MATH 549 2-24-10

Note Title 2/24/2010

Browder's thm: There is a framed closed mfd with Kervaire invariant 1 in dim $2^{j+1}-2$ $\iff$ h_j^2 is a permanent cycle in the Adams SS (It can happen only in these dims)

If h_j^2 is a permanent cycle, it rep's a map $S^{2^{j+1}-2} \xrightarrow{\theta_j} S^0$

$h_j \in \mathrm{Ext}^{1,2^j}$
$|h_j| = 2^j - 1$

θ_j is known to exist for $1 \le j \le 5$.

<u>Thm (HHR)</u> θ_j does <u>not</u> exist for $j \geq 7$.

Strategy of proof: Construct a ring-spectrum Ω (nonconnective) with 3 properties $S^0 \to \Omega$

(i) If $\exists\, \theta_j$, then its image in $\pi_* \Omega$ is nontrivial (DETECTION)

(ii) $\Sigma^{256} \Omega \simeq \Omega$ (PERIODICITY)

(iii) $\pi_{-2} \Omega = 0$ (GAP)

(ii) + (iii) $\Rightarrow$ $\pi_{254}\Omega = 0$. If $\exists\ \theta_7$, its image in this gp is nontrivial by (i)

How to construct Ω ??? We need 2 tools

1. MU-theory (complex cobordism) both to construct Ω and to prove (i).
2. Equivariant stable homotopy theory needed to prove (ii) and (iii).

An introduction to MU-theory

Let $G^{\mathbb{C}}_{n,k}$ denote the space of complex n-planes in $\mathbb{C}^{n+k}$.
It is a compact complex analytic manifold of complex dimension nk.
It is also a complex projective variety. $H^*(G^{\mathbb{C}}_{n,k};\mathbb{Z})$ is known.

One has maps $G^{\mathbb{C}}_{n,k} \longrightarrow G^{\mathbb{C}}_{n,k+1}$ induced by $\mathbb{C}^{n+k} \longrightarrow \mathbb{C}^{n+k+1}$

Let $BU(n) = \varinjlim_{k} G^{\mathbb{C}}_{n,k} = G^{\mathbb{C}}_{n,\infty}$

$BO(n)$ in real case

$=$ space of complex n-planes in $\mathbb{C}^{\infty}$.

There is a $\mathbb{C}^n$-bundle over $G^{\mathbb{C}}_{n,k}$ with total space is

$$E^{\mathbb{C}}_{n,k} = \{(z,x) \in \mathbb{C}^{n+k} \times G^{\mathbb{C}}_{n,k} : z \in x\} \simeq G^{\mathbb{C}}_{n,k}$$

There is a map $G^{\mathbb{C}}_{n,k} \to E^{\mathbb{C}}_{n,k} \xrightarrow[\simeq]{p} G^{\mathbb{C}}_{n,k}$ where

$x \mapsto (0,x)$

identity

$p^{-1}(x) \approx \mathbb{C}^n$. Let $M^{\mathbb{C}}_{n,k}$ be its one point compactification. (Thom space)

Let $MU(n) = \varinjlim_{k} M^{\mathbb{C}}_{n,k}$

Properties:

a) $H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1, c_2, \cdots c_n]$

where $c_i \in H^{2i}$ are Chern classes which can be defined geometrically.

b) $H^{k+2n}(MU(n);\mathbb{Z}) \cong H^k(BU(n);\mathbb{Z})$

Under the map $BU(n) \longrightarrow MU(n)$, the image in H^* is (c_n).

a) $H^*(BO(n); \mathbb{Z}/2) = \mathbb{Z}/2[w_1, \dots, w_n]$

where $w_i \in H^i$ is the ith Stiefel-Whitney class, which can be defined geometrically.

b) Under the map $BO(n) \to MO(n)$ the image in $H^*(\ ; \mathbb{Z}/2)$ is (w_n).

$$H^{k+n}(MO(n); \mathbb{Z}/2) \cong H^k(BO(n); \mathbb{Z}/2)$$

The $\underline{MU}$ prespectrum is defined by

$$\underline{MU}_{2n} = MU(n)$$

$$\underline{MU}_{2n+1} = \Sigma MU(n)$$

We need a map $\Sigma^2 MU(n) \longrightarrow MU(n+1)$

We have maps $G^{\mathbb{C}}_{k,n} = G^{\mathbb{C}}_{n,k} \longrightarrow G^{\mathbb{C}}_{n+1,k}$

induced by $\mathbb{C}^{n+k} \longrightarrow \mathbb{C}^{n+k+1}$

This leads to a map $BU(n) \longrightarrow BU(n+1)$

$BU(n)$ has a $\mathbb{C}^n$-bundle over it

$BU(n+1)$ " $\mathbb{C}^{n+1}$ "

Induced vector bundles

Let $p: E \to Y$ be a $\mathbb{C}^n$-bundle over Y. This means $p^{-1}(y) \cong \mathbb{C}^n$ $\forall y \in Y$ with certain conditions; and let $f: X \to Y$ be any map. Let

$$\tilde{E} = \{ (x, e) \in X \times E \, ; \, f(x) = p(e) \in Y \}$$

$\tilde{E}$ is a $\mathbb{C}^n$-bundle over X, the <u>bundle induced by f</u>.

