MATH 549 3-24-10

Note Title 3/24/2010

Our main computational tool: the slice spectral sequence

The classical Postnikov tower

Let X be a space or spectrum

$P^n X$ = nth Postnikov section of X

= the space or spectrum obtained by killing all homotopy gps above dimension n by attaching cells.

The fiber of the map $X \longrightarrow P^n X$ is $P_{n+1} X$, the n-connected cover of X.

$$\pi_i (P^n X) = \begin{cases} \pi_i X & \text{for } i \le n \\ 0 & \text{for } i > n \end{cases}$$

$$\pi_n (P_{n+1} X) = \begin{cases} \pi_i X & \text{for } i > n \\ 0 & \text{for } i \le n \end{cases}$$

P^n and P_{n+1} are functors.

Let $\underline{S}$ and $\underline{S}_{>n}$ be the categories of spectra and n-connected spectra.

Formal properties of P_{n+1} and $P^n : \underline{\mathcal{S}} \to \underline{\mathcal{S}}$

i) $P_{n+1} X \in \underline{\mathcal{S}}_{>n}$, i.e. $P_{n+1} : \underline{\mathcal{S}} \longrightarrow \underline{\mathcal{S}}_{>n}$

ii) For $A \in \underline{\mathcal{S}}_{>n}$ and $X \in \mathcal{S}$, the map of function spectra

$$\mathcal{S}\left(A, P_{n+1} X\right) \longrightarrow \mathcal{S}(A, X)$$

is a weak equivalence.

A map to X from an n-connected spectrum A is the same as a map to $P_{n+1} X$.

i.e. $P_{n+1} X \to X$ is the universal map to X

from an n-connected spectrum.

Similarly, the map $X \longrightarrow P^n X$ is universal among maps that are $\mathcal{S}_{>n}$-null, i.e. all maps to $P^n X$ from n-connected spectra are null.

Since $\underline{\mathcal{S}}_{>n} \subset \mathcal{S}_{>n-1}$, there is a nat trans $P^n \longrightarrow P^{n-1}$ whose fiber P^n_n

$P^n_n X$ = Eilenberg-MacLane spectrum $K(\pi_n(X), n)$

Note $\underline{S}_{>n}$ is a subcategory closed under hty colimits (wedges, suspensions, mapping cones) and smash products with suspension spectra.

Suppose $\underline{C} \subset \underline{S}$ is a subcategory with similar properties. Then The is a functor $P^{\underline{C}}$ analogous to P^n.

$P^{\underline{C}} X$ universal for a map from X to a $\underline{C}$-null spectrum.

$$P_{\underline{C}} X \longrightarrow X \longrightarrow P^{\underline{C}} X$$

We want an equiv analog of $\underline{\Delta}_{\geq n}$.
$\Delta_{\geq n}$ is "generated" by $\{ S^{n+1}, S^{n+2}, \dots \}$

Let $G = C_{2^n}$. Consider the following G-spectra, $H \subseteq G$

$h = |H|$
$g = |G|$
ρ_H = regular real rep of H

$$\hat{S}(m\rho_H) = G_+ \wedge_H S^{m\rho_H}$$

is a G-spectrum underlain by

$$\underbrace{S^{mh} \vee S^{mh} \cdots \vee S^{mh}}_{g/h}$$

Let $\mathcal{A} = \{ \hat{S}(m\rho_H), \Sigma^{-1}\hat{S}(m\rho_H) : m \in \mathbb{Z}, H \subset G \}$

Objects in $\mathcal{A}$ are called *slice cells*. If $H = \{e\}$, the cell is *free*, otherwise it is *isotropic*.

$$\dim S^{-1}(m\rho_H) = mh$$

$$\dim \Sigma^{-1}(m\rho_H) = mh - 1$$

Let $\mathcal{A}^G_{>n}$ = subcategory of G-spectra generated by all cells of $\dim > n$.

We can define functors P^n_G and P^G_{n+1} as before.

NOTE $P^G_{n+1} X$ need not be n-connected, i.e. $\pi^G_i P^G_{n+1} X$ for $i < n$ could be nonzero.

Recall $\pi^G_i (S^{m\rho_G}) = \pi_i (S^{m\rho_G})^G$

$$= \pi_i (S^m)$$

This can be $\neq 0$ for $i < gm$

We have this functors for each $n \in \mathbb{Z}$, so we get the slice tower

$$\begin{tikzcd}
\cdots \arrow[r] & P^{n+1}_G X \arrow[r] & P^n_G X \arrow[r] & P^{n-1}_G X \arrow[r] & \cdots \\
& G P^{n+1}_{n+1} X \arrow[u] & G P^n_n X \arrow[u] & G P^{n-1}_{n-1} X \arrow[u] &
\end{tikzcd}$$

where $\underrightarrow{\operatorname{holim}}\, P^n_G X = X$

and $\underleftarrow{\operatorname{holim}}\, P^n_G X = *$

Hence we get a spectral sequence

$$E_1^{n,*} = \pi_*\left(G P^n_n X\right) \Rightarrow \pi_* X$$

We want to study this for the G-spectrum

$$X = MU^{((g/2))} \qquad G = C_{2^j} \qquad g = |G| = 2^j$$

$$P^n_n X = n\text{th slice of } X.$$

Remark For $G = \{e\}$, this is uninteresting

$$\pi_i (P^n_n X) = \begin{cases} \pi_n X & \text{for } i = n \\ 0 & \text{for } i \neq n \end{cases}$$

In our case the SS has variants that compute $\pi_\star (X^H)$

Description for $G = C_2$, $X = MU_{\mathbb{R}} = MU$ with conjugation.

