Math 443 Final exam

December 16, 2019

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

Be sure to write your name on your bluebook.
INSERT THIS PAGE WITH THE HONOR PLEDGE SIGNED INTO YOUR BLUEBOOK.
Use a separate page (or pages) for each problem. Show all of your work.

1. **Euler characteristic question.** (20 POINTS) Let X be a finite CW-complex which is the union of two sub-CW-complexes A and B such that the intersection $A \cap B$ is also a sub-CW-complex. Show that the Euler characteristic $\chi(X)$ satisfies the formula

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Solution: Let x_i, a_i, b_i and c_i denote the number of i-cells in X, A, B and $A \cap B$ respectively. Each cell of X lies in either A or B or possibly both. This means that $x_i = a_i + b_i - c_i$; we subtract c_i so as not to count cells in the intersection twice. It follows that

$$\begin{aligned}
\chi(X) &= \sum_{i \geq 0} (-1)^i x_i \\
&= \sum_{i \geq 0} (-1)^i (a_i + b_i - c_i) \\
&= \sum_{i \geq 0} (-1)^i a_i + \sum_{i \geq 0} (-1)^i b_i - \sum_{i \geq 0} (-1)^i c_i \\
&= \chi(A) + \chi(B) - \chi(A \cap B).
\end{aligned}$$

2. **Another Euler characteristic question.** (20 POINTS) Let X be a graph with V vertices and E edges. Embed it in $\mathbf{R}^3$ (there is a theorem saying that any graph can be embedded in 3-space; there are some that cannot be embedded in the plane) and let Y be the space of all points within ϵ (a sufficiently small positive number) of the image of X. It is a 3-manifold bounded by a surface M. Find the Euler charcterisitic $\chi(M)$ and prove your answer.

HINT: Think of the building set in the lounge, the one with steel balls and black magnetic rods. We are going to build something with V balls and E rods. Find the Euler characteristic of the set of V 2-spheres bounding the V balls. Think about how the Euler characteristic of the surface changes each time you add a rod. *You may use the fact that*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

under suitable hypotheses on A and B.

Solution: The Euler characteristic of the disjoint union of V 2-spheres is $2V$. When we add an edge to the graph, we remove a disk from each of two (not necessarily distinct) spheres. This reduces χ by two. We then add a cyclinder by gluing its two boundary components to the two circles created by removing the two disks. This does not change χ, because both the cylinder and its boundary components have Euler characteristic zero. We do this E times, so $\chi(M) = 2V - 2E$.

3. **Infinite graph question.** (30 POINTS) Consider the infinite graph K in $\mathbf{R}^3$ with vertex set

$$\left\{(i,j,k) \in \mathbf{R}^3 \colon i,j,k \in \mathbf{Z}\right\} \cup \left\{\left(\frac{2i+1}{2}, \frac{2j+1}{2}, \frac{2k+1}{2}\right) \in \mathbf{R}^3 \colon i,j,k \in \mathbf{Z}\right\}$$

in which each vertex of the form (x,y,z) is connected by an edge to the eight neighboring vertices

$$\left\{\left(x \pm \frac{1}{2}, y \pm \frac{1}{2}, z \pm \frac{1}{2}\right)\right\}.$$

Thus the center of each edge is a point in the set

$$\left\{\left(i \pm \frac{1}{4}, j \pm \frac{1}{4}, k \pm \frac{1}{4}\right) : i,j,k \in \mathbf{Z}\right\}.$$

The two endpoints for such an edge with a given combination of signs are

$$(i,j,k) \qquad \text{and} \qquad \left(i \pm \frac{1}{2}, j \pm \frac{1}{2}, k \pm \frac{1}{2}\right)$$

with the same combination of signs in the second point. There is a corresponding surface M as in the previous two problems.

The group $G = \mathbf{Z}^3$ acts freely $\mathbf{R}^3$ by translation, with $(i,j,k) \in \mathbf{Z}^3$ sending $(x,y,z) \in \mathbf{R}^3$ to $(x+i, y+j, z+k)$. Hence it acts freely on both K and M. *Describe the finite orbit graph K/G and find the genus of the compact orbit surface M/G.* Both K/G and M/G are contained in the 3-dimensional torus $\mathbf{R}^3/G \cong S^1 \times S^1 \times S^1$, which is also a quotient of the unit cube.

Solution: The orbit graph has two vertices, the orbits of

$$(0,0,0) \qquad \text{and} \qquad \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

They are connected to each other by 8 edges, the orbits of the ones centered at the points

$$\left(\pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}\right).$$

hence $V = 2$ and $E = 8$. The result of problem 2 implies that $\chi(M) = 2V - 2E = -12$, so the genus of M is 7.

