MATH 549 2-3-10

Note Title 2/3/2010

Kervaire invariant problem

If M^{4k+2} is a smooth framed mfd, can $\Phi(M)$ be nonzero?

Early answers

YES for $k = 0, 1, 3$ $(j = 1, 2, 3)$

NO for $k = 2$ (Kervaire 1960)

This led to a nonsmoothable 10-mfd.

1967 Brown-Peterson

NO for $k > 0$ and k even

1969 Browder

YES only when $k = 2^{j-1} - 1$

($\cancel{so}$ $4k+2 = 2^{j+1} - 2$)

1970–1984 Barratt, Jones, Mahowald and Tangora

YES for k as above with $j = 4, 5$

2009 HHR

NO for $j \geq 7$ ($j = 6$ still open)

Browder gave a criterion for $k = 2^{j-1} - 1$. It involves the

Adams spectral sequence and Steenrod operations.

Some facts from homotopy theory

<u>Thm</u> Let A be an abelian (e.g. $\mathbb{Z}/2$) and n a positive integer. Then there is space $K(A,n)$ (Eilenberg - Mac Lane space) s.t.

$$H^n(X;A) \simeq [X, K(A,n)]$$

$\simeq$ homotopy classes of maps

$$X \longrightarrow K(A, n)$$

This set has a natural abelian group structure due to nice properties of $K(A, n)$.

Examples

$$K(\mathbb{Z}, 1) = S^1$$

$$K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$$

$$K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty$$

Let $K_n = K(\mathbb{Z}/2, n)$ and $H^*(X) = H^*(X; \mathbb{Z}/2)$

Suppose $\sigma \in H^{n+k}(K_n)$

[This gp is known in all cases]

$$H^{n+k}(K_n) = [K_n, K_{n+k}]$$

Let $x \in H^n(X) = [X, K_n]$

$$X \xrightarrow{x} K_n \xrightarrow{\sigma} K_{n+k}$$

The composition σx is an element in $H^{n+k}(X)$ i.e. σ induces a map $H^n(X) \longrightarrow H^{n+k}(X)$

This is a <u>cohomology operation</u>

It is natural in X, i.e. given

$$\begin{tikzcd}
W \arrow[r, "f"] & X \\
H^n W \arrow[d, "\sigma"] & H^n X \arrow[l, "f^*"'] \arrow[d, "\sigma"] \\
H^{n+k} W & H^{n+k} X \arrow[l, "f^*"']
\end{tikzcd}$$

diagram commutes

σ may or may not be a homomorphism

Facts about K_n

$$K_n \simeq \Omega K_{n+1} = \mathrm{Map}_*(S^1, K_{n+1})$$

$\Sigma K_n \xrightarrow{s_n} K_{n+1}$ related to ι_n

$H^n K_n = [K_n, K_n] \ni$ identity $= \iota_n$

$\quad \|$

$\mathbb{Z}/2$ generated by ι_n

$[\Sigma K_n, K_{n+1}] = H^{n+1}(\Sigma K_n)$

$= H^n(K_n) = \mathbb{Z}/2$

s_n induces a map

$$H^{n+k+1} K_{n+1} \xrightarrow{s_n^*} H^{n+k+1} \Sigma K_n = H^{n+k} K_n \ni \sigma$$

Thm $\sigma : H^n(X) \longrightarrow H^{n+k}(X)$ $\forall X$ is a hom iff σ is in image of s_n^*

If σ is in this image, it is also in the image of a similar map

$$H^{n+k+t} K_{n+t} \longrightarrow H^{n+k} K_n \quad \text{for any } t > 0$$

Such a σ is a stable cohomology operation.

