MATH 549 4-26-10

Note Title 4/26/2010

Toward the proof of Reduction Thm.

Last time we reduced to Slice Thm to the RT.

For each $k > 0$ we have an associative ring spectrum $S^0[S^{k\rho_2}]$ over C_2 and a map $\bar{x}_k : S^0[S^{k\rho_2}] \longrightarrow MU_{\mathbb{R}}$

Smashing these together for all $k > 0$ and applying the norm functor

N_2^g gives us a G-spectrum A

and an equiv map $A \to MU^{(g/2)}$

Recall (see notes from last time) we have a formula N_2^g (wedge)

$G = C_{2^n}$

$g = |G|$

Our spectrum A is a wedge of slice cells, and our map $A \xrightarrow{b} MU^{(g/2)}$ is a multiplicative refinement of $\pi_*^u(MU^{(g/2)})$. Consider the spectrum

$$MU^{(g/2)} \wedge_A S^0 =: R(\infty)$$

Here $MU^{(g/2)}$ is an A-module via f

S^0 '' via

the projection map $A \to S^0$ which is null on all positive dimensional summands of A.

$R(\infty)$ is obtained (equivariantly) from $MU^{(g/2)}$ by killing all of $\pi_*^u MU^{(g/2)}$ in positive dimensions

Reduction Thm The map $R(\infty) \to H\underline{\mathbb{Z}}$ is a G-equivalence.

Remarks

1) This implies the Slice Thm as explained last time

2) Statement is obvious if we forget the gp action

3) It was proved for $G = C_2$ by Hu-Kriz in 2001.

Why is $\pi_0^u [S^{k\rho_2}]$ not commutative?

is commutative only up to htpy

$$\begin{tikzcd} S^{k\rho_2} \wedge S^{l\rho_2} \arrow[d, "\text{swap}"'] \arrow[dr] & \\ S^{l\rho_2} \wedge S^{k\rho_2} \arrow[r] & S^{(k+l)\rho_2} \end{tikzcd}$$

Reduction Thm The map $R^G(\infty) \xrightarrow{b_G} H\underline{\mathbb{Z}}$ is a G-equivalence.

Proof is by induction on $|G|$

For any proper subgp $H \subset G$ it is easy to check (see section 4 of the HHR preprint online)

$i_H^* R^G(\infty) = R^H(\infty)$, so we can assume $i_H^* f_G$ is an H-equivalence

We will smash f_G with the isotropy separation sequence

$$EC_{2+} \longrightarrow S^0 \longrightarrow \tilde{E}C_2$$

G actions via $G \to C_2$

G acts trivially

equivariant map

$$\begin{tikzcd}
EC_{2+} \wedge R^G(\infty) \arrow[r] \arrow[d, "f'"] & R^G(\infty) \arrow[r] \arrow[d, "f"] & \tilde{E}C_2 \wedge R(\infty) \arrow[d, "f''"] \\
EC_{2+} \wedge H\underline{\mathbb{Z}} \arrow[r] & H\underline{\mathbb{Z}} \arrow[r] & \tilde{E}C_2 \wedge H\underline{\mathbb{Z}}
\end{tikzcd}$$

f' is an equivalence because EC_{2+} is a free C_2-complex. It suffices to show f'' is a G-equivalence.

Being a G-equiv means that f'' induces an ordinary on the fixed pt set for each subgp of G. We already know this for every proper subgp, so we need only show it for

$$\begin{array}{ccc} (\tilde{E}C_2 \wedge R(\infty))^G & \longrightarrow & (\tilde{E}C_2 \wedge H\underline{\mathbb{Z}})^G \\ \| & & \| \\ \Phi^G R(\infty) & \longrightarrow & \Phi^G H\underline{\mathbb{Z}} \end{array}$$

Facts about geometric fixed pts:

$$\pi_\star^G(\tilde{E}C_2 \wedge X) = a_\sigma^{-1} \pi_\star^G X$$

For $X = H\underline{\mathbb{Z}}$, previous computations show this is

$$\mathbb{Z}/2[u_{2\sigma}, a_\sigma^{\pm 1}] \qquad u_{2\sigma} \in \pi^G_{2-2\sigma}$$

$$a_\sigma \in \pi^G_{-\sigma}$$

$$b = a_\sigma^{-2} u_{2\sigma} \in \pi_2$$

Hence $\pi_* \Phi^G H\underline{\mathbb{Z}} = \mathbb{Z}/2[b]$

What about $\pi_* \Phi^G \mathbb{R}(\infty)$?

Recall $\Phi^G MU^{(\mathbb{g}/2)} = MO$

= unoriented cobordism spectrum

$\pi_* MO = \mathbb{Z}/2[h_2, h_4, h_5, h_6, \dots]$

$h_k \in \pi_k \quad , \quad k \neq 2^l - 1$

An alternate description of $R(\infty)$

$$\pi_*^u MU_{(2)}^{(g/2)} = \mathbb{Z}_{(2)}[\gamma^j m_k : k > 0,\ 0 \leq j < g/2]$$

i.e. there are $g/2$ generators in each even dimension.

Let $R(m)$ be the G-spectrum obtained from $MU^{(g/2)}$ by killing the first m sets of generators. $R(\infty) = \varinjlim R(m)$

What is $\Phi^G R(m)$?

