MATH 549 1-27-10

Note Title 1/27/2010

Recall the definition of the Arf-invariant

We have nonsingular skew-symmetric bilinear form λ on a free abelian gp H.

A quadratic refinement of the mod 2 reduction of λ is a map

$$q : H/2 \longrightarrow \mathbb{Z}/2 \quad \text{with}$$

$$q(a+b) = q(a) + q(b) + \lambda(a,b)$$

Let $\{a_i, b_i : 1 \leq i \leq g\}$ be a symplectic basis for λ. Then

$$\mathrm{Arf}(q) = \sum_{i=1}^{g} q(a_i)\, q(b_i) \in \mathbb{Z}/2$$

Exercise: Find $\mathrm{Arf}(q)$ when H has rank 4.

Kervaire's work 1960

Let M be a framed $(4k+2)$-manifold which $2k$-connected. (Any framed $(4k+2)$-mfd is cobordant to one that is $2k$-connected.)

This means $H_i M = \begin{cases} \mathbb{Z} & \text{if } i = 0, 4k+2 \\ \text{free ab gp} & \text{if } i = 2k+1 \\ 0 & \text{otherwise} \end{cases}$

$= \pi_{2k+1} M$

Let $H = H_{2k+1}(M;\mathbb{Z})$. It has a nonsingular skew symmetric bilinear form λ related to intersection #s and dual to cup product. Each element of λ is represented by an immersion

