MATH 549 3-17-10

Note Title 3/17/2010

Recall MU is a C_2-equivariant spectrum under complex conjugation.

More about equiv spectra.

① $RO(G)$-graded homotopy gps

$RO(G)$ = real orthogonal representation ring of G.

$G = C_2$: There are 2 irreducible reps
1 ($\mathbb{R}$ with trivial action
σ ($\mathbb{R}$ with sign representation)

$RO(C_2)$ = free abelian gp on $\{1, \sigma\}$

$\sigma^2 = 1$

$G = C_4$: 3 irred reps : $1, \sigma, \lambda$

$\lambda = \mathbb{R}^2$ with rotation by $\pi/2$

$RO(C_4) = \mathbb{Z}\{1, \sigma, \lambda\}$

$G = C_8$: 5 irred reps : $1, \sigma, \lambda_1, \lambda_2, \lambda_3$

where $\lambda_j = \mathbb{R}^2$ with rotation by $\pi j/4$

The spectrum $\underline{S}^{-V}$ is defined as a prespectrum by

$E(W) = S^{W-V}$ for $V \subset W$ and

$W-V$ = orthogonal compliment of V in W.

Let $\underline{S}^{W-V} = \underline{S}^{-V} \wedge \underline{S}^{W}$ for any V and W

$\pi^{G}_{W-V}(\underline{X}) = [\underline{S}^{W-V}, \underline{X}]^{G}$ for a G-spectrum $\underline{X}$.

Think of the index $W-V$ as an element in $RO(G)$. Hence we have homotopy graded by $RO(G)$ rather than $\mathbb{Z}$. These are collectively denoted by

$$\pi_{\star}^{G}(X) \supset \pi_{*}^{G}(X)$$

Remark: For $n \in \mathbb{Z}$, $\pi_n^G(X) = [S^n, X]^G = [S^n, X^G]$
$$= \pi_n(X^G).$$

Example ① Let V be a representation for which $V^G = 0$. Then there is an equiv map $S^0 \xrightarrow{a_V} S^V$ $\qquad S^{V^G} \xrightarrow{a_V} S^V$

$a_V \in \pi_0^G(S^V) = \pi_{-V}^G(S^0)$ $\qquad a_V \in \pi_{|V^G|-V}^G(S^0)$

② Let W be an oriented rep., i.e. a hom $G \to SO(n)$ where $n = |W|$

$$S^W \xrightarrow{u_W} S^{|W|} \wedge H\mathbb{Z}$$

$$u_W \in \pi^G_{|W|-W}(H\mathbb{Z})$$

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2) How to get G-space/spectra from H-spaces/spectra for $H \subset G$.

a) For an H-space X, $G \times_H X$ is a G-space. The underlying space is a disjoint union of $|G/H|$ copies of X.

For an H-spectrum X, $G_+ \wedge_H X$ is a G-spectrum underlain is the wedge of $|G/H|$ of X.

b) For an H-space X, $\mathrm{Map}^H(G, X)$ is a G-space underlain by $X^{|G/H|}$. The analogous spectrum construction is denoted by $N_H^G X$, the (H to G) norm of X. It is a G-spectrum underlain by $X^{(|G/H|)}$.

e.g. [Notational convention: $|G| = g$

$N_H^G X = N_h^g X$]

Let X be MU as a C_2-spectrum

$H = C_2$ and $G = C_8$

$N_2^8 MU = MU^{(4)}$ $\qquad$ $N_2^4 MU = MU^{(2)}$

For a generator γ of G

$$\gamma(x_1 \wedge x_2 \wedge x_3 \wedge x_4) \qquad x_i \in MU$$

$$= \overline{x_4} \wedge x_1 \wedge x_2 \wedge x_3$$

$$\gamma^4(\quad) = \overline{x_1} \wedge \overline{x_2} \wedge \overline{x_3} \wedge \overline{x}^4$$

