

MATH 549 4-14-10

Note Title

4/14/2010

Toward the detection theorem

We need the Adams-Novikov SS

Recall the classical Adams SS (2/22/10)

$$X = X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \dots$$

$f_0 \downarrow$ $f_1 \downarrow$ $f_2 \downarrow$

L_0 L_1 L_2

An Adams
resolution

①

with

1) Each L_i is a wedge of $\Sigma^? H/2$

2) $H^*(f_i)$ is onto

3) X_{i+1} is the fibers of f_i

One way to achieve 1) and 2) is let $L_0 = H/2^{-1} X_0$
 This is the canonical Adams resolution.

① leads to a free A -resolution of $H^* X$,

$$0 \leftarrow H^* X \leftarrow H^* L_0 \leftarrow H^* \Sigma L_1 \leftarrow H^* \Sigma^2 L_2 \leftarrow \dots$$

In the canonical case we get

$$0 \leftarrow H^* X \leftarrow A \otimes H^* X \leftarrow A \otimes \bar{A} \otimes H^* X \leftarrow A \otimes \bar{A}^{\otimes 2} \otimes H^* X \leftarrow \dots$$

where $0 \rightarrow \bar{A} \rightarrow A \xrightarrow{\epsilon} \mathbb{Z}/2 \rightarrow 0$

$\text{ker } \epsilon$

tensor this with $\bar{A}^{\otimes s} \otimes H^* X$
 for $s \geq 0$ and SPLICED

For technical reasons it is convenient
 to reformulate in terms of H_X

$A_{\mathbb{Z}/2} = \text{Hom}_{\mathbb{Z}/2}(A, \mathbb{Z}/2) = \text{dual Steenrod algebra}$

Its structure was described by Milnor

The module structure $A \otimes H^*X \rightarrow H^*X$

is dual to $A_{\mathbb{Z}/2} \otimes H_{\mathbb{Z}/2}X \leftarrow H_{\mathbb{Z}/2}X$

making $H_{\mathbb{Z}/2}X$ is a comodule over the coalgebra $A_{\mathbb{Z}/2}$. The SES

$$0 \rightarrow \tilde{A} \rightarrow A \xrightarrow{\epsilon} \mathbb{Z}/2 \rightarrow 0$$

is dual to

$$0 \leftarrow \tilde{A}_{\mathbb{Z}/2} \leftarrow A_{\mathbb{Z}/2} \xleftarrow{\eta} \mathbb{Z}/2 \leftarrow 0$$

← unit

The Adams E_2 -Term is

$$\begin{aligned} H_2^{\text{Ext}} &= \text{Ext}_A^{\text{Ext}}(H^* X, \mathbb{Z}/2) \\ &= \text{Ext}_{A_X}^{\text{Ext}}(\mathbb{Z}/2, H_X X) \end{aligned}$$

This is Ext in category of comodules over A_X .

Big Leap: Replace $H/2$ by a ring spectrum R with nice properties to be named later.

$$X = X_0 \xleftarrow{g_0} X_1 \xleftarrow{g_1} X_2 \xleftarrow{g_2} \dots$$

$$\begin{array}{ccc} \beta_0 \downarrow & \beta_1 \downarrow & \beta_2 \downarrow \\ L_0 & L_1 & L_2 \end{array}$$

Cartan-Eilenberg
resolution (2)

with

$$L_0 = R \cap X_0, \quad X_{n+1} \text{ is the fiber of } f_n$$

R has a unit

$$\begin{array}{ccc} S^0 & \xrightarrow{i} & R \\ X & \xrightarrow{X_{n+1}} & R \cap X \end{array} \quad \text{unit}$$

e.g.

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & L_0 \\ X_{n+1} & = & \overline{R} \cap X_0 \end{array}$$

where $\overline{R} \rightarrow S^0 \xrightarrow{i} R$

$\overline{R} = \text{fiber of } i$

We get a spectral sequence with

$$E_2 = \text{Ext}_{R_*} (R_*, R_*(X))$$

Note: For $R = H/2$,

$$R_*(R) = H_*(H/2) = A_*$$

$$\text{In general, } R_*(X) = \pi_*(R \smallfrown X)$$

$$R_* = \pi_* R$$

What kind of object is $R_*(R)$?

When $R = H/2$, $R_*(R) = A_*$ is a Hopf algebra ^{graded comm $\mathbb{Z}/2$ -}

i.e. a cogroup object in the category of graded comm $\mathbb{Z}/2$ -algebras.

This means for any such algebra C ,

$\text{Hom}(A_*, C)$ has a natural group structure

This requires a map $A_* \rightarrow A_* \otimes A_*$ which induces

$$\text{Hom}(A_* \otimes A_*, C) \rightarrow \text{Hom}(A_*, C)$$

$$\text{Hom}(A_*, C) \times \text{Hom}(A_*, C)$$

Hom here is in the category of graded comm algebras over $\mathbb{Z}/2$.

