MATH 542 4-28-10

Note Title 4/28/2010

Reference section 7 of HHR preprint online

<u>Reduction Thm</u>

$$\begin{array}{ccc} R(\infty) & \xrightarrow{f} & H\underline{\mathbb{Z}} \\ \| & & \\ MU^{(g/2)} \underset{A}{\wedge} S^0 & & \end{array}$$

is a G-equiv

$G = C_{2^n}$ $g = |G|$

A = certain wedge of spheres.

This implies slice Theorem (4/22)

Suffices to show $\underline{\Phi}^G f$ is equiv (4/26)

We saw that

$\pi_\star \Phi^G R(\infty)$ and $\pi_\star \Phi^G H\mathbb{Z} = \mathbb{Z}/2[b]$

are abstractly iso $\qquad b = a_\sigma^{-2} u_{2\sigma} \in \pi_2$

Some equivariant tools.

Let $G' \subset G$ be the subgp of index 2

W = rep of G

The transfer

$$\pi_W^{G'} X \xrightarrow{\mathrm{Tr}} \pi_W^{G} X$$

$$\| $$

$$\pi_W^{G}(C_{2+} \wedge X) = \pi_W^{G}(G_+ \underset{G'}{\wedge} X) \qquad C_2 = G/G'$$

induced by the equiv map $C_{2+} \longrightarrow S^0$

Lemma 7.16 Let $\varepsilon = \pm 1$ be $\det(\gamma)$ for the rep W, where γ is a generator of G.

$$\begin{array}{ccc} S^{|W|} & \xrightarrow{a} & X \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\gamma} \\ S^{|W|} & \xrightarrow{\varepsilon\gamma a} & X \end{array}$$

$$\begin{array}{ccccc} [G/G'_+ \wedge S^W, X]^G = \pi^{G'}_W X & \xrightarrow{T_r} & \pi^G_W X \\ \downarrow & & \downarrow \\ \pi^u_{|W|}(X) \oplus \pi^u_{|W|} X \supset \pi^u_{|W|} X & \xrightarrow{1+\varepsilon\gamma} & \pi^u_{|W|} X \end{array}$$

$$(\varepsilon\gamma b, \varepsilon\gamma a) \xleftarrow{\gamma} (a, b) \longmapsto a + b$$

$$(\gamma^2 a, \varepsilon\gamma a) \longleftarrow (a, \varepsilon\gamma a) \longmapsto (1+\varepsilon\gamma) a$$

γ^2 gen's G'

If $\gamma^2 a = a$ then $(a, \varepsilon\gamma a)$ is invariant. QED

Remark 2.29 Isotropy separation seq

$$E C_{2+} \longrightarrow S^0 \longrightarrow \tilde{E} C_2$$

G acts via G/G'

G action trivial

For any proper subgp $H \subset G$, $\tilde{E}C_2$ is H-contractible. Let T be a G-CWcx with $T^G =: T_0$. T is obtained from T_0 by attaching cells of the form $G_+ \wedge_H D^n$. It follows that the map $T_0 \longrightarrow T$ induces an iso

$$[T, \tilde{E}C_2 \wedge X]^G \xrightarrow{\simeq} [T_0, \tilde{E}C_2 \wedge X]^G$$

e.g. $T = S^{W}$

$$\pi^G_W \widetilde{EC_2} \wedge X \xrightarrow{\sim} \pi^G_{W_0} \widetilde{EC_2} \wedge X$$

$$W_0 = W^G \qquad \pi^G_{W_0}\widetilde{EC_2}\wedge X = \pi_{|W_0|} \Phi^G X$$

Back to main story

$$\pi_* \Phi^G R(\infty) \longrightarrow \pi_* \Phi^G H\underline{\mathbb{Z}} = \mathbb{Z}/2[b]$$

$b \in \pi_2$

suffices to show this is onto

<u>Lemma 7.5</u> If for each $l \geq 1$, $b^{2^{(l-1)}} \in \pi_{2^l}$ is in the image of

$$\pi_{2^l} \Phi^G MU^{(g/2)} / G \cdot \bar{r}_{2^l - 1} \longrightarrow \pi_{2^l} \Phi^G H\underline{\mathbb{Z}}$$

then our map is onto.

Pf Recall $\pi_*^u MU^{(g/2)} = \mathbb{Z}[\gamma^j \bar{r}_k : k \geq 0, 0 \leq j < g/2]$

$\dim r_k = 2k$

$$R(\infty) = \bigwedge_{k \geq 0}^{MU^{(g/2)}} MU^{(g/2)}(G \cdot \bar{r}_k)$$

There is a map

$$\Phi^{G}\left(MU^{(g/2)}/(G \cdot \bar{r}_{2^{\ell}-1}) \longrightarrow R(\infty) \longrightarrow H\mathbb{Z}\right)$$

This means that if $b^{2^{(\ell-1)}}$ is in the image of this map, any power of b in the image of $\Phi^{G}(R(\infty) \longrightarrow H\mathbb{Z})$. QED

