Math 443 — Final exam

December 19, 2022

Name:

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ____

Problems begin below, and there are two blank pages to write your answer on following each of three problems.

1. **Infinite graph question.** (40 POINTS.) Consider the infinite graph K in $\mathbf{R}^3$ with vertex set
$$\{(i,j,k) \in \mathbf{R}^3 \colon i,j,k \in \mathbf{Z}\} \cup \left\{\left(\frac{2i+1}{2}, \frac{2j+1}{2}, \frac{2k+1}{2}\right) \in \mathbf{R}^3 \colon i,j,k \in \mathbf{Z}\right\}$$
in which each vertex of the form (x,y,z) is connected by an edge to the eight neighboring vertices
$$\left\{\left(x \pm \frac{1}{2}, y \pm \frac{1}{2}, y \pm \frac{1}{2}\right)\right\}.$$
Thus the center of each edge is a point in the set
$$\left\{\left(i \pm \frac{1}{4}, j \pm \frac{1}{4}, k \pm \frac{1}{4}\right) : i,j,k \in \mathbf{Z}\right\}.$$
The two endpoints for such an edge with a given combination of signs are
$$(i,j,k) \qquad \text{and} \qquad \left(i \pm \frac{1}{2}, j \pm \frac{1}{2}, k \pm \frac{1}{2}\right)$$
with the same combination of signs in the second point.

   Let L be the set of points within ϵ of K, for some positive $\epsilon < 1/4$. It is a noncompact compact 3-manifold with boundary in $\mathbf{R}^3$. Its boundary M is a noncompact surface.

   The group $G = \mathbf{Z}^3$ acts freely $\mathbf{R}^3$ by translation, with $(i,j,k) \in \mathbf{Z}^3$ sending $(x,y,z) \in \mathbf{R}^3$ to $(x+i, y+j, z+k)$. Hence it acts freely on both K and M. *Describe the finite orbit graph K/G and find the genus of the compact orbit surface M/G.* Both K/G and M/G are contained in the 3-dimensional torus $\mathbf{R}^3/G \cong S^1 \times S^1 \times S^1$, which is also a quotient of the unit cube.

Solution: The orbit graph has two vertices, the orbits of

$$(0,0,0) \qquad \text{and} \qquad \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

They are connected to each other by 8 edges, the orbits of the ones centered at the points

$$\left(\pm\frac{1}{4},\pm\frac{1}{4},\pm\frac{1}{4}\right).$$

hence $V = 2$ and $E = 8$. Thus the surface is fiormed by taking two copies of S^2 and attaching them with 8 tubes. Each tube, and each cicle where it meets either sphere, has Euler characteristic 0. It follows that the Euler characteristic of the surface is the same as that of the two spheres with 8 disks removed from each. Hence it is

$$2(2-8) = -12 = 2-2g,$$

so the genus g is 7.

Suppose we take the cube $[-1/2,1/2]^3$ as a fundamental domain for the group action on $\mathbf{R}^3$. Then the point $(0,0,0)$ is its center and each vertex maps to the orbit of $(1/2,1/2,1/2)$. The edges of K/G correspond to the 8 lines connecting the center of the cube to the cube's vertices.

Workspace for problem 1 continued.

Workspace for problem 1 continued.

2. **Complete bipartite graph question.** (30 POINTS.) A *bipartite graph* is one in which the vertices fall into two disjoint sets, say red and blue vertices, and each edge connects a red vertex to a blue one. It is *complete* if there is a unique edge connecting each red vertex to each blue one.

   Let $K_{m,n}$ denote the complete bipartite graph with m red vertices and n blue ones. Hence it has mn edges.

   Show that if $K_{m,n}$ can be embedded in a closed oriented surface of genus g, then

$$g \geq \frac{(m-2)(n-2)}{4}.$$

   In particular, $g > 0$, so the graph is nonplanar, for $m = n = 3$. $K_{3,3}$ is known as the houses and utilities graph.

