MATH 549 3-22-10

Note Title 3/22/2010

Will define equivariant spectra as functors to $\underline{\mathcal{T}}$ = category pointed top spaces from a certain category to be named later

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Def: A monoidal category $\underline{\mathcal{V}}$ is one in which the class of objects is a monoid (group without inverses). The binary operation is $\otimes$ and identity is $\underline{1}$.

Examples

1) Abelian gps with $\oplus$, $\underline{1} = \{e\}$ (not closed)

2) " $\otimes$, $\underline{1} = \mathbb{Z}$

3) Topological spaces, $\times$, $\underline{1} = pt.$

4) Pointed topological spaces, $\wedge$ (smash product), $\underline{1} = S^0$.

A monoidal category $(\underline{V}, \otimes, \underline{1}) = V$ is <u>symmetric</u> if there are natural isos between $A \otimes B$ and $B \otimes A$. It is <u>closed</u> if the functor $A \otimes (-)$ has a right adjoint, so that

$\underline{V}(A,X) = \underline{V}(\underline{1}, X^A)$. X^A = "internal hom"

Enriched categories

A category $\underline{C}$ is enriched over $\underline{D}$ if $\underline{C}(X,Y)$ is an object in $\underline{D}$. (e.g. $\underline{D} = \underline{Top}$, $\underline{C}$ is a topological category). A functor $F: \underline{C} \longrightarrow \underline{C}'$ of categories over $\underline{D}$ is

i) a map $F: ob(\underline{C}) \longrightarrow ob(\underline{C}')$

ii) $\forall X, Y$ we get $\underline{C}(X,Y) \xrightarrow{F} \underline{C}'(FX, FY)$ a morphism in $\underline{D}$, satisfying some conditions

Suppose $\underline{D}$ is a closed symmetric monoidal (CSM) category $(\mathcal{V}_0, \otimes, \underline{1})$. For each object X in $\underline{C}$ we have an identity morphism

$$\underline{1} \longrightarrow \underline{C}(X, X)$$

and for each triple X, Y, Z

$$\underline{C}(Y, Z) \otimes \underline{C}(X, Y) \longrightarrow \underline{C}(X, Z)$$

(analog of morphism composition)

Such an enriched $\underline{C}$ is underlain by an ordinary category $\underline{C}_0$ with the same objects as in $\underline{C}$ and morphism sets

$\underline{C}_0(X,Y) = V_0(\underline{1}, \underline{C}(X,Y))$

A functor $F : \underline{C} \longrightarrow \underline{C}'$ is a map

ob $\underline{C} \xrightarrow{F}$ ob $\underline{C}'$ and a morphism

$\underline{C}(X,Y) \xrightarrow{F} \underline{C}'(FX, FY)$

compatible with units + composition

Orthogonal G-spectra

$\mathcal{T}$ = category of pointed compactly generated weak Hausdorff spaces

It is a CSM category with smash product and $\underline{1} = S^0$.

A topological category is a category enriched over $(\mathcal{T}, \wedge, S^0) = \underline{\mathcal{T}}$

$\underline{\mathcal{T}}_G = (\mathcal{T}_G, \wedge, S^0)$

$\mathcal{T}_G$ = category of pointed spaces with left G-action

The "category of G-spaces"

an ordinary category

a topological category

a category enriched over the ordinary category of G-spaces.

The set of cont maps between G-spaces X and Y has a G-action via conjugation

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rrr, bend right, "g(f)"'] & X \arrow[r, "f"] & Y \arrow[r, "g^{-1}"] & Y \end{tikzcd}$$

The equivariant maps are the ones fixed by this action.

The indexing category

G = gp.

Let V be a finite dim orth rep of G.

$O(V)$ = gp of orthogonal maps $V \to V$

It is a G-space

$O(V)^G = \{$ equiv orth maps $V \longrightarrow V\}$

Let W be another orth rep of G

$O(V,W) =$ Stiefel mfd of orthogonal embeddings $V \hookrightarrow W$

$O(W)$ acts transitively on the left

$$O(W) \times O(V,W) \longrightarrow O(V,W)$$

Choosing an embedding $V \hookrightarrow W$ leads to an identification

$$O(V,W) = O(W) / O(W-V)$$

We define a category $\mathcal{J}^G$ enriched over G-spaces as follows.

Objects are reps V as above

$$\mathcal{J}^G(V,W) = \mathrm{Thom}\left(O(V,W);\, W-V\right)$$

$O(V,W)$ has a vector bundle where the fiber at x is $W - x(V)$, the orth compliment of $x(V)$ in W.

Total space of vector bundle

$$\{ (w, x) \in W \times O(V,W) : w \perp x(V) \}$$

Given reps U, V and W

$$O(V,W) \times O(U,V) \longrightarrow O(U,W)$$

This induces correspond maps on Thom spaces

$$\mathcal{J}^G(V,W) \times \mathcal{J}^G(U,V) \longrightarrow \mathcal{J}^G(U,W)$$

Let $\mathcal{J} = \mathcal{J}^{\{e\}}$

Prop. Let $\underline{B}G$ denote the one object category with morphism set G.

Let $\mathcal{J}^{BG} :=$ the category of functors $BG \longrightarrow \mathcal{J}$

objects in $\mathcal{J}^{BG}$ are functors $BG \to \mathcal{J}$

morphism " are natural transformation

Then $\mathcal{L}^{BG}$ is equivalent to $\mathcal{L}^{G}$.

<u>Def</u> An orthogonal G-spectrum is a functor $\mathcal{L}^{G} \longrightarrow \mathcal{T}_{G}$ enriched over G-spaces and equivariant maps

Informally G-spectrum X is a collection of G-spaces X_V for $V \in \mathcal{L}^{G}$ and for each equiv emb $V \longrightarrow W$

$S^{W-V} \wedge X_V \longrightarrow X_W$. These are required to compatible with composition in $\mathcal{I}^G$ and to vary continuously with the embedding $V \longrightarrow W$. More formally

$$\mathrm{Thom}(O(V,W); V-W) \wedge X_V \longrightarrow X_W$$

(Abelian gps, $\otimes$, $\mathbb{Z}$)

$$A \otimes X = \mathrm{Hom}(\mathbb{Z}, A \otimes X)$$

(ab, $\otimes$, $\{e\}$)

$$A \otimes X = \mathrm{Hom}(e, ?)$$

Thom $(O(V,W);W-V)$ as a set is

$$\bigvee_{V\to W} S^{W-V}$$