We need to generalize this to

Def. A Hopf algebraoid over K is a comonoidal object in the category of comm graded K -algebras

A group is a category with one object in which every morphism is invertible. (group elements are morphisms)

A groupoid is a small category in which every morphism is invertible.

Def Let X be a G -set
There is a groupoid whose object set is X and morphisms $x \rightarrow y$ are elements $g \in G$ with $g(x) = y$.
This is a split groupoid.

Example $X =$ set of formal group laws
over a ring R

$G_1 =$ gp of ^{functionally} invertible power
series over R

$$= \{ f(x) \in R[[x]] : f(0) = 0, f'(0) \text{ is a unit} \}$$

If $F(x, y)$ is a FGL, so is $f \circ F(\tilde{f}(x), \tilde{f}(y))$
 $\tilde{f}(F)$

This defines a G_1 -action on X .

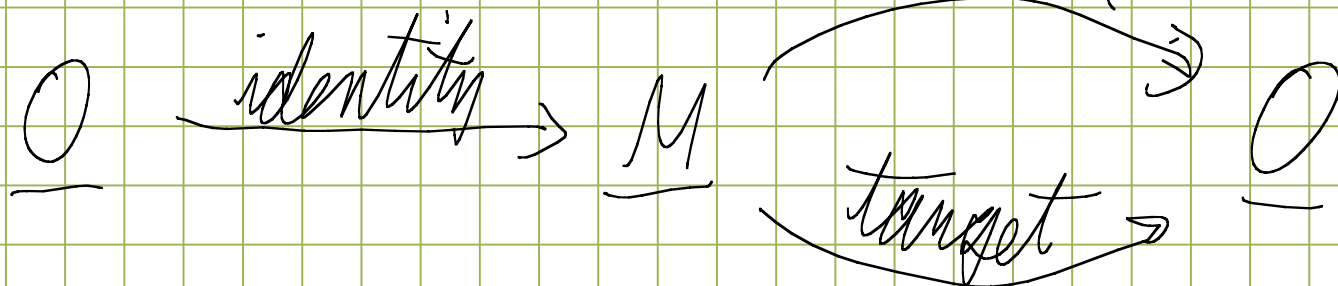
For a nice ring spectrum R ,
 $R_*(R)$ is a Hopf algebroid over
 $\pi_0(R)$. Explanation later

What is a comonoid object?

Any small category \mathcal{C} has
a set of objects \underline{O}

and " morphism \underline{M}

with structure maps



subset $\xrightarrow{\text{composition}}$ \underline{M}
 \cap
 $\underline{M \times M}$

If \underline{C} is a groupoid we have $\underline{M} \xrightarrow{\text{inverses}} \underline{M}$

For $R = MU$, then $MU_x(MU)$ is
related to the split groupoid described
above.

$$R_x(R) = \pi_x(R \smile R)$$

What is a comonoid object in the
category of K -algebras

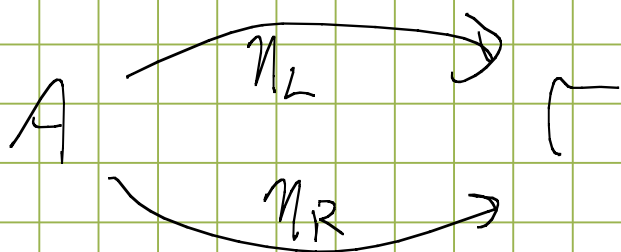
We need K -algebras A and Γ

s.t. $\text{Hom}(A, -) =$ object set of
a species

$\text{Hom}(\Gamma, -) =$ morphism set
of a species.

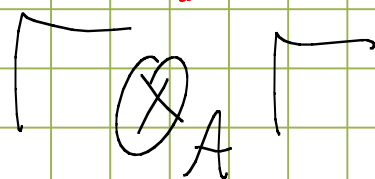
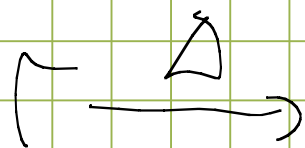
These have 5 structure maps
as above. source + target
are represented by 2 maps

source
target



left and
right units

use η_R and η_L



composition

quotient of $\Gamma \otimes_K \Gamma$

For a ring spectrum R

$$A = \pi_* (R)$$

$$S^0 \xrightarrow{i} R$$

$$\Gamma = \pi_* (R \circ R)$$

$$\begin{array}{ccc}
 S^0 \wr R & \xrightarrow{i \wr R} & R \wr R \\
 \parallel & & \\
 R & & \\
 \parallel & & \\
 R \wr S^0 & \xrightarrow{R \wr i} &
 \end{array}$$

left/right
unit

$$\Gamma \otimes_A \Gamma = \pi_* (R \wr R \wr R)$$

$$R \wr R = R \wr S^0 \wr R \xrightarrow{R \wr i \wr R} R \wr R \wr R$$

The inverse map
is induced by

$$\begin{array}{ccc}
 \Gamma & \longrightarrow & \Gamma \\
 R \wr R & \xrightarrow{I} & R \wr R \\
 x \wr y & \longrightarrow & y \wr x
 \end{array}$$