$$\cong \int_{\mathcal{D}\times\mathcal{D}} \mathcal{D}(A \oplus B, D) \otimes \mathcal{D}(D_1, A) \otimes \mathcal{D}(D_2, B)$$
$$\cong \int_{\mathcal{D}\times\mathcal{D}} \mathcal{D}(D_1, A) \otimes \mathcal{D}(D_2, B) \otimes \mathcal{D}(A \oplus B, D)$$
$$\cong \int_{\mathcal{C}} \mathcal{C}((D_1, D_2), (A, B)) \otimes F((A, B))$$

where $\mathcal{C} := \mathcal{D} \times \mathcal{D}$ and $F((A, B)) := \mathcal{D}(A \oplus B, D)$

$$\cong F((D_1, D_2)) \qquad \text{by Proposition 3.2.25}$$
$$\cong \mathcal{D}(D_1 \oplus D_2, D) = (よ^{D_1 \oplus D_2})_D. \qquad \square$$

The following is proved by Mandell *et al* in [MMSS01, 22.1] in the case of topological categories.

Proposition 3.3.15. Lax symmetric monoidal functors and commutative algebras. *The category of (commutative) monoids in $[\mathcal{D}, \mathcal{V}]$ is isomorphic to that of lax (symmetric) monoidal functors $\mathcal{D} \to \mathcal{V}$ (Definition 2.6.19).*

Proof Let $R : \mathcal{D} \to \mathcal{V}$ be lax (symmetric) monoidal. Then, in the notation of Definition 2.6.19, we have a unit map $\iota : \mathbf{1} \to R(\mathbf{0})$ and a natural transformation μ from $R(-) \otimes R(-)$ to $R(- \oplus -)$. By the definition of the tensored Yoneda functor $F^{\mathbf{0}}$ and the Yoneda functor $\mathbf{1} = よ^{\mathbf{0}}$ of Yoneda Lemma 2.2.10, the maps ι and μ determine and are determined by the maps $\eta : \mathbf{1} \to R$ and $m : R \otimes R \to R$ of Definition 2.6.58 that give R the structure of a (commutative) monoid. $\square$

3.4 Simplicial sets and simplicial spaces

The category of simplicial sets is a convenient combinatorial substitute for that of topological spaces and a widely used tool in homotopy theory. A thorough modern account can be found in [GJ99].

3.4A The category of finite ordered sets

Let $\mathbf{\Delta}$ be the category of finite ordered sets $[n] = \{0, 1, \ldots, n\}$ and order preserving maps. It is an easy exercise to show that any such map can be written as a composite of the following ones:

- the **face maps** $d_i : [n-1] \to [n]$ for $0 \leqslant i \leqslant n$, where d_i is the order preserving monomorphism that does not have i in its image and
- the **degeneracy maps** $s_i : [n+1] \to [n]$ for $0 \leqslant i \leqslant n$, where s_i is the order preserving epimorphism sending i and $i+1$ to i.

These satisfy the **simplicial identities**:

(i) $d_i d_j = d_{j-1} d_i$ for $i < j$
(ii) $d_i s_j = s_{j-1} d_i$ for $i < j$
(iii) $d_i s_j = id$ for $i = j$ and for $i = j + 1$
(iv) $d_i s_j = s_j d_{i-1}$ for $i > j + 1$
(v) $s_i s_j = s_j s_{i-1}$ for $i > j$.

Definition 3.4.1. *A* **simplicial set** X *is a functor* $\mathbf{\Delta}^{op} \to \mathcal{S}et$. *It is common to denote its value on* $[n]$ *by* X_n *and call it the* **set of** n**-simplices** *of* X. *A simplicial set* X *thus consists of a collection of sets* X_n *for* $n \geqslant 0$, *along with face maps* $d_i : X_n \to X_{n-1}$ *and degeneracy maps* $s_i : X_n \to X_{n+1}$ *for* $0 \leqslant i \leqslant n$ *satisfying the identities (i)–(v) above. A simplex is* **nondegenerate** *if it is not in the image of any degeneracy map* s_i. *The category* $\mathcal{S}et_{\mathbf{\Delta}}$ *of simplicial sets is the category of such functors with natural transformations as morphisms.*