We also get a map between the

one point compactifications (Thom spaces) of $\tilde{E}$ and E.

<u>Example</u> $BU(n) \xrightarrow{f} BU(n+1)$

$BU(n+1)$ has a $\mathbb{C}^{n+1}$-bundle γ^{n+1}

$BU(n)$ " $\mathbb{C}^{n}$ " γ^{n}

and a $\mathbb{C}^{n+1}$-bundle $f^*\gamma^{n+1}$

We find $f^*(\gamma^{n+1}) = \gamma^n \oplus \varepsilon$

where ε is the trivial $\mathbb{C}^1$-bundle.

its Thom space is $\Sigma^2 MU(n)$

$= \Sigma^2$ (Thom space for γ^n)

It induces the desired map

$$\Sigma^2 MU(n) \longrightarrow MU(n+1)$$

This is the $(2n+1)$th structure map for the spectrum MU.

MO is the prespectrum defined by $MO_n = MO(n)$ with similar structure map.

Properties of MU and MO.
Both are ring spectra

$$\pi_* MU = \mathbb{Z}[x_1, x_2, x_3, \dots] \quad x_i \in \pi_{2i}$$

$$\pi_* MO = \mathbb{Z}/2[y_2, y_4, y_5, y_6, y_8, \dots]$$

$$y_i \in \pi_i, \quad i \neq 2^j - 1$$

$$H_*(MU; \mathbb{Z}) = \mathbb{Z}[b_1, b_2, \dots] \quad b_i \in H_{2i}$$

$$H_*(MO; \mathbb{Z}/2) = \mathbb{Z}/2[\bar{b}_1, \bar{b}_2 \dots] \quad \bar{b}_i \in H_i$$

[Recall if E is a prespectrum

$$H_i(E) = \varinjlim_n H_{n+i}(E_n)$$

$$\pi_i(E) = \varinjlim_n \pi_{n+i}(E_n)$$
]

Given a space or spectrum X, we define $MU_n(X) = \pi_n(X \wedge MU)$, the MU-homology or complex bordism of X, and $MU^n X = [X, \Sigma^n MU]$, the MU-cohomology $\cong [\Sigma^{-n}X, MU]$ or complex cobordism of X. Similar defs for MO.

These have similar formal properties to H_* and H^*. In particular for a space X, $MU^* X$ has cup products.

$$MU_*(S^0) \cong \pi_*(MU \wedge S^0) = \pi_*(MU)$$

$$\cong \text{as before}$$

$$MU^n(S^0) = [S^0, \Sigma^n MU] = [S^{-n}, MU]$$

space $\qquad$ spectrum

$$= \pi_{-n}(MU)_*$$

$$MU^*(pt) = MU^*(S^0) = \mathbb{Z}[x_1, x_2, \dots] \qquad x_i \in MU^{-2i}(S^0),$$

For a space X, $MU^*(X)$ is an algebra over $MU^*(pt)$.

Let $X = \mathbb{C}P^\infty$ = infinite dimensional complex projective space

$$= BU(1),$$

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[x] \quad \text{where } x \in H^2(-)$$

$$= H^*(pt)[x]$$

$$MU^*(\mathbb{C}P^\infty; \mathbb{Z}) = MU^*(pt)[[x]] \quad \text{where } x \in MU^2(-)$$

$$MU^*(\mathbb{C}P^n) = MU^*(pt)[x] / (x^{n+1})$$

$$\mathbb{C}P^\infty = \varinjlim \mathbb{C}P^n$$

$$MU^*(\mathbb{C}P^\infty) = \varprojlim MU^*(\mathbb{C}P^n)$$

$$= MU^*(pt)[[x]]$$

e.g. $\quad x + \hat{x}_1 x^2 + \hat{x}_2 x^3 + \cdots \in MU^2(\mathbb{C}P^\infty)$

$$|x| = 2 \qquad |\hat{x}_i| = -2i \qquad |\hat{x}_i x^{i+1}| = 2$$