Math 443 **Final exam**

December 13, 2021

Name:

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ____________________

Problems begin below, and there are two blank pages to write your answer on following each of five problems.

1. **3-manifold homology question question.** (30 POINTS.) Find the homology of the 3-manifold obtained by "attaching k handles" to the 3-sphere S^3. "Attaching a handle" to a 3-manifold means M the following:

   - Remove two disjoint open disks from M, thus obtaining a manifold M' bounded by two copies of S^2.
   - The cylinder $S^2 \times I$ is another 3-manifold with the same boundary.
   - Form a new closed 3-manifold N by identifying the boundaries of M' and $S^2 \times I$.

   One can use the Mayer-Vietoris sequence to compute H_*M' in terms of H_*M, and H_*N in terms of H_*M'. You can assume that $H_3X = 0$ for a connected 3-manifold with boundary X, and that $H_3Y = \mathbf{Z}$ for a connected 3-manifold without boundary Y.

   Starting with S^3, do the above k times to obtain a 3-manifold M_k. Equivalently, one could remove $2k$ disjoint open disks from S^3 and identify the resulting boundary with that of k copies of $S^2 \times I$.

   Note that the 2-dimensional analog of this process leads from S^2 to a surface of genus k.

Solution: Let $A \subset S^3$ denote the complement of $2k$ open disks, and let $B \subset S^3$ denote their closure. This makes $A \cap B$ the disjoint union of $2k$ copies of S^2. It follows that the groups in the Mayer-Vietoris sequence are as in the following table.

i	$H_i(A\cap B)$	$H_iA \oplus H_iB$	H_iS^3
3	0	0	$\mathbf{Z}$
2	$\mathbf{Z}^{2k}$	H_2A	0
1	0	H_1A	0
0	$\mathbf{Z}^{2k}$	$H_0A \oplus \mathbf{Z}^{2k}$	$\mathbf{Z}$

From this we conclude that

$$H_iA = \begin{cases} \mathbf{Z}^{2k-1} & \text{for } i = 2 \\ 0 & \text{for } i = 1 \\ \mathbf{Z} & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now let C denote the disjoint union of k copies of $S^2 \times I$, so $M_k = A \cup C$ with $A \cap C = A \cap B$. Then the corresponding table is

i	$H_i(A \cap C)$	$H_iA \oplus H_iC$	H_iM_k
3	0	0	$\mathbf{Z}$
2	$\mathbf{Z}^{2k}$	$\mathbf{Z}^{2k-1} \oplus \mathbf{Z}^k$	H_2M_k
1	0	$0 \oplus 0$	H_1M_k
0	$\mathbf{Z}^{2k}$	$\mathbf{Z} \oplus \mathbf{Z}^k$	$\mathbf{Z}$

From this we conclude that

$$H_iM_k = \begin{cases} \mathbf{Z} & \text{for } i = 3 \\ \mathbf{Z}^k & \text{for } i = 2 \\ \mathbf{Z}^k & \text{for } i = 1 \\ \mathbf{Z} & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Workspace for problem 1 continued.

Workspace for problem 1 continued.

2. **Infinite graph question.** (30 POINTS.) Consider the infinite graph K in $\mathbf{R}^3$ with vertex set
$$\left\{(i,j,k)\in\mathbf{R}^3\colon i,j,k\in\mathbf{Z}\right\}\cup\left\{\left(\frac{2i+1}{2},\frac{2j+1}{2},\frac{2k+1}{2}\right)\in\mathbf{R}^3\colon i,j,k\in\mathbf{Z}\right\}$$
in which each vertex of the form (x,y,z) is connected by an edge to the eight neighboring vertices
$$\left\{\left(x\pm\frac{1}{2},y\pm\frac{1}{2},y\pm\frac{1}{2}\right)\right\}.$$
Thus the center of each edge is a point in the set
$$\left\{\left(i\pm\frac{1}{4},j\pm\frac{1}{4},k\pm\frac{1}{4}\right)\colon i,j,k\in\mathbf{Z}\right\}.$$
The two endpoints for such an edge with a given combination of signs are
$$(i,j,k)\qquad\text{and}\qquad\left(i\pm\frac{1}{2},j\pm\frac{1}{2},k\pm\frac{1}{2}\right)$$
with the same combination of signs in the second point.

Let L be the set of points within ϵ of K, for some positive $\epsilon<1/4$. It is a noncompact compact 3-manifold with boundary in $\mathbf{R}^3$. Its boundary M is a noncompact surface.