More generally a **simplicial object** X **in a category** $\mathcal{C}$ *is a functor* $X : \mathbf{\Delta}^{op} \to \mathcal{C}$. *It is common to write it as* $X_\bullet$ *to emphasize its simplicial nature. We denote the category of simplicial objects in* $\mathcal{C}$ *by* $\mathcal{C}_{\mathbf{\Delta}}$.

Similarly a **cosimplicial object** Y **in a category** $\mathcal{C}$, *sometimes denoted by* $Y^\bullet$, *is a* $\mathcal{C}$ *valued functor on* $\mathbf{\Delta}$ *whose value on* $[n]$ *is denoted by* Y^n. *It consists of a collection of objects* Y^n *in* $\mathcal{C}$ *for* $n \geqslant 0$, *along with coface maps* $d^i : Y^{n-1} \to Y^n$ *and codegeneracy maps* $s^i : Y^{n+1} \to Y^n$ *for* $0 \leqslant i \leqslant n$ *satisfying identities dual to (i)–(v) above. We denote the category of cosimplicial objects in* $\mathcal{C}$ *by* $\mathcal{C}^{\mathbf{\Delta}}$. *In particular, a* **cosimplicial space** *is an object in the category* $\mathcal{T}op^{\mathbf{\Delta}}$ *of functors* $\mathbf{\Delta} \to \mathcal{T}op$.

For an object C *in* $\mathcal{C}$, *we denote by* $cs_*(C)$ *the* **constant simplicial object at** C, *the functor* $\mathbf{\Delta}^{op} \to \mathcal{C}$ *sending each object to* C *and each morphism to* 1_C. *The* **constant cosimplicial object at** C, $cc_*(X)$ *is similarly defined.*

Simplicial sets are ubiquitous in homotopy theory, but cosimplicial sets are rarely considered. Cosimplicial spaces are more common.

Definition 3.4.2. *The cosimplicial space* $\Delta^\bullet$, **the cosimplicial standard simplex**, *is the functor* $[n] \mapsto \Delta^n$, *where the* **standard** n**-simplex** Δ^n *is the space*

$$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbf{R}^{n+1} \colon t_i \geqslant 0 \text{ and } \sum_i t_i = 1 \right\}.$$

It is homeomorphic to the n*-disk* D^n. *Its* **boundary** $\partial\Delta^n$ *is the set of points with at least one coordinate equal to 0; it is homeomorphic to* S^{n-1}. *The* i**th face** Δ^n_i *for* $0 \leqslant i \leqslant n$ *is the set of points with* $t_i = 0$; *it is homeomorphic to* D^{n-1}. *The* i**th horn** Λ^n_i *is the complement of the interior of the* i*th face in the boundary, the set of points with at least one vanishing coordinate and with* $t_i > 0$. *It is also homeomorphic to* D^{n-1}. *It is an* **inner horn** *if* $0 < i < n$; *otherwise it is an* **outer horn**.

The cosimplicial standard simplicial set $\Delta[\bullet]$ *(called the cosimplicial standard simplex in [Hir03, Definition 15.1.15]) is the functor* $[n] \mapsto \Delta[n]$, *where the simplicial set* $\Delta[n]$ *(also called the* **standard** n**-simplex***) is given by*

$$\Delta[n]_k = \mathbf{\Delta}([k], [n]).$$

The singular chain complex for Y is obtained from the free abelian groups on these sets by defining a boundary operator in terms of the face maps d_i.

