MATH 549 4-14-10

Note Title 4/14/2010

Toward the detection theorem

We need the Adams-Novikov SS

Recall the classical Adams SS (2/22/10)

$$\begin{array}{ccccccc} X = X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \cdots \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\ L_0 & & L_1 & & L_2 & & \end{array}$$

An Adams resolution

1

with

1) Each L_s is a wedge of $\Sigma^i H/2$

2) $H^*(f_s)$ is onto

3) X_{s+1} is the fiber of f_s

One way to achieve 1) and 2) is let $L_s = H/2 \wedge X_s$

This is the canonical Adam resolution.

1 leads to a free A-resolution of H^*X,

$$0 \leftarrow H^*X \leftarrow H^*L_0 \leftarrow H^*\Sigma L_1 \leftarrow H^*\Sigma^2 L_2 \leftarrow \cdots\cdots$$

In the canonical case we get

$$0 \leftarrow H^*X \leftarrow A \otimes H^*X \leftarrow A \otimes \bar{A} \otimes H^*X \leftarrow A \otimes \bar{A}^{\otimes 2} \otimes H^*X \leftarrow \cdots$$

where $0 \to \underset{\ker \varepsilon}{\overset{\shortparallel}{\bar{A}}} \to A \xrightarrow{\varepsilon} \mathbb{Z}/2 \to 0$

tensor this with $\bar{A}^{\otimes s} \otimes H^*X$ for $s \geq 0$ and SPLICE.

For technical reasons it is convenient to reformulate in terms of H_*

$A_* = \mathrm{Hom}_{\mathbb{Z}/2}(A, \mathbb{Z}/2)$ = dual Steenrod algebra

Its structure was described by Milnor.

The module structure $A \otimes H^* X \longrightarrow H^* X$

is dual to $A_* \otimes H_* X \longleftarrow H_* X$

making $H_* X$ is a comodule over the coalgebra A_*. The SES

$$0 \longrightarrow \bar{A} \longrightarrow A \xrightarrow{\varepsilon} \mathbb{Z}/2 \longrightarrow 0$$

is dual to

unit

$$0 \longleftarrow \bar{A}_* \longleftarrow A_* \xleftarrow{\eta} \mathbb{Z}/2 \longleftarrow 0$$

The Adams E_2-term is

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^* X, \mathbb{Z}/2)$$

$$= \mathrm{Ext}_{A_*}^{s,t}(\mathbb{Z}/2, H_* X)$$

This is Ext in category of comodules over A_*.

Big leap: Replace $H/2$ by a ring spectrum R with nice properties to be named later.

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \cdots \arrow[l, "g_2"'] \\ L_0 & L_1 & L_2 & \end{tikzcd}$$

R Adams resolution

(2)

with

$L_s = R \wedge X_s$, X_{s+1} is the fiber of f_s

R has a unit $S^0 \xrightarrow{i} R$ unit

$X \xrightarrow{X \wedge i} R \wedge X$

e.g. $X_s \xrightarrow{f_s} L_s$

$X_{s+1} = \overline{R} \wedge X_s$

where $\overline{R} \to S^0 \xrightarrow{i} R$ $\quad \overline{R} =$ fiber of i

We get a spectral sequence with

$$E_2 = \mathrm{Ext}_{R_*(R)}\left(R_*, R_*(X)\right)$$

Note: For $R = H/2$,

$$R_*(R) = H_*(H/2) = A_*$$

In general, $R_*(X) = \pi_*(R \wedge X)$

$$R_* = \pi_* R$$

What kind of object is $R_*(R)$?

When $R = H/2$, $R_*(R) = A_*$ is a graded comm $\mathbb{Z}/2$-Hopf algebra i.e. a cogroup object in the category of graded comm $\mathbb{Z}/2$-algebras.