Suppose $m \neq 2^k - 1$. Then there is

a cofibre sequence

$$\Sigma^{m} \Phi^{G} R(m-1) \xrightarrow{h_m} \Phi^{G} R(m-1) \longrightarrow \Phi^{G} R(m)$$

Recall $R(0) = MU^{(g/2)}$ and $\Phi^{G} R(0) = MO$

$\Phi^{G} R(m)$ is a relative of MO.

For $m = 2^{\ell} - 1$ then we get

$$\begin{array}{ccccc} \Sigma^{m} \Phi^{G} R(m-1) & \xrightarrow{0} & \Phi^{G} R(m-1) & \longrightarrow & \Phi^{G} R(m) \\ & & & & \| \\ & & & & \Phi^{G} R(m-1) \vee \Sigma^{2^{\ell}} \Phi R(m-1) \end{array}$$

$$
\begin{tikzcd}
m=1: \quad \Sigma MO \arrow[r, "0"] & MO \arrow[r] \arrow[d, equal] & \Phi^G R(1) \arrow[d, equal] \\
& \Phi^G R(0) & MO \vee \Sigma^2 MO
\end{tikzcd}
$$

$$
\begin{tikzcd}
m=2 \quad \Sigma^2 \Phi^G R(1) \arrow[r, "h_2"] & \Phi^G R(1) \arrow[r] & \Phi^G R(2) \arrow[d, equal] \\
& & MO/(h_2) \vee \Sigma^2 MO/(h_2)
\end{tikzcd}
$$

$$
\begin{tikzcd}
m=3 \quad \Sigma^3 \Phi^G R(2) \arrow[r, "0"] & \Phi^G R(2) \arrow[r] & \Phi^G R(3) \arrow[d, equal] \\
& & MO/(h_2) \wedge (S^0 \vee S^2) \wedge (S^0 \vee S^4)
\end{tikzcd}
$$

$$\Phi^G R(4) = MO/(h_2, h_4) \wedge (\quad \text{same} \quad)$$

$$\pi_*(MO) = \mathbb{Z}/2[h_2, h_4, h_5, h_6, h_8 \cdots]$$

We get $\pi_* \Phi^G R(\infty) = \mathbb{Z}/2$ in every even dim ≥ 0.

Conclusion: $\pi_* \Phi R(\infty)$ and $\pi_* \Phi^G H\mathbb{Z}$ are abstractly isomorphic. We still need to show the iso is induced by the map.

Remark If this is going to work, $R(m)$ cannot be a ring spectrum! If it were, then $\Phi^G R(m)$ would be one. But (we hope) the map

$$\Phi^G R(m) \longrightarrow \Phi^G R(\infty) \longrightarrow \Phi^G H\mathbb{Z}$$

has image in π_* which is not a subring.

How to define generators of $\pi_* MU^{(g/2)}$ in a nice way.

Pick a C_2-map $MU_{\mathbb{R}} \longrightarrow MU^{(g/2)}$

$$H_* MU = \mathbb{Z}[m_k : k > 0] \qquad m_k \in H_{2k}$$

$$\log x = x + \sum_{k>0} m_k x^k$$

$$\log x \in \pi_* MU \otimes \mathbb{Q} \quad [[x]]$$

$$\in H_*(MU) \quad [[x]]$$

$$\pi_* MU \longrightarrow H_* MU \longrightarrow H_* MU \otimes \mathbb{Q}$$
$$\| $$
$$\pi_* MU \otimes \mathbb{Q}$$

$$H_*^u MU^{(g/2)} = \mathbb{Z}[\gamma^j m_k : k > 0, \ 0 \leq j < g/2]$$

with $\gamma^{g/2} m_k = (-1)^k m_k$

$$I = \ker \pi_*^u MU^{(g/2)} \longrightarrow \mathbb{Z}$$
$$I_H = \ker H_*^u MU^{(g/2)} \longrightarrow \mathbb{Z}$$

$Q_* = I/I^2$ = indecomposables in $\pi_*^u MU^{(g/2)}$

$QH_* = I_H/I_H^2$ = indecomps in $H_*^u MU^{(g/2)}$

Lemma (Milnor)

$$Q_{2k} \xrightarrow{\simeq} QH_{2k} \quad \text{if } k \neq 2^l - 1$$

For $k = 2^l - 1$ we have SES

$$0 \to Q_{2k} \longrightarrow QH_{2k} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

$$\gamma^j m_k \longmapsto 1$$

Lemma A Let $\mu = \sum_{0 \le j < g/2} a_j \gamma^j m_k \in QH_{2k}$

$k \neq 2^\ell - 1$

$a_j \in \mathbb{Z}_{(2)}$

The map

$$\mathbb{Z}_{(2)}[G] \longrightarrow QH_{2k} \cong Q_{2k}$$

$$1 \longmapsto \mu$$

factors thru a map

$$\mathbb{Z}_{(2)}[G] / (\gamma^{g/2} - (-1)^k) \longrightarrow QH_{2k}$$

which is an iso if

$\sum a_j \equiv 1 \mod 2$.

Lemma B For $k = 2^\ell - 1$ and μ as above,

the map

$$\mathbb{Z}_{(2)}[G] \longrightarrow \mathbb{Q}\, H_{2(2^\ell-1)}$$

factors

$$1 \longmapsto y$$

thru

$$\mathbb{Z}_{(2)}[G]/(\gamma^{g/2}+1) \longrightarrow \mathbb{Q}_{2(2^\ell-1)}$$

is an iso

iff $y = (1-\gamma)\,r$ with r as before