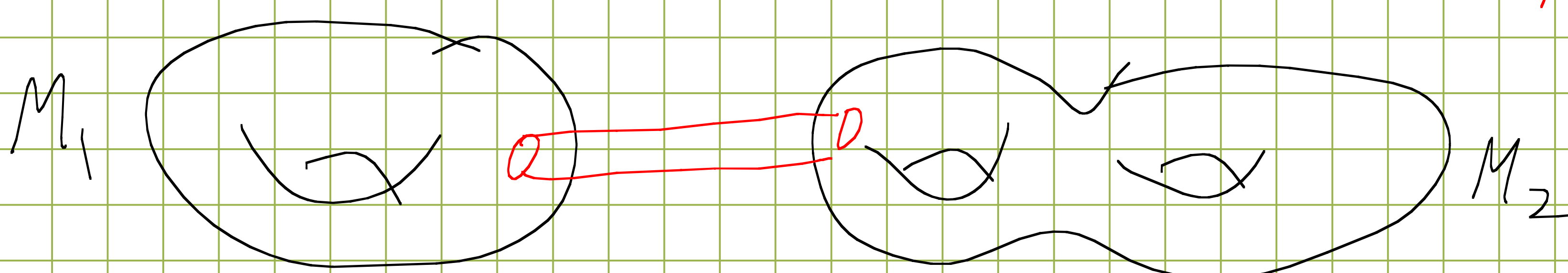
MATH549 2-1-10

Note Title 2/1/2010

The work of Kervaire-Milnor

Θ_n = gp of mfds homeomorphic to S^n modulo diffeomorphism
(gp operation is connected sum)
$n \geq 5$

Given 2 mfds M_1^n and M_2^n



Connected sum $M_1 \# M_2$

If M_1 and M_2 are n-spheres, so is $M_1 \# M_2$.

Any n-sphere Σ^n has a framing, so we get a hom

$$\Theta_k \longrightarrow \pi_{k+n}(S^n) / \operatorname{Im} J \qquad n \gg 0$$

$$\| $$

$$(\operatorname{coker} J)_k$$

A Σ^n is in the kernel if it bounds a framed mfd.

Let bP_k = the gp of k-spheres that bound framed mfds

We have an exact sequence

$$0 \to bP_k \to \Theta_k \to (\operatorname{coker} J)_k$$

Suppose $\Sigma^k = \partial N^{k+1}$ where N^{k+1} is framed. What can we do with N to make it nicer?

ANSWER: SURGERY

Suppose N is $(j-1)$-connected and we have a generator x of $\pi_j N$ with $j \leq k/2$. We know that x is rep'd by an embedding $S^j \hookrightarrow N^{k+1}$ with trivial normal bundle, i.e. S^j has a closed nbhd $\approx S^j \times D^{k+1-j}$.

SIMPLE FACT: $D^{k+2} \approx D^{j+1} \times D^{k+1-j}$

$$S^{k+1} = \partial D^{k+2} = \partial D^{j+1} \times D^{k+1-j} \cup D^{j+1} \times \partial D^{k+1-j}$$

$$= S^{j} \times D^{k+1-j} \cup D^{j+1} \times S^{k-j}$$

Intersection $= S^j \times S^{k-j}$

SURGERY: remove $S^k \times D^{k+1-j}$,

leaving a boundary of the form $S^j \times S^{k-j}$,

and add $D^{j+1} \times S^{k-j}$. We replace N

by mfd N' in which x has

been killed.

We can kill each generator of π_j and replace every N by a j-connected framed mfd.

We have $\Sigma^k = \partial N^{k+1}$, N framed

Using surgery we can convert N to $(k/2)$-connected mfd.

① Suppose k is even

$$\Sigma^{2k} = \partial N^{2k+1}$$

Using surgery we can make N k-connected, i.e. $\pi_j N = 0$ for $j \leq k$

This means N is contractible.

Hence (using Poincaré conjecture)

$N \cong D^{2k+1}$ and $\Sigma^{2k} \cong S^{2k}$

and $bP_{2k} = 0$

$2k > 4$

② $k \equiv 1 \mod 4$

$$\Sigma^{4k+1} = \partial N^{4k+2}, \quad N \text{ framed}$$

We can assume (via surgery) that N is $2k$-connected. We get a Kervaire invariant $\Phi(N) \in \mathbb{Z}/2$.

For $k=2$, Kervaire showed any framed smooth closed mfd M^{4k+2} has $\Phi(M) = 0$. This is false for $k = 0, 1$ or 3.

For general k there are 2 possibilities:

Kervaire's statement

every framed smooth closed M^{4k+2} has $\Phi(M) = 0$, and Σ^{4k+1} is not S^{k+1}.

TRUE $\Leftrightarrow$ every framed $(4k+2)$-mfd is cobordant to a sphere

Thus we have an exact sequence

$$0 \to \Theta_{4k+2} \to (\operatorname{cok} J)_{4k+2} \to bP_{4k+1} \to \Theta_{4k+1} \to (\operatorname{cok} J)_{4k+1} \to 0$$

$$bP_{4k+1} = \mathbb{Z}/2$$

The middle gp is generated by Kervaire's handcuff. If Kervaire's stmt is true, it has a nontrivial image in Θ_{4k+1}; if not, Kervaire's boundary is a standard sphere. We can glue on a D^{4k+2} and get a framed mfd not cobordant to a sphere, so the map $\Theta_{4k+2} \to \operatorname{cok} J_{4k+2}$ is not onto.

Suppose we are in dim $4k-1$

$$\begin{array}{c}
0 \to \Theta_{4k} \to (\mathrm{coker}\, J)_{4k} \to bP_{4k-1} \to \Theta_{4k-1} \to (\mathrm{coker}\, J)_{4k-1} \to 0 \\
\phantom{0 \to \Theta_{4k} \to (\mathrm{coker}\, J)_{4k} \to}\ \| \\
\phantom{0 \to \Theta_{4k} \to (\mathrm{coker}\, J)_{4k} \to}\ ?
\end{array}$$

$k \geq 2$

$\Sigma^{4k-1} = \partial N^{4k}$. Using surgery,

we can assume N is $(2k-1)$-connected.

Consider the topological mfd

$\hat{M} = N \cup_{\Sigma^{4k-1}} D^{4k}$, $\hat{M}$ is $(2k-1)$-connected