Math 443 **Final exam**

December 13, 2021

Name:

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

Problems begin below, and there are two blank pages to write your answer on following each of five problems.

1. **3-manifold homology question question.** (30 POINTS.) Find the homology of the 3-manifold obtained by "attaching k handles" to the 3-sphere S^3. "Attaching a handle" to a 3-manifold means M the following:

   - Remove two disjoint open disks from M, thus obtaining a manifold M' bounded by two copies of S^2.
   - The cylinder $S^2 \times I$ is another 3-manifold with the same boundary.
   - Form a new closed 3-manifold N by identifying the boundaries of M' and $S^2 \times I$.

   One can use the Mayer-Vietoris sequence to compute H_*M' in terms of H_*M, and H_*N in terms of H_*M'. You can assume that $H_3X = 0$ for a connected 3-manifold with boundary X, and that $H_3Y = \mathbf{Z}$ for a connected 3-manifold without boundary Y.

   Starting with S^3, do the above k times to obtain a 3-manifold M_k. Equivalently, one could remove $2k$ disjoint open disks from S^3 and identify the resulting boundary with that of k copies of $S^2 \times I$.

   Note that the 2-dimensional analog of this process leads from S^2 to a surface of genus k.

Workspace for problem 1 continued.

Workspace for problem 1 continued.

2. **Infinite graph question.** (30 POINTS.) Consider the infinite graph K in $\mathbf{R}^3$ with vertex set

$$\left\{(i,j,k) \in \mathbf{R}^3 \colon i,j,k \in \mathbf{Z}\right\} \cup \left\{\left(\frac{2i+1}{2}, \frac{2j+1}{2}, \frac{2k+1}{2}\right) \in \mathbf{R}^3 \colon i,j,k \in \mathbf{Z}\right\}$$

in which each vertex of the form (x,y,z) is connected by an edge to the eight neighboring vertices

$$\left\{\left(x \pm \frac{1}{2}, y \pm \frac{1}{2}, y \pm \frac{1}{2}\right)\right\}.$$

Thus the center of each edge is a point in the set

$$\left\{\left(i \pm \frac{1}{4}, j \pm \frac{1}{4}, k \pm \frac{1}{4}\right) : i,j,k \in \mathbf{Z}\right\}.$$

The two endpoints for such an edge with a given combination of signs are

$$(i,j,k) \qquad \text{and} \qquad \left(i \pm \frac{1}{2}, j \pm \frac{1}{2}, k \pm \frac{1}{2}\right)$$

with the same combination of signs in the second point.

Let L be the set of points within ϵ of K, for some positive $\epsilon < 1/4$. It is a noncompact compact 3-manifold with boundary in $\mathbf{R}^3$. Its boundary M is a noncompact surface.

The group $G = \mathbf{Z}^3$ acts freely $\mathbf{R}^3$ by translation, with $(i,j,k) \in \mathbf{Z}^3$ sending $(x,y,z) \in \mathbf{R}^3$ to $(x+i, y+j, z+k)$. Hence it acts freely on both K and M. *Describe the finite orbit graph K/G and find the genus of the compact orbit surface M/G.* Both K/G and M/G are contained in the 3-dimensional torus $\mathbf{R}^3/G \cong S^1 \times S^1 \times S^1$, which is also a quotient of the unit cube.

Workspace for problem 2 continued.

Workspace for problem 2 continued.

3. **Euler characteristic question.** (20 POINTS.) Let X be a finite graph with V vertices and E edges. Embed it in $\mathbf{R}^3$ (there is a theorem saying that any graph can be embedded in 3-space; there are some that cannot be embedded in the plane) and let Y be the space of all points within ϵ (a sufficiently small positive number) of the image of X. It is a 3-manifold bounded by a surface M. Find the Euler charcterisitic $\chi(M)$ and prove your answer.

   HINT: Think of the building set in the lounge, the one with steel balls and black magnetic rods. We are going to build something with V balls and E rods. Find the Euler characteristic of the set of V 2-spheres bounding the V balls. Think about how the Euler characteristic of the surface changes each time you add a rod. *You may use the fact that*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

   under suitable hypotheses on A and B.

Workspace for problem 3 continued.

Workspace for problem 3 continued.

4. **Complete bipartite graph question.** (20 POINTS.) A *bipartite graph* is one in which the vertices fall into two disjoint sets, say red and blue vertices, and each edge connects a red vertex to a blue one. It is *complete* if there is a unique edge connecting each red vertex to each blue one.

   Let $K_{m,n}$ denote the complete bipartite graph with m red vertices and n blue ones. Hence it has mn edges.

   Show that if $K_{m,n}$ can be embedded in a closed oriented surface of genus g, then

$$g \geq \frac{(m-2)(n-2)}{4}.$$

   In particular, $g > 0$, so the graph is nonplanar, for $m = n = 3$. $K_{3,3}$ is known as the houses and utilities graph.

Workspace for problem 4 continued.

Workspace for problem 4 continued.

5. **Brouwer Fixed Point question.** (20 POINTS) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk D^2 to itself has a fixed point. You may assume $\pi_1 S^1 = \mathbf{Z}$.

Workspace for problem 5 continued.

Workspace for problem 5 continued.