Definition 3.4.3. The geometric realization $|X|$ **(or** $\mathcal{R}e(X)$**) of a simplicial set** X *is the coend (Definition 2.4.5)*

$$|X| := \int_{\mathbf{\Delta}} X_n \times \Delta^n.$$

This means the topological space $|X|$ *is the quotient of the union of all of the simplices of* X*,*

$$\coprod_n X_n \times \Delta^n,$$

obtained by gluing them together appropriately. Equivalently it is the quotient of a similar disjoint union using only the nondegenerate simplices of X*. In particular the space* Δ^n *is* $|\Delta[n]|$ *for the simplicial set* $\Delta[n]$ *of Definition 3.4.2.*

The **geometric realization** $|X|$ **of a simplicial space** X *is similarly defined as a quotient of the union of the spaces* $X_n \times \Delta^n$*, whose topologies are determined by those of the spaces* X_n *as well the spaces* Δ^n*.*

Remark 3.4.4. *Following common practice, we are using the term "standard* n*-simplex" for both the topological space* Δ^n *and the simplicial set* $\Delta[n]$ *of Definition 3.4.2 in hopes that the distinction between the two will be clear from the context. Note that* $|\Delta[n]| \cong \Delta^n$*, so* $|\Delta[\bullet]| \cong \Delta^\bullet$*.*

Remark 3.4.5. The realization of a bisimiplicial set. *It follows from the definitions that the coend*

$$\int_{\mathbf{\Delta}} X_n \times \Delta[n]$$

is the simplicial set X *itself. Now suppose that* X *is a* **bisimplicial set**, *meaning a simplicial object in the category of simplicial sets or equivalently set valued functor on* $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$*. Then in the coend above, each* X_n *is itself a simplicial set, and the coend is another simplicial set* $|X|$*. Hirschhorn [Hir03, Definition 15.11.1] calls this the* **realization** *of the bisimplicial set* X*. In [Hir03, Theorem 15.11.6] he shows that it is naturally isomorphic to the diagonal simplicial set*

$$\mathbf{\Delta}^{op} \xrightarrow{\ diag\ } \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \xrightarrow{\ X\ } \mathcal{S}et. \qquad (3.4.6)$$

Definition 3.4.7. The singular functor. *For a topological space Y the simplicial set $\mathcal{S}ing(Y)$ (the* **singular complex** *of Y) is given by letting $\mathcal{S}ing(Y)_n$ be the set of* **all** *continuous maps $\Delta^n \to Y$. The face and degeneracy operators are defined in terms of the coface and codegeneracy operators on $\mathbf{\Delta}$.*

The following is proved by May in [May67, 14.1].

Proposition 3.4.8. $|X|$ as a CW complex. *The geometric realization $|X|$ of a simplicial set X is a CW complex with one n-cell for each nondegenerate n-simplex of X.*

Similarly we have a map

$$\coprod_n X_n \to \int_{\mathbf{\Delta}} X_n,$$

which is the set $\pi_0|X|$ of path connected components of $|X|$. Thus collapsing each Δ^n to a point in Definition 3.4.3 gives a map

$$|X| = \int_{\mathbf{\Delta}} \Delta^n \times X_n \xrightarrow{\quad \epsilon \quad} \int_{\mathbf{\Delta}} X_n = \pi_0|X|. \tag{3.4.9}$$

A simplicial space X, i.e., a functor $X : \mathbf{\Delta}^{op} \to \mathcal{T}op$, has a geometric realization $|X|$ defined as in Definition 3.4.3, but with the not necessarily discrete topology of X_n taken into account.

For a simplicial set X, $|X^{[n]}|$ is the n-skeleton of the CW complex $|X|$.

The following was proved by Kan in [Kan58a].

Proposition 3.4.10. The equivalence of $\mathcal{S}et_{\mathbf{\Delta}}$ and $\mathcal{T}op$ and of their pointed analogs. *As a functor from $\mathcal{S}et_{\mathbf{\Delta}}$ to $\mathcal{T}op$, geometric realization of Definition 3.4.3 is the left adjoint of $\mathcal{S}ing$, the singular functor of Definition 3.4.7. The adjunction*

$$|\cdot| : \mathcal{S}et_{\mathbf{\Delta}} \underset{\perp}{\rightleftarrows} \mathcal{T}op : \mathcal{S}ing$$

and its pointed analog are equivalences of categories.