$$\gamma^8 = e$$

How we construct Ω

Start with $MU^{(4)}$ as a C_8-spectrum

Choose an element $x \in \pi_{19\rho}^{C_8}(MU^{(4)})$

where ρ = regular rep of C_8

$= 1 + \sigma + \lambda_1 + \lambda_2 + \lambda_3$

$MU^{(4)}$ is an E_∞-ring spectrum. We invert x, i.e. form the telescope

$$MU^{(4)} \xrightarrow{x} \Sigma^{-19\rho} MU^{(4)} \xrightarrow{x} \Sigma^{-38\rho} MU^{(4)} \longrightarrow \cdots$$

$$\tilde{\Omega} = x^{-1} MU^{(4)} \qquad \Omega = \tilde{\Omega}^{C_8} \simeq \tilde{\Omega}^{hC_8}$$

$$S^{19\rho} \xrightarrow{x} MU^{(4)} =: E$$

$$\Sigma^{19\rho} E \longrightarrow E \wedge E \xrightarrow{\text{mult}} E$$

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3 Flavors of fixed points

Let X be a G-space

$$X^G = \mathrm{Map}^G(pt, X)$$

Let EG be a contractible free G-space

$EC_2 = S^\infty$ with antipodal action

$$X^{hG} = \mathrm{Map}^G(EG, X)$$

The map $EG \to pt$ induces

$$X^G \longrightarrow X^{hG}$$

<u>Example</u> Suppose G acts trivially on X

Then $X^G = X$ $\qquad BG = EG/G =$ orbit space

$$\mathrm{Map}^G(EG, X) = \mathrm{Map}(BG, X)$$

Sullivan Conjecture (Miller's Thm) says when G is a finite gp and X is a finite complex, then $X^G \xrightarrow{\simeq} X^{hG}$

Let $G = C_2$ and $X = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$

$$\mathrm{Map}(BG, K(\mathbb{Z},2)) = \mathrm{Map}(\mathbb{R}P^\infty, K(\mathbb{Z},2))$$

π_0 of this is $H^2(\mathbb{R}P^\infty; \mathbb{Z})$.

It is possible to make similar definitions in spectra.

Fixed points behave badly in the stable category.

$$(X \wedge Y)^G \neq X^G \wedge Y^G$$

$$(\Sigma^\infty X)^G \neq \Sigma^\infty (X^G)$$

Let S^0 have trivial G-action

$\pi_0\left((S^0)^G\right) = A(G) =$ Burnside ring

tom Dieck

To form S^0 as a G-spectrum

$E(V) = S^V_0$

<u>Geometric fixed points</u>.

Replace $\{E(V)\}$ by F with

1) $F(V^G) = E(V)^G$

Alternate definition: Suppose G is a cyclic 2-group. The isotropy separation sequence is

$$EC_{2+} \longrightarrow S^0 \longrightarrow \tilde{E}C_2$$

G acts via $G \to C_2$ — trivial — cofiber

Def X = G-spectrum

$$\Phi^G X = (\tilde{E}C_2 \wedge X)^G = \text{geometric fixed pts of } X$$

Convenient properties

1) $\Phi^G(X \wedge Y) \simeq \Phi^G(X) \wedge \Phi^G(Y)$

2) $\Phi^G(\Sigma^\infty X) \simeq \Sigma^\infty(X^G)$ for a G-space X

3) A map $X \xrightarrow{f} Y$ is a G-equiv if $\Phi^H(f)$ is an equiv for each $H \subset G$.

Cor $\Phi^{C_2} MU \simeq MO$ $\quad G = C_2$

$MU(n)^G \simeq MO(n)$

$$\pi_{\star}^{G}(\widetilde{EC_2} \wedge X) = a_\sigma^{-1}\, \pi_{\star}^{G}(X)$$

$\pi_* \Phi^G X$ = integer graded part of

$$a_\sigma^{-1}\, \pi_{\star}^{G}(X)$$