This means for any such algebra C,

$\mathrm{Hom}(A_*, C)$ has a natural group structure

This requires a map $A_* \longrightarrow A_* \otimes A_*$ which induces

$$\mathrm{Hom}(A_* \otimes A_*, C) \longrightarrow \mathrm{Hom}(A_*, C)$$

$$\| $$

$$\mathrm{Hom}(A_*, C) \times \mathrm{Hom}(A_*, C)$$

Hom here is in the category of graded comm algebras over $\mathbb{Z}/2$.

We need to generalize this to

Def. A Hopf algebroid over K is a cogroupoid object in the category of comm graded K-algebras

A <u>group</u> is a category with one object in which every morphism is invertible. (group elements are morphisms)

A groupoid is a small category in which every morphism is invertible.

Def Let X be a G-set. There is a gpoid whose object set is X and morphisms $x \to y$ are elements $g \in G$ with $g(x) = y$. This is a split groupoid.

<u>Example</u> X = set of formal group laws over a ring R

G = gp of functionally invertible power series over R

$= \{ f(x) \in R[[x]] : f(0) = 0,\ f'(0) \text{ is a unit} \}$

If $F(x,y)$ is a FGL, so is $f\, F(f^{-1}(x), f^{-1}(y))$

$$f\, F(f^{-1}(x), f^{-1}(y)) = f(F)$$

This defines a G-action on X.

For a nice ring spectrum R,
$R_*(R)$ is a Hopf algebroid over
$\pi_0(R)$. Explanation later

What is a cogroupoid object?
Any small category $\underline{C}$ has
a set of objects $\underline{O}$
and " morphism $\underline{M}$
with structure maps

$$\underline{O} \xrightarrow{\text{identity}} \underline{M} \underset{\text{target}}{\overset{\text{source}}{\rightrightarrows}} \underline{O}$$

$$\begin{tikzcd} \text{subset} \arrow[r, "\text{composition}"] \arrow[d, phantom, "\cap"] & \underline{M} \\ \underline{M} \times \underline{M} & \end{tikzcd}$$

If $\underline{C}$ is a gpoid we have $\underline{M} \xrightarrow{\text{inverse}} \underline{M}$

For $R = MU$, then $MU_*(MU)$ is related to the split gpoid described above.

$$R_*(R) = \pi_*(R \wedge R)$$

What is a cogroupoid object in the category of K-algebras

We need K-algebras A and Γ s.t. $\mathrm{Hom}(A, -)$ = object set of a gpoid

$\mathrm{Hom}(\Gamma, -)$ = morphism set of a gpoid.

These have 5 structure maps as above. source + target are represented by 2 maps source

target

$$\begin{tikzcd} A \arrow[r, bend left, "\eta_L"] \arrow[r, bend right, "\eta_R"'] & \Gamma \end{tikzcd}$$

left and right units

use η_R and η_L

$$\Gamma \xrightarrow{\Delta} \Gamma \otimes_A \Gamma$$

composition

$$\Gamma \otimes_A \Gamma = \text{quotient of } \Gamma \otimes_k \Gamma$$

For a ring spectrum R

$$A = \pi_*(R) \qquad \Gamma = \pi_*(R \wedge R)$$

$$S^0 \xrightarrow{i} R$$

$$\begin{tikzcd}
S^0 \wedge R \arrow[dr, "i \wedge R"] \arrow[d, equal] & \\
R \arrow[d, equal] & R \wedge R \\
R \wedge S^0 \arrow[ur, "R \wedge i"'] &
\end{tikzcd}$$

left/right unit

$$\Gamma \otimes_A \Gamma = \pi_*(R \wedge R \wedge R)$$

$$R \wedge R = R \wedge S^0 \wedge R \xrightarrow{R \wedge i \wedge R} R \wedge R \wedge R$$

The inverse map $\Gamma \longrightarrow \Gamma$ is induced by $R \wedge R \xrightarrow{T} R \wedge R$

$$x \wedge y \longrightarrow y \wedge x$$