MATH 549 4-12-10

Note Title 4/12/2010

Recall in the slice SS for $MU^{(4)}$

$$d_{1+8(2^k-1)}\left(u_{2^k\sigma}\right) = a_\sigma^{2^k} f_{2^k-1} \quad \text{for } k \ge 1$$

where σ = sign rep for $G = C_8$

$$\begin{tikzcd} S^{2^k-1} \arrow[r] \arrow[rr, bend left, "f_{2^k-1}"] & S^{(2^k-1)\rho_8} \arrow[r, "\bar{\Delta}_k^{(8)}"] & MU^{(4)} \end{tikzcd}$$

$g = |G| = 8$

γ

$$\bar{\Delta}_k^{(g)} = m_{2^k-1}\,\gamma(m_{2^k-1})\,\gamma^2(m_{2^k-1})\,\gamma^3(m_{2^k-1})$$

$$\pi_*^u MU^{(4)} = \mathbb{Z}\left[\gamma^j(m_i) : i > 0,\ 0 \le j \le 3\right]$$

$$m_i \in \pi_{2i}^u$$

Inverting $\bar{\Delta}_k^{(8)}$ makes $u_{2^{k+1}\sigma} \simeq u_{2\sigma}^{2^k}$ a permanent cycle

Goal: Invert something to some power of $u_{2\rho_8}$ a permanent cycle.

We will define a $D \in \pi^{G}_{19\rho_8} MU^{(4)}$

$\tilde{\Omega} = D^{-1} MU^{(4)}$

$\tilde{\Omega}^{hC_8}$ is 256-periodic

Since we are inverting an elt in $\pi^{G}_{\star\rho_8}$)

the resulting telescope has the gap property, i.e. $\pi_k \tilde{\Sigma}^{C_8} = 0$ for $-4 < k < 0$.

Lemma. Let $H \subset G = C_8$ and V an orient rep of H, so $u_V \in \pi^H_{|V|-V} H\underline{\mathbb{Z}}$

(1) $$S^{|V|} \xrightarrow{u_V} S^V \wedge H\underline{\mathbb{Z}}$$

Then $u_{V''} N_H^G(u_V) = u_{V'}$.

where $V' = \mathrm{ind}_H^G V$ and $V'' = \mathrm{ind}_H^G |V|$

Proof Applying norm functor to (1) gives

$S^{|V'|}$

$$S^{V''} \longrightarrow S^{V'} \wedge (H\underline{\mathbb{Z}})^{(g/h)} \longrightarrow S^{V'} \wedge H\underline{\mathbb{Z}}$$

$$\begin{tikzcd}
S^{|V''|} \arrow[r, "u_{V''}"] \arrow[rr, bend right, "u_{V'}"'] & S^{V''} \wedge H\underline{\mathbb{Z}} \arrow[r, "N_h^g(u_V)"] & S^{V'} \wedge H\underline{\mathbb{Z}}
\end{tikzcd}$$

(QED)

Corollary $V = 2\rho_4$ = reg rep of C_4

$$u_{2\rho_8} = N_4^8(u_{2\rho_4})\, u_{8\sigma_8} \qquad \sigma_8 = \text{sign rep of } G$$

$$u_{2\rho_4} = N_2^4(u_{2\rho_2})\, u_{4\sigma_4} = N_2^4(u_{2\sigma_2})\, u_{4\sigma_4}$$

$$N_4^8(u_{2\rho_4}) = N_2^8(u_{2\sigma_2})\, N_4^8(u_{4\sigma_4})$$

$$M_{2\rho_8} = N_2^8(M_{2\sigma_2})\, N_4^8(M_{4\sigma_4})\, M_{8\sigma_8}$$

We make a power of $N_4^8(M_{2\sigma_4})$ a perm cycle by inverting $\overline{\Delta}_k^{(4)}$ for some k.

We can a power of $M_{2\rho_8}$ a perm cycle by inverting $N_2^8(\Delta_{k_1}^{(2)})\, N_4^8(\Delta_{k_2}^{(4)})\, \Delta_{k_3}^{(8)}$

We need to choose k_m so that the resulting $\tilde{\Omega}^{hC_8}$ detects the θ_j's.

It turns out the we need $8 \mid 2^m k_m$

for $m = 1, 2, 3$. (To be explained later)

Let $k_1 = 4$, $k_2 = 2$, $k_3 = 1$

Inverting $\bar{\Delta}_4^{(2)}$ make $u_{32 6_2}$ a perm cycle

" $\bar{\Delta}_2^{(4)}$ " $u_{8 6_4}$ "

" $\bar{\Delta}_1^{(8)}$ " $u_{4 6_8}$ "

Let $D = \bar{\Delta}_1^{(8)} N_4^8(\bar{\Delta}_2^{(4)}) \; N_2^8(\bar{\Delta}_4^{(2)}) \in \pi_{19\rho_8}^{C_8} MU^{(4)}$

Inverting D makes $u_{32\rho_8} = u_{2\rho_8}^{16}$ a permanent cycle.