Example where $k = n$

$$H^n X \xrightarrow{Sq^n} H^{2n} X$$
$$x \longmapsto x^2$$
$$(x+y) \longmapsto (x+y)^2 = x^2 + y^2$$

Since it is additive it is stable, i.e. related to similar operations

$$H^{n+t}(X) \xrightarrow{Sq^n} H^{2n+t}(X) \quad \text{for all } t \geq 0.$$

Steenrod squaring operation.

e.g. $H^* K_1 = \mathbb{Z}/2[x] \qquad x \in H^1$

Properties of the Sq^n:

i) $Sq^0 = 1 = \text{identity}$

ii) Cartan formula

$$Sq^n(xy) = \sum_{0 \le i \le n} Sq^i(x)\, Sq^{n-i}(y)$$

iii) Adem relation. Let $0 < a < 2b$

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$$

e.g. $Sq^1 Sq^{2n} = Sq^{2n+1}$

$Sq^1 Sq^{2n+1} = 0$

Sq^n is decomposable unless (for $n>0$)

$n = 2^k$ for some k

<u>Def</u> A monomial $Sq^{i_1} Sq^{i_2} \cdots Sq^{i_m}$ is <u>admissible</u> if

$i_1 \geq 2i_2, \; i_2 \geq 2i_3, \; \ldots \quad i_{m-1} \geq 2i_m$

Using the Adem relation any monomial is a sum of admissible ones.

Thm Any stable operation can be written in terms of the Sq^n's.

We get an algebra $\mathcal{A}_2$ of stable mod 2 cohomology operations. This is the mod 2 Steenrod algebra.

The admissible monomials form a basis for $\mathcal{A}_2$.

$H^*(X)$ is (naturally) a module over $\mathcal{A}_2$ subject to the <u>unstable condition</u>

$$Sq^n x = \begin{cases} x^2 & \text{if } x \in H^n \\ 0 & \text{if } |x| < n \end{cases}$$

$$Sq^1 Sq^2 Sq^1 = Sq^2 Sq^2 \qquad Sq^1 Sq^1 = 0$$

$$Sq^2 = Sq^3 \;\Rightarrow\; Sq^1 Sq^2 Sq^1 = Sq^3 Sq^1$$

$$Sq^1 Sq^2 = Sq^3$$

$Sq^2 Sq^1$ is admissible.

Milnor's formulation

Consider $\mathcal{A}_* = \mathrm{Hom}_{\mathbb{Z}/2}(\mathcal{A}_2, \mathbb{Z}/2)$

The Cartan formula leads to a coproduct

$$\mathcal{A}_2 \longrightarrow \mathcal{A}_2 \otimes \mathcal{A}_2$$

$$Sq^n \longrightarrow \sum_{0 \le i \le n} Sq^i \otimes Sq^{n-i}$$

This extends to an algebra map

Structure of $\mathcal{A}_*$

$\mathbb{Z}/2[\xi_1, \xi_2, \xi_3 \cdots]$

$|\xi_i| = 2^i - 1$

In $\mathcal{A}$ we have

multiplication $\mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$
not commu

comultiplication $\mathcal{A} \otimes \mathcal{A} \longleftarrow \mathcal{A}$
commutative

Dually we have

not cocommutative $\quad A_* \otimes A_* \longleftarrow A_*$

commutative $\quad A_* \otimes A_* \longrightarrow A_*$

$$\sum_{0 \leq i \leq n} \xi_{n-i}^{2^i} \otimes \xi_i \longleftarrow \xi_n$$

where $\xi_0 = 1$

This is equivalent to Adem's formula.

A question: Is there a map

$$S^{n+2^i-1} \xrightarrow{f} S^n \quad \text{for } n \gg 0$$

[the cofiber is $S^n \cup e^{n+2^i} = X$

$$\begin{array}{ccc} H^n X & \xrightarrow{Sq^{2^i}} & H^{n+2^i} X \\ \| & & \| \\ \mathbb{Z}/2 & & \mathbb{Z}/2 \end{array}$$
]

such that Sq^{2^i} acts nontrivially in $H^* X$?

Examples where the action is nontrivial:

$i=0 \qquad S^1 \xrightarrow{2} S^1$

$i=1 \qquad S^3 \xrightarrow{\eta} S^2$

$i=2 \qquad S^7 \xrightarrow{\nu} S^4$

$i=3 \qquad S^{15} \xrightarrow{\sigma} S^8$

Hopf map 1930

Thm (Adams 1961) You cannot do this for $i > 3$.