6^{-1} \pi_\star MU^{(4)}$$

$$\pi^G_*(\qquad) = \pi_* MO$$

As before we can show that inverting

$\Delta_k^{(8)} = m_{2^k-1}(1) \cdots m_{2^k-1}(4)$ makes

$u_{2^{k+1}\sigma}$ a permanent cycle

$$u_{2^k\sigma} \to f_{2^k-1} = a_\sigma^{2^k} \Delta_k^{(8)}$$

We need to make some power of

$u_{2\rho_8}$ a permanent cycle

What is ρ_8 and $RO(C_8)$?

The irreducible orthogonal reps of C_8 are

1 trivial action on $\mathbb{R}$

σ_8 sign "

λ_j rotation of $\mathbb{R}^2$ by angle $\pi j/4$

$$j = 1, 2, 3$$

$$\rho_8 = 1 + \sigma + \lambda_1 + \lambda_2 + \lambda_3$$

If V is a rep of a subgp H, then $\mathrm{ind}_H^G V$, the induced rep of G is

$$W = V \otimes_{\mathbb{R}[H]} \mathbb{R}[G]$$

$$\dim W = (|G|/|H|) \dim V$$

What is $RO(C_4)$?

1 $\quad$ $\mathbb{R}$ with trivial action

σ_4 $\quad$ $\mathbb{R}$ " sign "

λ $\qquad \mathbb{R}^2$ with rotation by $\pi/2$

$\rho_4 = 1 + \sigma_4 + \lambda$

Also $\rho_2 = 1 + \sigma_2$

$\mathrm{Ind}_2^4 (1) = 1 + \sigma_4 \qquad \mathrm{Ind}_2^4 (\sigma_2) = \lambda$

$$\left.\begin{array}{l} \mathrm{Ind}_4^8 (1) = 1 + \sigma_8 \\ \mathrm{Ind}_4^8 (\sigma_4) = \lambda_2 \\ \mathrm{Ind}_4^8 (\lambda) = \lambda_1 + \lambda_3 \end{array}\right\} \mathrm{Ind}_4^8 (\rho_4) = \rho_8$$