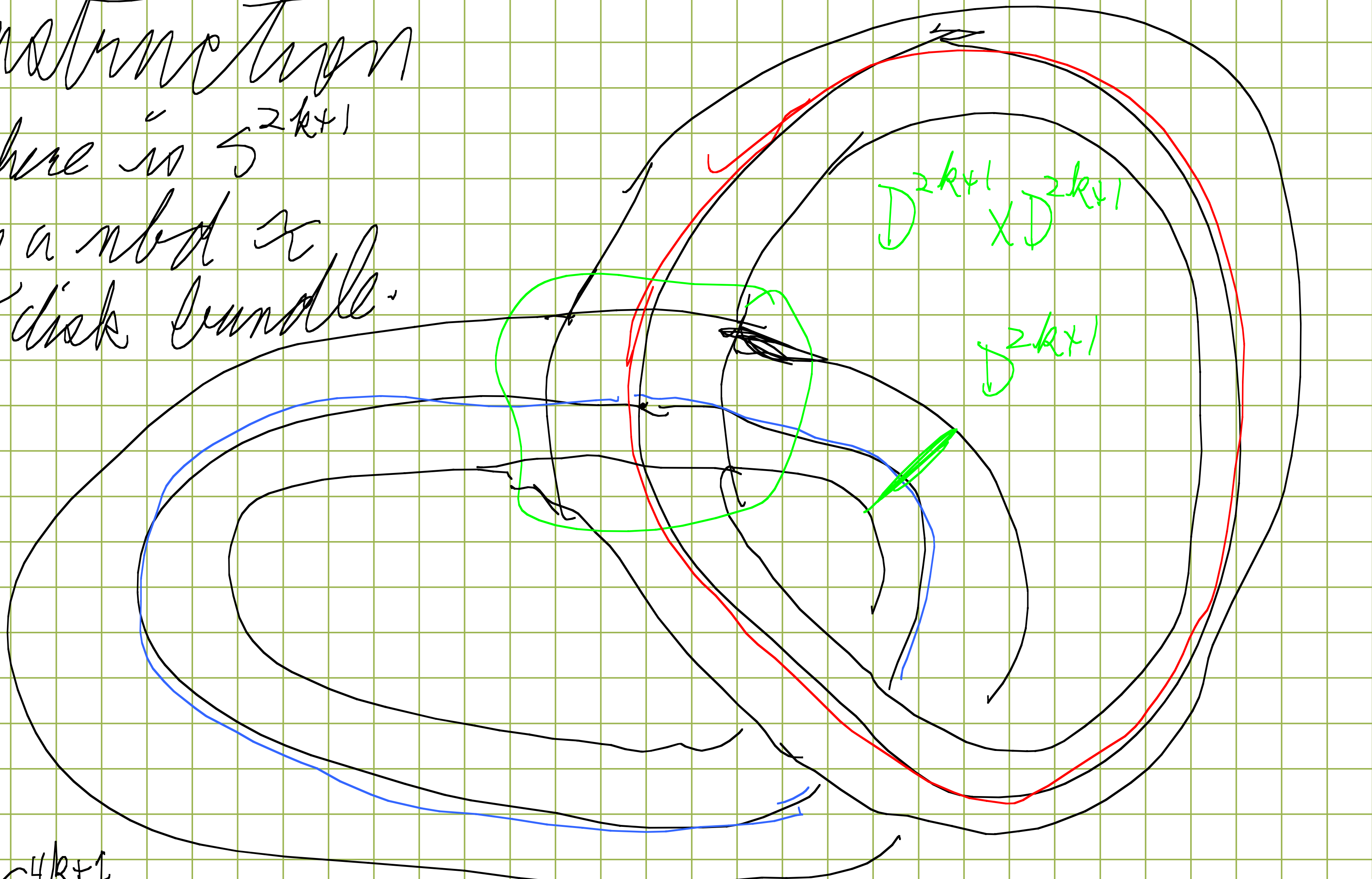
$$S^{2k+1} \hookrightarrow M^{4k+2}$$

It has a normal $\mathbb{R}^{2k+1}$-bundle in M which is stably trivial.

Construction

Each sphere is S^{2k+1}

Each has a nbd $\cong$ tangent disk bundle.



N^{4k+2}

$\partial N^{4k+2} \cong S^{4k+1}$

This is a $(4k+2)$-manifold N with boundary homeomorphic to S^{4k+1}

Consider the diagonal embedding

$$S^{2k+1} \longrightarrow S^{2k+1} \times S^{2k+1}$$

Its normal bundle is iso to the tangent bundle of S^{2k+1}.

Consider the topological manifold M obtained by attaching a D^{4k+2} to N along the boundary. We have

$$H_i M = \begin{cases} \mathbb{Z} & \text{if } i = 0, 4k+2 \\ \mathbb{Z} \oplus \mathbb{Z} = H & i = 2k+1 \\ 0 & \text{otherwise} \end{cases}$$

H has a nonsing skew symm bilin form

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

A it has a quadratic refinement q related to the normal bundle of each sphere and $\mathrm{Arf}(q) = 1$

Let M^{4k+2} be a $2k$-connected framed mfd as before. Then there is a quadratic refinement q on $H = H_{2k+1}(M; \mathbb{Z}/2)$ defined in terms of the normal bundle on each S^{2k+1} representing an element of H.

The Kervaire invariant $\Phi(M)$ is $\mathrm{Arf}(q)$.

Using the N^{4k+2} above we get a mfd M with $\Phi(M) = 1$

Kervaire proved that for $k=2$ and M smooth, then $\Phi(M)=0$.
Therefore the M^{10} above does not have a smooth structure, and ∂N^{10} is not diffeomorphic to S^9.

Question: What happens for other k?

M is smooth for $k = 0, 1$ and 3.

In those cases S^{2k+1} has trivial tangent bundle and $M \simeq S^{2k+1} \times S^{2k+1}$

It can be framed so it represents a nontrivial elt in $\pi_{4k+2+n}(S^n)$ for $n >> 0$.

~1967 Brown-Peterson proved for $k > 0$ and even, if M is smooth then $\Phi(M) = 0$.

~1969 Browder showed $\Phi(M)=0$ for smooth M unless $k = 2^{j-1}-1$ $\quad j>0$

$$|M| = 4k+2 = 2^{j+1}-2$$

We know there are simple examples for $j = 1, 2, 3,$

Browder gave an additional criterion in terms of the Adams spectral sequence.

Work of Kervaire – Milnor on exotic spheres 1963

1956 Milnor found some exotic 7-spheres

1960 Kervaire an exotic 9-sphere

Their methods were different.

Studied the problem in terms of framed cobordism

An manifold homeo to S^k can

always be framed.
Pontryagin showed the framed cobordism is iso to $\pi_{k+n} S^n$ for $n \gg 0$.

$$J : \pi_k SO(n) \longrightarrow \pi_{n+k}(S^n)$$

is related to different framings on the standard sphere S^k. This means we should replace framed cobordism gp by coker J.

Let Θ_k = gp of diffeomorphism classes of manifolds $\simeq S^k$ [assume $k \geq 5$] under connected sum

$$bP_k \longrightarrow \Theta_k \longrightarrow (\operatorname{coker} J)_k$$

still unknown much studied

bP_k $\parallel$ exotic spheres bounding framed mfds.

determined by K-M

Let Σ^k be a mfd homeo to S^k

Suppose $\Sigma^k = \partial N^{k+1}$, N framed.

They need surgery to simplify N without changing ∂N.

<u>Consider $\pi_1 N$.</u> Can show it is generated by embedded circles with trivial normal bundles in N

Suppose N is $(i-1)$ connected (for $i < k/2$)

$\pi_1 N$ is generated by embedded S^1's with trivial normal bundle.

This π_1 can be surgically removed.

NEXT TIME