To show the hyp of 7.5 is met

consider $N_2^g \overline{r}_{2^\ell-1} \in \pi_V^G MU^{(g/2)}$ where $V=(2^\ell-1)\rho_G$

This has trivial image in

$$\pi_V^G \tilde{E}C_2 \wedge MU^{(g/2)} \xrightarrow[\cong]{2,2g} \pi^G_{2^\ell-1} \tilde{E}C_2 \wedge MU^{(g/2)}$$

$$= \pi_{2^\ell-1} \Phi^G MU^{(g/2)}$$

$$= \pi_{2^\ell-1} MO$$

$\overline{r}_{2^\ell-1}$ can be chosen so $\Phi N \overline{r}_{2^\ell-1} = 0$

$$EC_{2+} \wedge MU^{(g/2)} \longrightarrow MU^{(g/2)} \longrightarrow \tilde{E}C_2 \wedge MU^{(g/2)}$$

$$\pi_V^{G} EC_{2^+} \wedge MU^{g/2} \ni y_\ell \longmapsto N_{M_{2^\ell-1}} \in \pi_V^{G} \quad \longmapsto 0$$

Prop 7.9. Any choice of y_ℓ has nontrivial image in $\pi_V^G EC_{2^+} \wedge H\underline{\mathbb{Z}}$.

Proof that $7.9 \Rightarrow$ hypotheses of 7.5

Let $M_\ell = MU^{(g/2)} / G \cdot \overline{\mu}_{2^\ell-1}$

$$\begin{tikzcd}
EC_{2^+} \wedge MU^{(g/2)} \arrow[r] \arrow[d] & MU^{(g/2)} \arrow[r] \arrow[d] & \tilde{E}C_2 \wedge MU^{(g/2)} \arrow[d] \\
EC_{2^+} \wedge M_\ell \arrow[r] \arrow[d] & M_\ell \arrow[r] \arrow[d] & \tilde{E}C_2 \wedge M_\ell \arrow[d] \\
EC_{2^+} \wedge H\underline{\mathbb{Z}} \arrow[r] & H\underline{\mathbb{Z}} \arrow[r] & \tilde{E}C_2 \wedge H\underline{\mathbb{Z}}
\end{tikzcd}$$

$y_\ell \longmapsto 0$

$\neq 0$ by 7.9

$\tilde{y}_\ell \in \pi_{\star V}^{G}$

$\neq 0$

$\tilde{y}_\ell$ has nontrivial image in

$$\pi^G_{1+V} \widetilde{EC_2} \wedge H\underline{\mathbb{Z}} \underset{7.29}{=} \pi^G_{2\ell} \widetilde{EC_2} \wedge H\underline{\mathbb{Z}}$$

$$= \pi_{2\ell} \tilde{\Phi}^G H\underline{\mathbb{Z}} \ni b^{2^{k-1}}$$

$$= \mathbb{Z}/2$$

Hence we get hypothesis of 7.5 QED

How to prove 7.9

$$\pi^G_V EC_{2+} \wedge MU^{(g/2)} \longrightarrow \pi^G_V EC_{2+} \wedge H\underline{\mathbb{Z}}$$

These gps are covered by induction and can be studied via the slice SS. It is convenient to factor the map

$$MU^{((G/2))} \longrightarrow R(2^\ell - 2) \longrightarrow H\underline{\mathbb{Z}}$$

$R(2^\ell - 2) =$ quotient of $MU^{((G/2))}$

Will study slice tower for $EC_{2+} \wedge R(2^\ell - 2)$

We get a SS

$$E_2^{s,t} = \pi^G_{V+t-s}\left(EC_{2+} \wedge P^{|V|+t}_{|V|+t} R(2^\ell - 2)\right)$$

$$\Downarrow$$

$$\pi^G_{V+t-s}\left(EC_{2+} \wedge R(2^\ell - 2)\right)$$

This SS is known quantity

It is depicted on another slide.

We get a short exact seq

$$\begin{tikzcd}
0 \arrow[r] & \pi_V^G EC_{2+} \wedge P_{|V|} R(2^\ell - 2) \arrow[d] & \\
y_\ell \in & \pi_V^G EC_{2+} \wedge R(2^\ell - 2) \arrow[d] & \\
& \pi_V^G EC_{2+} \wedge H\underline{\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

$$\pi_V^G EC_{2+} \wedge H\underline{\mathbb{Z}} = \mathbb{Z}/2$$

We need to show y_ℓ is not in the subgp

Prop 7.12 The image of

$$\begin{tikzcd}
\pi_V^G EC_{2^+} \wedge P_{|V|} R(2^\ell-2) \arrow[r] \arrow[d] & \pi_V^G EC_{2^+} \wedge R(2^\ell-2) \arrow[dl] \arrow[r, phantom, "\ni"] & y_\ell \arrow[dd] \\
\pi_V^G R(2^\ell-2) \arrow[d] & & \\
\pi_{|V|}^u R(2^\ell-2) & & i_0^* N_2^g \overline{M}_{2^\ell-1}
\end{tikzcd}$$

is contained in that of $(1-\gamma)$, while that of y_ℓ is not. Hence y_ℓ is not in the subgroup.