Let $\tilde{\Omega} = D^{-1} MU^{(4)}$

$$S^0 \xrightarrow{D} \Sigma^{-19\rho_8} MU^{(4)}$$

$$\begin{tikzcd} MU^{(4)} \arrow[r] \arrow[rr, bend right, "D"'] & \Sigma^{-19\rho_8} MU^{(4)} \wedge MU^{(4)} \arrow[r] & \Sigma^{-19\rho_8} MU^{(4)} \end{tikzcd}$$

D can be iterated

$$MU^{(4)} \xrightarrow{D} \Sigma^{-19\rho_8} MU^{(4)} \xrightarrow{D} \Sigma^{-38\rho_8} MU^{(4)} \to \cdots$$

$$\tilde{\Omega} = \operatorname{colim} \Sigma^{-19m\rho_8} MU^{(4)}$$

$$S^{256} \xrightarrow{u_{32\rho_8}} S^{32\rho_8} \wedge H\underline{\mathbb{Z}}$$

Let $\Delta_1^{(8)} = u_{2\rho_8} \left(\bar{\Delta}_1^{(8)}\right)^2 \in E_2^{0,16} \tilde{\Omega}$

$$\left(\Delta_1^{(8)}\right)^{16} = u_{32\rho_8}\left(\bar{\Delta}_1^{(8)}\right)^{32} \in E_2^{-0,256}\,\tilde{\Omega}$$

It is a perm cycle. We have a map

$$\Sigma^{256}\,\tilde{\Omega} \xrightarrow{\pi} \tilde{\Omega}$$

$\Delta_1^{(8)}$ is a factor of D.

Restricting to the trivial gp makes $u_{32\rho_8}$ a unit. The map π induces an equiv after restriction to C_1.

Hence we get an equiv

$$\Sigma^{256}\,\tilde{\Sigma}^{\mathfrak{h}C_8} \xrightarrow{\sim} \tilde{\Sigma}^{\mathfrak{h}C_8}$$

This is the periodicity for $\tilde{\Sigma}^{\mathfrak{h}C_8}$.

Our fixed pt thm says:

If X is a G-spectrum with $\Phi^G X = pt$,

then $X^G \simeq X^{hG}$

Then $D = \bar{\Delta}_1^{(8)} \cdots$

$$\bar{\Delta}_1^{(8)} = r_1\, \gamma(r_1)\, \gamma^2(r_1)\, \gamma^3(r_1)$$

$$\begin{array}{ccc} S^{\rho_8} & \xrightarrow{\overline{\Delta}_1^{(8)}} & MU^{((4))} \\ & & \Big\downarrow \Phi^G \\ S^1 & \longrightarrow & MO \end{array} \qquad \pi_1 MO = 0$$

$$\Phi^G(\overline{\Delta}_1^{(8)}) = 0$$

so $\Phi^G(\tilde{\Omega}) = *$

Hence our fixed point theorem says $\tilde{\Omega}^{C_8} = \tilde{\Omega}^{hC_8}$

We know

1) $\Sigma^{256}\,\widetilde{\Omega}^{hC_8} \simeq \Omega^{hC_8}$

2) The slice SS for $\pi_*(\widetilde{\Omega}^{C_8})$ has the gap property

3) $\widetilde{\Omega}^{hC_8}$ has the detection property

$\Omega = \widetilde{\Omega}^{hC_8} = \widetilde{\Omega}^{C_8}$

Preview of detection thm

It is know that the homotopy fixed point SS for $\pi_*(MU^{((4))})^{hC_8}$ is the same as the Adams-Novikov

DITTO $\tilde{\Omega}^{hC_8}$

$$E_2 = H^*(C_8; \pi_* \tilde{\Omega})$$

We have $\theta_j \in E_2^{2, 2^{j+1}}(S^0)$ in the ANSS for S^0

We will map $H^*(C_8; T \times \tilde{\Omega})$ to a simpler gadget related to a certain formal A-module where $A = \mathbb{Z}_2[S]/(S^4+1)$

In A, let $\pi = S-1$

$A/(\pi) = \mathbb{Z}/2$, i.e. (π) is the maximal ideal and

$\pi^4 = 2 \cdot$ unit. There is a FGL $/A[\pi^{-1}]$

with $\log x = \sum_{n=0}^{\infty} \frac{x^{2^n}}{\pi^n} (w^{2^n-1})$

θ_j will map to an easily $H^2(?)$ related to this FGL.