In particular for an arbitrary space X one has a weak homotopy equivalence $|\mathcal{S}ing(X)| \to X$ whose source is a CW complex. For this reason, e.g., in [BK72] (the "yellow monster"), the terms "space" and "simplicial set" are sometimes used interchangeably.

Definition 3.4.11. Topological and simplicial categories.

(i) *When $\mathcal{V} = (\mathcal{T}op, \times, *)$, we say that a $\mathcal{V}$-category is a* **topological category***. We denote the category of topological categories by $\mathcal{CAT}_{\mathcal{T}op}$ and that of small topological categories by $\mathcal{C}at_{\mathcal{T}op}$.*

(ii) *When $\mathcal{V} = (\mathcal{T}, \wedge, S^0)$, we say that a $\mathcal{V}$-category is a* **pointed topological category***. We denote the category of pointed topological categories by $\mathcal{CAT}_{\mathcal{T}}$ and that of small pointed topological categories by $\mathcal{C}at_{\mathcal{T}}$..*

(iii) When $\mathcal{V} = (\mathcal{S}et_{\boldsymbol{\Delta}}, \times, *)$*, we say that a* $\mathcal{V}$*-category is a* **simplicial category**. *We denote the category of simplicial categories by* $\mathcal{CAT}_{\boldsymbol{\Delta}}$ *and that of small simplicial categories by* $\mathcal{C}at_{\boldsymbol{\Delta}}$.

(iv) When $\mathcal{V} = (\mathcal{S}et_{\boldsymbol{\Delta}*}, \wedge, S^0)$*, we say that a* $\mathcal{V}$*-category is a* **pointed simplicial category**. *We denote the category of simplicial categories by* $\mathcal{CAT}_{\boldsymbol{\Delta}*}$ *and that of small pointed simplicial categories by* $\mathcal{C}at_{\boldsymbol{\Delta}*}$.

We will see below in Corollary 5.6.16 that every topological model category is also a simplicial one.

The adjunction

$$|\cdot| : \mathcal{S}et_{\boldsymbol{\Delta}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{T}op : \mathcal{S}ing$$

leads to

$$|\cdot| : \mathcal{CAT}_{\boldsymbol{\Delta}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{CAT}_{\mathcal{T}op} : \mathcal{S}ing$$

(see Definition 3.4.11) in the obvious way. Given a simplicial category $\mathcal{C}$, we define the topological category $|\mathcal{C}|$ to have the same objects as $\mathcal{C}$ with morphism spaces

$$|\mathcal{C}|(X, Y) = |\mathcal{C}(X, Y)|,$$

and given a topological category $\mathcal{D}$, we define the simplicial category $\mathcal{S}ing(\mathcal{D})$ to have the same objects as $\mathcal{D}$ with simplicial morphisms sets

$$\mathcal{S}ing(\mathcal{D})(X, Y) = \mathcal{S}ing(\mathcal{D}(X, Y)).$$

3.4B The nerve of a small category

Definition 3.4.12. The nerve and classifying space of a small (topological) category. *For a small category* J*, the nerve* $N(J)$ *is the simplicial set given by*

$$N(J)_n = \mathcal{C}at([n], J)$$

where $[n]$ *here denotes the linearly ordered set* $\{0, \ldots, n\}$ *regarded as a category. The* **classifying space** BJ *is the geometric realization of the nerve,* $|N(J)|$.

For a small topological category D*, the similarly defined nerve* $N(D)$ *is a simplicial space whose geometric realization (see Definition 3.4.3) is the classifying space* BD.

In other words, $N(J)_n$ is the set of diagrams in J of the form

$$j_0 \to j_1 \to \cdots \to j_{n-1} \to j_n. \tag{3.4.13}$$

Of the $n+1$ face maps $N(J)_n \to N(J)_{n-1}$, $n-1$ are obtained by composing each of the $n-1$ pairs of adjacent arrows above, and the other two are obtained