Recall

$\pi_* MU = \mathbb{Z}[x_1, x_2, \cdots]$ $\quad x_i \in \pi_{2i}$ (Milnor, Novikov 1962)

$\Phi^{C_2} MU = MO$

$\pi_* MO = \mathbb{Z}/2[y_2, y_4, y_5 \cdots]$ $\quad y_i \in \pi_i$, $i \neq 2^j - 1$

(Thom 1950s)

x_i is represented by a map

$\bar{x}_i : S^{i\rho_2} \longrightarrow MU$

In $RO(C_2)$, $\rho_2 = 1 + \sigma$

We have $a_6 : S^0 \to S^6$

$u_{26} : S^2 \to S^{26} \wedge H\underline{\mathbb{Z}}$

Slice thm for $MU_{\mathbb{R}}$: The odd slices are contractible. The $(2n)$th slice is a wedge of copies $S^{n\rho} \wedge H\underline{\mathbb{Z}}$, with one summand for each degree n monomial in $\mathbb{Z}[\bar{x}_1, \bar{x}_2, \ldots]$.

$|u_6| = 1-6$

Indexing convention: Adams convention $E_r^{s,t}$ is related to $\pi_{t-s}(X)$

where $s \in \mathbb{Z}$, $t \in RO(G)$.

$\overline{x}_1 \in E_1^{0, i\rho}$ $\quad (u_{\sigma_i} \overline{x}_1)^2 \in E_1^{0, 4i}$

We need to know $\pi_*(S^{n\rho} \wedge H\mathbb{Z})$

for all $n \in \mathbb{Z}$

$\pi_*(S^{n\rho} \wedge H\mathbb{Z}) \simeq H_*(S^{n\rho})$

We will study this by examining

$S^{n\rho}$ as a G-CW complex

e.g. $n = 2$. As a G-complex it has

cellular chain cx

$\gamma \in C_2$

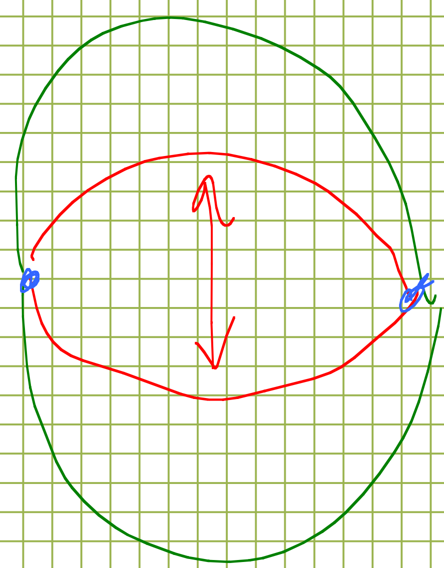
$$C: \quad \underset{2}{\mathbb{Z}} \xleftarrow{\varepsilon} \underset{3}{\mathbb{Z}[C_2]} \xleftarrow{1-\gamma} \underset{4}{\mathbb{Z}[C_2]}$$

$$1 \longleftarrow \gamma$$

chain cx of $\mathbb{Z}[C_2]$-modules

$$(S^{2\rho})^G = S^2 \qquad S^{2\rho} = S^2 \cup (e^3 \amalg e^3) \cup (e^4 \amalg e^4)$$

$S^{2\sigma}$



green 2-cells free C_2-action

red 1-cells free C_2-action

blue 0-cells fixed

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z}\{1+\gamma\} \xleftarrow{0} \mathbb{Z}\{1+\gamma\}$$

$\gamma \in C_2$ $\quad C:\quad \mathbb{Z} \xleftarrow{\Sigma} \mathbb{Z}[C_2] \xleftarrow{1-\gamma} \mathbb{Z}[C_2]$ $\quad$ chain cx of $\mathbb{Z}[C_2]$-modules

$$1 \xleftarrow{\quad} 1\gamma$$

$$2 \qquad\qquad 3$$

$$H_* C = \overline{H}_*(S^4)$$

$$\pi_*^{C_2}(S^{2\rho} \wedge H\underline{\mathbb{Z}}) = H_*\left(\mathrm{Hom}_{\mathbb{Z}[C_2]}(\mathbb{Z}, C)\right)$$

$$(1+\gamma)(1-\gamma) = 1-\gamma^2 = 0$$