Suppose we take the cube $[-1/2, 1/2]^3$ as a fundamental domain for the group action on $\mathbf{R}^3$. Then the point $(0,0,0)$ is its center and each vertex maps to the orbit of $(1/2, 1/2, 1/2)$. The edges of K/G correspond to the 8 lines connecting the center of the cube to the cube's vertices.

4. (20 POINTS) **Nonplanar graph question.** Let K be the houses and utilities graph. It has six vertices, x_1, x_2, x_3, y_1, y_2, and y_3. Each x_i is connected to each y_j by an edge, so there are nine edges. Use an Euler characteristic argument to prove that K cannot be embedded in the plane. HINT: SHOW THAT EACH FACE MUST BE BOUNDED BY AT LEAST 4 EDGES.

Solution: The vertices of a face must be alternately houses and utilities since each edge connects a house to a utility. Hence each face has an even number of edges. The number cannot be two because we cannot have two edges connecting the same house to the same utility, so it must be at least four.

A spherical polyhedron with 6 vertices and 9 edges must have 5 faces in order to have Euler characteristic 2. The hint implies that $E \geq 2F$ since each edge belongs to 2 faces. This is a contradiction.

5. (20 POINTS) **Covering space question.** Let $\zeta = e^{2\pi i/3}$, let $\tilde{X}$ be the complement of the set

$$\left\{z_0 = 0, z_1 = 1, z_2 = \zeta, z_3 = \zeta^2\right\}$$

in $\mathbf{C}$ (the complex numbers), and let X be the complement of the set $\{0, 1\}$ in $\mathbf{C}$. Let $p : \tilde{X} \to X$ be defined by $p(z) = z^3$. Using the point $\tilde{x}_0 = 1/2 \in \tilde{X}$ as a base point, we define four closed paths ω_k for $0 \leq k \leq 3$ in $\tilde{X}$ as follows:

$$\begin{aligned}
\omega_0(t) &= e^{2\pi i t}/2 && \text{for } 0 \leq t \leq 1 \\
\omega_1(t) &= 1 - (e^{2\pi i t}/2) && \text{for } 0 \leq t \leq 1 \\
\omega_2(t) &= \begin{cases} e^{2\pi i t}/2 & \text{for } 0 \leq t \leq 1/3 \\ \zeta(1 - (e^{6\pi i t}/2)) & \text{for } 1/3 \leq t \leq 2/3 \\ e^{-2\pi i t}/2 & \text{for } 2/3 \leq t \leq 1 \end{cases} \\
\omega_3(t) &= \begin{cases} e^{-2\pi i t}/2 & \text{for } 0 \leq t \leq 1/3 \\ \zeta^2(1 - (e^{6\pi i t}/2)) & \text{for } 1/3 \leq t \leq 2/3 \\ e^{2\pi i t}/2 & \text{for } 2/3 \leq t \leq 1 \end{cases}
\end{aligned}$$

(I suggest you draw a picture of these paths.)

(a) (5 POINTS) Find $\pi_1(\tilde{X}, \tilde{x}_0)$ and describe the elements in it represented by the 4 closed paths ω_k.

Solution: Since $\tilde{X}$ is the complement of 4 points in the plane, its π_1 is the free group on 4 generators, say a_k for $0 \leq k \leq 3$. The four paths each go around one of them in a counterclockwise direction, so each ω_k represents one of the generators a_k.

(b) (5 POINTS) Show that p is a 3-sheeted covering.

Solution: The preimage of every every point in X is a set of three points in $\tilde{X}$.

(c) (5 POINTS) Let $x_0 = p(\tilde{x}_0) \in X$ and find $\pi_1(X, x_0)$. Describe the elements in it represented by the 4 closed paths $p\omega_k$. You may assume that the image under p of a circle of radius 1/2 about a cube root of unity is a simple closed curve going counterclockwise around 1 and not going around 0.

Solution: Since X is the complement of 2 points in the plane, its π_1 is the free group on 2 generators, say x and y corresponding to 0 and 1. Then drawing suitable pictures shows that

$$\begin{aligned} p(a_0) &= x^3 \\ p(a_1) &= y \\ p(a_2) &= xyx^{-1} \\ p(a_3) &= x^{-1}yx \end{aligned}$$

(d) (5 POINTS) Find a homomorphism $\varphi : \pi_1(X, x_0) \to C_3$ whose kernel contains $p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

Solution: Let $\gamma \in C_3$ be a generator, and define φ by $\varphi(x) = \gamma$ and $\varphi(y) = e$.