The group $G=\mathbf{Z}^3$ acts freely $\mathbf{R}^3$ by translation, with $(i,j,k)\in\mathbf{Z}^3$ sending $(x,y,z)\in\mathbf{R}^3$ to $(x+i,y+j,z+k)$. Hence it acts freely on both K and M. *Describe the finite orbit graph K/G and find the genus of the compact orbit surface M/G.* Both K/G and M/G are contained in the 3-dimensional torus $\mathbf{R}^3/G\cong S^1\times S^1\times S^1$, which is also a quotient of the unit cube.

Solution: The orbit graph has two vertices, the orbits of
$$(0,0,0)\qquad\text{and}\qquad\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$
They are connected to each other by 8 edges, the orbits of the ones centered at the points
$$\left(\pm\frac{1}{4},\pm\frac{1}{4},\pm\frac{1}{4}\right).$$
hence $V=2$ and $E=8$. The result problem 1 implies that $\chi(M)=2V-2E=-12$, so the genus of M is 7.

Suppose we take the cube $[-1/2,1/2]^3$ as a fundamental domain for the group action on $\mathbf{R}^3$. Then the point $(0,0,0)$ is its center and each vertex maps to the orbit of $(1/2,1/2,1/2)$. The edges of K/G correspond to the 8 lines connecting the center of the cube to the cube's vertices.

Workspace for problem 2 continued.

Workspace for problem 2 continued.

3. **Euler characteristic question.** (20 POINTS.) Let X be a finite graph with V vertices and E edges. Embed it in $\mathbf{R}^3$ (there is a theorem saying that any graph can be embedded in 3-space; there are some that cannot be embedded in the plane) and let Y be the space of all points within ϵ (a sufficiently small positive number) of the image of X. It is a 3-manifold bounded by a surface M. Find the Euler charcterisitic $\chi(M)$ and prove your answer.

   HINT: Think of the building set in the lounge, the one with steel balls and black magnetic rods. We are going to build something with V balls and E rods. Find the Euler characteristic of the set of V 2-spheres bounding the V balls. Think about how the Euler characteristic of the surface changes each time you add a rod. *You may use the fact that*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

   under suitable hypotheses on A and B.

Solution: The Euler characteristic of the disjoint union of V 2-spheres is $2V$. When we add an edge to the graph, we remove a disk from each of two (not necessarily distinct) spheres. This reduces χ by two. We then add a cyclinder by gluing its two boundary components to the two circles created by removing the two disks. This does not change χ, because both the cylinder and its boundary components have Euler characteristic zero. We do this E times, so $\chi(M) = 2V - 2E$.

Workspace for problem 3 continued.

Workspace for problem 3 continued.

4. **Complete bipartite graph question.** (20 POINTS.) A *bipartite graph* is one in which the vertices fall into two disjoint sets, say red and blue vertices, and each edge connects a red vertex to a blue one. It is *complete* if there is a unique edge connecting each red vertex to each blue one.

Let $K_{m,n}$ denote the complete bipartite graph with m red vertices and n blue ones. Hence it has mn edges.

Show that if $K_{m,n}$ can be embedded in a closed oriented surface of genus g, then

$$g \geq \frac{(m-2)(n-2)}{4}.$$

In particular, $g > 0$, so the graph is nonplanar, for $m = n = 3$. $K_{3,3}$ is known as the houses and utilities graph.

Solution: If $K_{m,n}$ is embedded in such a surface, we get a polyhedron with $V = m + n$ vertices, $E = mn$ edges and F faces. If we add the number of edges on each face, we get $2mn$ since each edge is shared by two faces two faces. Each face must have at least four edges, so $2mn \geq 4F$ and $F \leq mn/2$. Thus the Euler characteristic of the surface is

$$\begin{aligned} 2 - 2g = V - E + F &= m + n - mn + F \\ &\leq m + n - mn + mn/2 = m + n - mn/2 \\ 2 - m + n + mn/2 &\leq 2g \\ g &\geq \frac{2 - m + n + mn/2}{2} = \frac{(m-2)(n-2)}{4} \end{aligned}$$

Workspace for problem 4 continued.

Workspace for problem 4 continued.

5. **Brouwer Fixed Point question.** (20 POINTS) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk D^2 to itself has a fixed point. You may assume $\pi_1 S^1 = \mathbf{Z}$.

Solution: See page 32 of Hatcher.

Workspace for problem 5 continued.

Workspace for problem 5 continued.