Solution: If $K_{m,n}$ is embedded in such a surface, we get a polyhedron with $V = m + n$ vertices, $E = mn$ edges and F faces. If we add the number of edges on each face, we get $2mn$ since each edge is shared by two faces two faces. Each face must have at least four edges, so $2mn \geq 4F$ and $F \leq mn/2$. Thus the Euler characteristic of the surface is

$$\begin{aligned}
2 - 2g &= V - E + F = m + n - mn + F \\
&\leq m + n - mn + mn/2 = m + n - mn/2 \\
2 - m + n + mn/2 &\leq 2g \\
g &\geq \frac{2 - m + n + mn/2}{2} = \frac{(m-2)(n-2)}{4}
\end{aligned}$$

Workspace for problem 2 continued.

Workspace for problem 2 continued.

3. (30 POINTS) **Covering space question.** Let $\zeta = e^{2\pi i/3}$, let $\tilde{X}$ be the complement of the set

$$\{z_0 = 0, z_1 = 1, z_2 = \zeta, z_3 = \zeta^2\}$$

in $\mathbf{C}$ (the complex numbers), and let X be the complement of the set $\{0, 1\}$ in $\mathbf{C}$. Let $p : \tilde{X} \to X$ be defined by $p(z) = z^3$. Using the point $\tilde{x}_0 = 1/2 \in \tilde{X}$ as a base point, we define four closed paths ω_k for $0 \le k \le 3$ in $\tilde{X}$ as follows:

$$\begin{aligned}
\omega_0(t) &= e^{2\pi i t}/2 && \text{for } 0 \le t \le 1 \\
\omega_1(t) &= 1 - (e^{2\pi i t}/2) && \text{for } 0 \le t \le 1 \\
\omega_2(t) &= \begin{cases} e^{2\pi i t}/2 & \text{for } 0 \le t \le 1/3 \\ \zeta(1 - (e^{6\pi i t}/2)) & \text{for } 1/3 \le t \le 2/3 \\ e^{-2\pi i t}/2 & \text{for } 2/3 \le t \le 1 \end{cases} \\
\omega_3(t) &= \begin{cases} e^{-2\pi i t}/2 & \text{for } 0 \le t \le 1/3 \\ \zeta^2(1 - (e^{6\pi i t}/2)) & \text{for } 1/3 \le t \le 2/3 \\ e^{2\pi i t}/2 & \text{for } 2/3 \le t \le 1 \end{cases}
\end{aligned}$$

(I suggest you draw a picture of these paths.)

(a) (5 POINTS) Find $\pi_1(\tilde{X}, \tilde{x}_0)$ and describe the elements in it represented by the 4 closed paths ω_k.

> **Solution:** Since $\tilde{X}$ is the complement of 4 points in the plane, its π_1 is the free group on 4 generators, say a_k for $0 \le k \le 3$. The four paths each go around one of them in a counterclockwise direction, so each ω_k represents one of the generators a_k.

(b) (5 POINTS) Show that p is a 3-sheeted covering.

> **Solution:** The preimage of every every point in X is a set of three points in $\tilde{X}$.

(c) (5 POINTS) Let $x_0 = p(\tilde{x}_0) \in X$ and find $\pi_1(X, x_0)$. Describe the elements in it represented by the 4 closed paths $p\omega_k$. You may assume that the image under p of a circle of radius 1/2 about a cube root of unity is a simple closed curve going counterclockwise around 1 and not going around 0.

> **Solution:** Since X is the complement of 2 points in the plane, its π_1 is the free group on 2 generators, say x and y corresponding to 0 and 1. Then drawing suitable pictures shows that
>
> $$\begin{aligned} p(a_0) &= x^3 \\ p(a_1) &= y \\ p(a_2) &= xyx^{-1} \\ p(a_3) &= x^{-1}yx \end{aligned}$$

(d) (5 POINTS) Find a homomorphism $\varphi : \pi_1(X, x_0) \to C_3$ whose kernel contains $p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

Solution: Let $\gamma \in C_3$ be a generator, and define φ by $\varphi(x) = \gamma$ and $\varphi(y) = e$.

Workspace for problem 3 continued.

Workspace for problem 3 continued.