Math 443 **Final exam**

May 8, 2012

Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

1. (20 POINTS) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk D^2 to itself has a fixed point. You may assume $\pi_1 S^1 = \mathbf{Z}$.

   Solution: See page 32 of Hatcher.

2. (20 POINTS) Prove or disprove the Borsuk-Ulam theorem for the torus, which says the following. For every map $f : S^1 \times S^1 \to \mathbf{R}^2$, there is a point (x, y) in $S^1 \times S^1$ such that $f(-x, -y) = f(x, y)$. Here we regard S^1 as the unit circle in the complex numbers in order to define $-z$ for $z \in S^1$.

   Solution: The theorem is false. Let f be the composite of projection p_1 onto the first coordinate followed by an embedding i of S^1 into the plane. Then we have

$$f(-x, -y) = i(-x) \neq i(x) = f(x, y).$$

3. (30 POINTS) Let M_g be a closed oriented surface of genus g. Its homology is as follows.

$$H_i(M_g) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ \mathbf{Z}^{2g} & \text{for } i = 1 \\ \mathbf{Z} & \text{for } i = 2 \\ 0 & \text{for } i > 2 \end{cases}$$

Let $M_{g,k}$ be M_g with k disjoint open disks removed. Compute $H_*(M_{g,k})$ for $k > 0$ and prove your answer.

Solution:

We use the Mayer-Vietoris sequence in which $A = M(g,k)$, B is k copies of D^2 and $C = A \cap B$ is k copies of S^1. Then we have

$$\begin{array}{ccccccc}
 & 0 & & H_2(M_{g,k}) & & \mathbf{Z} & \\
 & \| & & \| & & \| & \\
\cdots \longrightarrow & H_2(C) & \longrightarrow & H_2(A) \oplus H_2(B) & \longrightarrow & H_2(M_g) & \xrightarrow{\partial_2} \\
 & \mathbf{Z}^k & & H_1(M_{g,k}) & & \mathbf{Z}^{2g} & \\
 & \| & & \| & & \| & \\
\longrightarrow & H_1(C) & \longrightarrow & H_1(A) \oplus H_1(B) & \longrightarrow & H_1(M_g) & \xrightarrow{\partial_1} \\
 & \mathbf{Z}^k & & H_0(M_{g,k}) \oplus \mathbf{Z}^k & & \mathbf{Z} & \\
 & \| & & \| & & \| & \\
\longrightarrow & H_0(C) & \longrightarrow & H_0(A) \oplus H_0(B) & \longrightarrow & H_0(M_g) & \longrightarrow 0
\end{array}$$

Now $M_{g,k}$ is not a closed manifold, so $H_2(M_{g,k}) = 0$ and ∂_2 is one-to-one. It is path connected so $H_0(M_{g,k}) = \mathbf{Z}$ and $\partial_1 = 0$. It follows that

$$H_1(M_{g,k}) = \mathbf{Z}^{2g+k-1}.$$

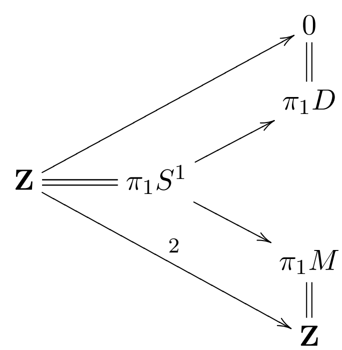
4. (20 POINTS) Let K be the houses and utilities graph. It has six vertices, x_1, x_2, x_3, y_1, y_2, and y_3. Each x_i is connected to each y_j by an edge, so there are nine edges. Use an Euler characteristic argument to prove that K cannot be embedded in the plane. HINT: SHOW THAT EACH FACE MUST BE BOUNDED BY AT LEAST 4 EDGES.

Solution: The vertices of a face must be alternately houses and utilities since each edge connects a house to a utility. Hence each face has na even number of edges. The number cannot be two because we cannot have two edges connecting the same house to the same utility, so it must be at least four.

A spherical polyhedron with 6 vertices and 9 edges must have 5 faces in order to have Euler characteristic 2. The hint implies that $E \geq 2F$ since each edge belongs to 2 faces. This is a contradiction.

5. (20 POINTS) Let M be the Möbius band and D the 2-dimensional disk. Let X be the quotient of $M \cup D$ obtained by identifying the bounding circles of M and D via a homeomorphism between them. Find $\pi_1(X)$.

Solution: According to the van Kampen theorem, $\pi_1 X$ is the pushout of fundamental groups in the diagram



This pushout is $\mathbf{Z}/2$ and X is the real projective plane.